Cayley-automatic semigroups

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Notation

A : a finite set of symbols.

A^* : the set of all (finite) words formed from the symbols in A (including the *empty word* ε).

If we take non-empty words (i.e. we omit ε) then we get A^+.

A^+ is a semigroup (under concatenation).

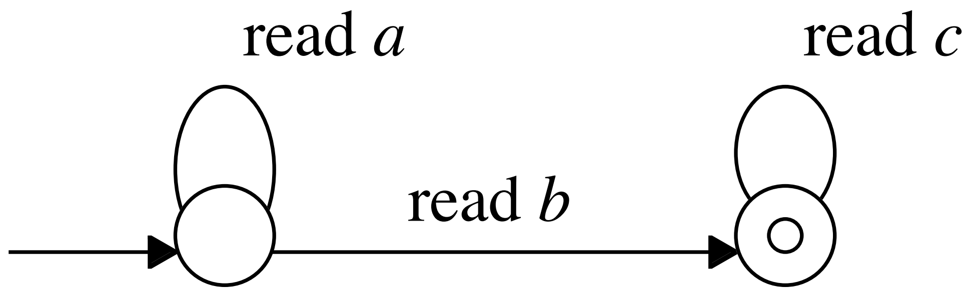
A^* is a monoid with identity ε.

If M is a monoid (respectively S is a semigroup) generated by a finite set A then there is a natural homomorphism $\varphi : A^* \to M$ (respectively $\varphi : A^+ \to S$).

A *language* is a subset of A^* (for some finite set A).

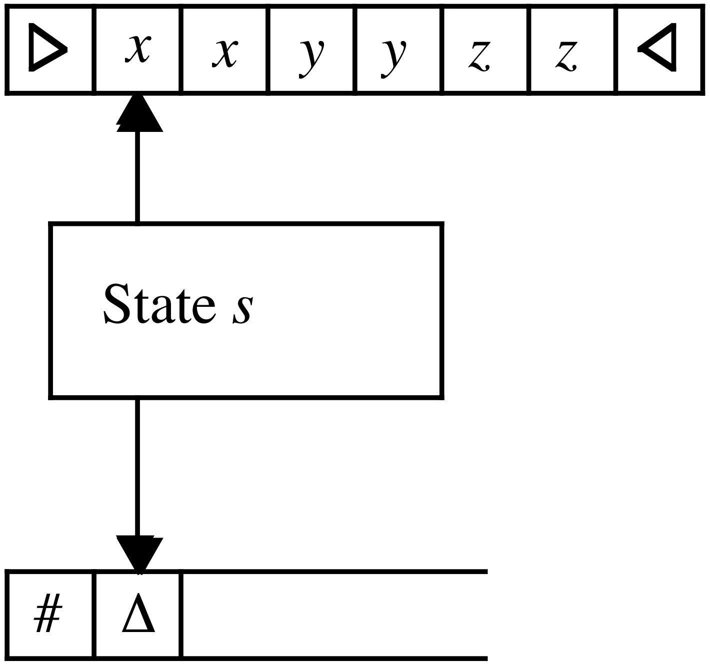
Regular languages are the languages accepted by *finite automata*.

A word α is *accepted* by an automaton M if α maps the start state to an accept state. For example, the finite automaton below accepts the language $\{a^n b c^m : n, m \in \mathbf{N}\}$:



Allowing nondeterminism here does not increase the set of languages accepted.

We can also consider a general model of computation such as a *Turing machine*.



Here we have some memory (in the form of a “work tape”) as well as the input.

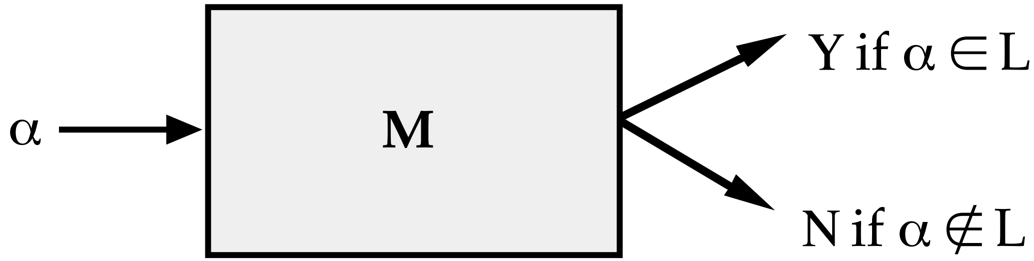
A Turing machine with a given input will either

(i) terminate (if it enters a halt state); or

(ii) hang (no legal move defined); or

(iii) run indefinitely without terminating.

We will take a *decision-making Turing machine* (one that always terminates and outputs true or



false) here (we are considering the class of *recursive languages*).

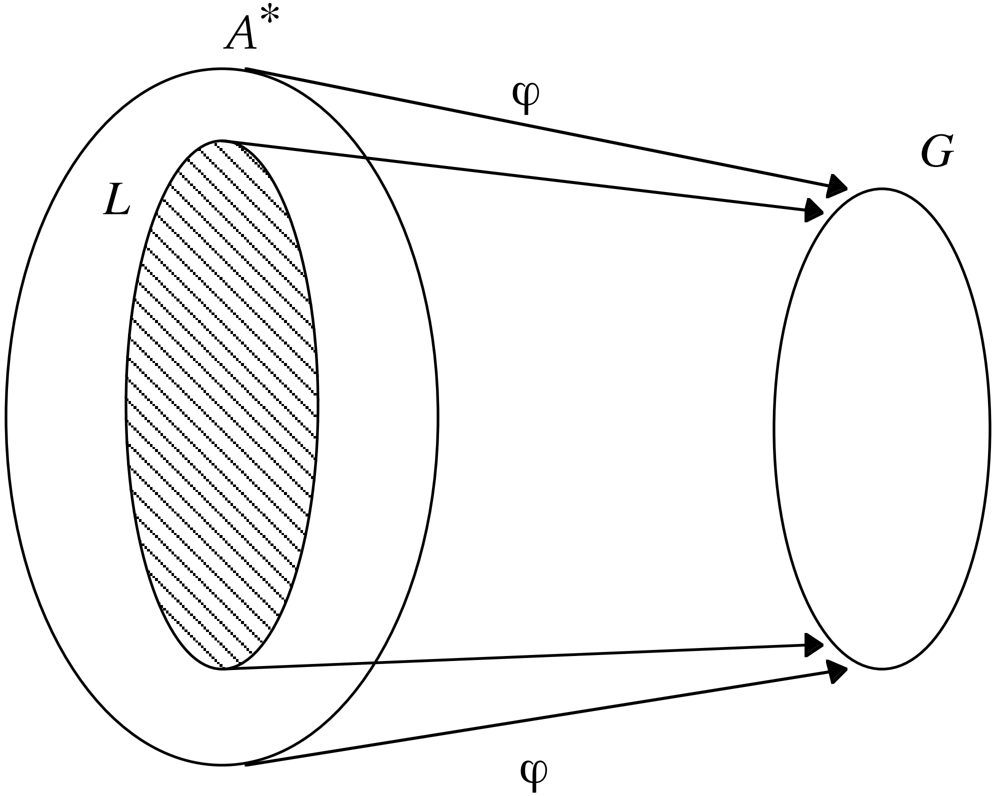
A structure $\mathcal{S} = (D, R_1, R_2, \ldots, R_n)$ consists of:

- a set D, called the *domain* of $\mathcal{S}$;
- relations $R_1, R_2, \ldots, R_n$ such that, for each i with $1 \le i \le n$, there exists $r = r_i \ge 1$ with R_i a subset of D^r; r is called the *arity* of the relation R_i.

A structure $\mathcal{S} = (D, R_1, R_2, \ldots, R_n)$ is said to be *computable* if:

- there is a set of symbols A such that $D \subseteq A^*$ and there is a decision-making Turing machine for D;
- for each R_i of arity r there is a decision-making Turing machine that, on input $(a_1, a_2, \ldots, a_r)$, outputs *true* if $a_i \in D$ for each i and if $(a_1, a_2, \ldots, a_r) \in R_i$ and outputs *false* otherwise.

Automatic groups



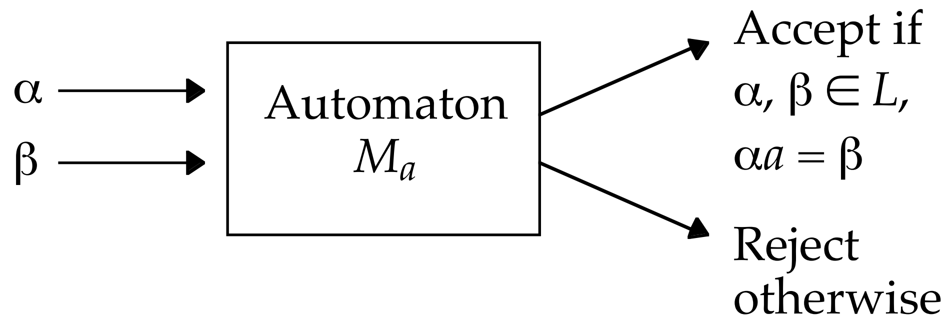
L is a regular subset of A^* (or A^+). The general idea is that "multiplication in the group G is recognized by automata".

When we talk about "accepting" a pair (or, more generally, a tuple) of words, we are "padding" the shorter words with a new symbol (say \$) to make the words all the same length:

a_1	a_2	a_3	....	a_n	\$	....	\$
b_1	b_2	b_3	....	b_n	b_{n+1}	....	b_m

We are thus reading the different words "synchronously".

For automatic groups, for each $a \in A$, there is a finite automaton M_a such that



Automaticity generalizes naturally to semigroups (but not to other structures in an obvious way).

There are many interesting examples of automatic groups and semigroups. For example, Zelmanov asked (Lisbon, 2011) whether plactic monoids were automatic and this was solved (in the affirmative) by A. J. Cain, R. D. Gray and A. Malheiro.

Another notion called *FA-presentability* was introduced by B. Khoussainov & A. Nerode; this applies to more general structures.

Automatic groups

D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson & W. P. Thurston, *Word Processing in Groups* (Jones and Bartlett, 1992).

φ is a homomorphism from A^+ to G; φ maps L bijectively to G.

For each $a \in A$ there is a finite automaton M_a such that M_a accepts a pair (α, β) of words if $\alpha, \beta \in L$ & $\alpha a = \beta$ (as group elements) and which rejects the pair (α, β) otherwise.

FA-presentable groups

B. Khoussainov & A. Nerode, Automatic presentations of structures, *in* D. Leivant (ed.), *Logic and Computational Complexity* (LNCS **960**, Springer-Verlag, 1995), 367-392.

A regular language L and a bijective mapping φ from L to G.

There is a finite automaton M such that M accepts a triple (α, β, γ) of words if $\alpha, \beta, \gamma \in L$ and $\alpha\beta = \gamma$ (as group elements) and which rejects the triple (α, β, γ) otherwise.

A structure $S = (D, R_1, R_2, \dots, R_n)$ is said to be *FA-presentable* if:

- there is a regular language L and a bijective map $\varphi : L \to D$;
- for each relation R_i of arity r, there is a finite automaton that accepts a tuple $(a_1, a_2, \dots, a_r)$ if and only if $a_p \in L$ for all p and $(a_1, a_2, \dots, a_r) \in R_i$.

If S is an FA-presentable structure then the first-order theory of S is decidable. B. Khoussainov & A. Nerode

There are not many classes where we have a complete characterization of FA-presentable structures:

An ordinal α is FA-presentable if and only if $\alpha < \omega^{\omega}$. C. Delhommé

An integral domain is FA-presentable if and only if it is finite.

B. Khoussainov, A. Nies, S. Rubin & F. Stephan

An infinite Boolean algebra is FA-presentable if and only if it of the form $\mathcal{B}^n$ (some $n \in \mathbf{N}$), where $\mathcal{B}$ is the Boolean algebra of finite and cofinite subsets of $\mathbf{N}$. B. Khoussainov, A. Nies, S. Rubin & F. Stephan

A finitely generated group is FA-presentable if and only if it is virtually abelian. G. P. Oliver & R. M. Thomas

Consequence: if a finitely generated group is FA-presentable then it is automatic (but the converse is false). What about semigroups?

A finitely generated commutative semigroup:

- need not be automatic. M. Hoffmann & R. M. Thomas
- is FA-presentable. A. J. Cain, N. Ruskuc, G. P. Oliver & R. M. Thomas

So a finitely generated FA-presentable semigroup need not be automatic.

A finitely generated cancellative semigroup is FA-presentable if and only if it embeds in a (finitely generated) virtually abelian group. A. J. Cain, N. Ruskuc, G. P. Oliver & R. M. Thomas

There is a finitely generated non-automatic semigroup that is a subsemigroup of a virtually abelian group; so a finitely generated cancellative FA-presentable semigroup need not be automatic. A. J. Cain

Let Γ be a graph with vertex set V and edge set E. Form a semigroup S with elements $V \cup \{e, 0\}$ $(e, 0 \notin V)$ and multiplication

$$uv = e \text{ if } u, v \in V \ \&\ \{u, v\} \in E; \qquad uv = 0 \text{ if } u, v \in V \ \&\ \{u, v\} \notin E;$$
$$se = es = s0 = 0s = 0 \text{ for all } s \in S.$$

The resulting semigroup S is FA-presentable if and only if the graph Γ is FA-presentable (and isomorphism is preserved). The isomorphism problem for FA-presentable graphs is undecidable.

B. Khoussainov, A. Nies, S. Rubin & F. Stephan

Given a group G with a finite set of generators $A = \{a_1, \dots , a_n\}$, we form a new structure $\mathcal{G} = (G, R_1, \dots , R_n)$ where $(g, h) \in R_i$ if and only if $ga_i = h$; this is called the *Cayley graph* of G with respect to A.

If G is an automatic group then we have an encoding of the elements of G as words in A^* such that there are finite automata recognizing multiplication by elements of A.

So, if G is an automatic group then the Cayley graph $\mathcal{G}$ is FA-presentable (but the converse is false).

G finitely generated FA-presentable $\Rightarrow$ G automatic

$\Rightarrow$ $\mathcal{G}$ FA-presentable

We say that a finitely generated group G is *CGA* (*Cayley graph automatic*) if its Cayley graph $\mathcal{G}$ is FA-presentable.

This generalizes naturally to finitely generated semigroups.

S finitely generated FA-presentable	$\Rightarrow$	S CGA
S automatic	$\Rightarrow$	S CGA

What can we say about CGA groups and semigroups?

If G is a CGA group then the word problem for G can be solved in quadratic time. O. Kharlampovich, B. Khoussainov & A. Miasnikov

This result generalizes to CGA semigroups.

A. J. Cain, R. Carey, N. Ruskuc & R. M. Thomas

Cayley graph automaticity for groups is preserved under:

- finite extensions;
- direct products;
- free products;
- finitely generated regular subgroups;
- certain semidirect products;
- certain amalgamated free products;

O. Kharlampovich, B. Khoussainov & A. Miasnikov

- wreath products with $\mathbf{Z}$; D. Berdinsky & B. Khoussainov

So CGA groups are not necessarily finitely presented.

Some interesting examples of CGA groups:

finitely generated nilpotent groups of class at most 2;

O. Kharlampovich, B. Khoussainov & A. Miasnikov

Baumslag-Solitar groups $< a, t : t^{-1}a^m t = a^n >$

D. Berdinsky & B. Khoussainov

The conjugacy problem is undecidable for CGA groups.

The isomorphism problem is undecidable for CGA groups.

A. Miasnikov & Z Sunic

CGA semigroups. Joint work with A. J. Cain, R. Carey & N. Ruskuc

Cayley graph automaticity for semigroups is preserved under:

- subsemigroups of finite Rees index;
- finitely generated regular subsemigroups;
- direct products (if the product is finitely generated);
- certain semidirect products;
- free products;
- zero unions;
- finitely generated Rees matrix semigroups.

Some complete classifications (for example, when a strong semilattice of semigroups is a CGA semigroup).

Many open questions here – work in progress!

A structure $S = (D, R_1, \ldots, R_n)$ is said to be unary *FA-presentable* if:

- there is a regular language L over an alphabet consisting of one symbol and a bijective map $\varphi : L \to D$;
- for each relation R_i of arity r, there is a finite automaton that accepts a tuple $(a_1, a_2, \ldots, a_r)$ if and only if $a_p \in L$ for all p and $(a_1, a_2, \ldots, a_r) \in R_i$.

Which structures are unary FA-presentable?

Cancellative unary FA-presentable semigroups are finite.

(This generalizes a previous result for groups by A. Blumensath.)

Finitely generated unary FA-presentable semigroups are finite.

(In general, unary FA-presentable semigroups are locally finite.)

A. J. Cain, N, Ruskuc & R. M. Thomas

What about unary CGA semigroups?

A cancellative semigroup is unary CGA if and only if it embeds into a virtually cyclic group.

A. J. Cain, R. Carey, N. Ruskuc & R. M. Thomas

Thank you!