Generating sets for powers of finite algebras and the complexity of quantified constraints

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Acidissima gens, optima consulens, pessima faciens

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Let us study the growth rate of generating sets for direct powers of an algebra $\mathbb{A}$.

For $\mathbb{A}$ we have a function $f_{\mathbb{A}} : \mathbb{N} \to \mathbb{N}$, giving the cardinality of the minimal generating sets of the sequence

- $\mathbb{A}, \mathbb{A}^2, \mathbb{A}^3, \ldots$ as
- $f(1), f(2), f(3), \ldots$.

We say $\mathbb{A}$ has the g-GP if $f(m) \leq g(m)$ for all m.

(PGP) polynomial, when $f_{\mathbb{A}} = O(i^c)$, for some c; and

(EGP) exponential, when exists b so that $f_{\mathbb{A}} = \Omega(b^i)$.

History

Theorem (Wiegold 1987)

Let $\mathbb{B}$ be a finite semigroup. If $\mathbb{B}$ is a monoid then $\mathbb{B}$ has the (linear) PGP. Otherwise, $\mathbb{B}$ has the EGP.

Proof of PGP.

If $\mathbb{B}$ is a monoid with identity 1 and $|B| = n$, then

$$\begin{gathered}(B, 1, \ldots, 1, 1)\\(1, B, \ldots, 1, 1)\\\vdots\\(1, 1, \ldots, B, 1)\\(1, 1, \ldots, 1, B)\end{gathered}$$

is a generating set for $\mathbb{B}^m$ of size mn.

Theorem (Wiegold 1987)

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Proof of EGP.

Otherwise, without an identity, $\mathbb{B}$ and $\mathbb{B}^m$ have the properties that

$$|x \cdot B| \leq n-1, \text{ for each } x \in B.$$
$$|z \cdot B^m| \leq (n-1)^m, \text{ for each } z \in B^m.$$

Thus, a subset of B^m of size r can generate no more $r + r(n-1)^m$ elements in $\mathbb{B}^m$. Thus, a generating set must be of size $\geq \left(\frac{2n}{2n-1}\right)^m$.

Constraint Satisfaction Problems

The *constraint satisfaction problem* (CSP) is a popular formalism in Artificial Intelligence in which one is given

- a triple $(V, D, \mathcal{C})$ of variables, domain, constraints

and in which one asks for an assignment of the variables to the domain that satisfies the constraints.

A popular parameterisation involves fixing D and restricting

- the constraint language $\mathcal{C}$.

This can be formulated combinatorially as CSP($\mathcal{C}$) with

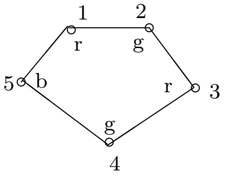
- Input: a structure $\mathcal{A}$.
- Question: does $\mathcal{A}$ have a homomorphism to $\mathcal{C}$?

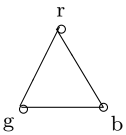
or logically as CSP($\mathcal{C}$) with

- Input: a sentence ϕ of $\{\exists, \wedge, =\}$-FO.
- Question: does $\mathcal{C} \models \phi$?

Example

$\mathrm{CSP}(\mathcal{K}_3)$, or $\mathrm{CSP}(\{r, g, b\}; \neq)$, is *Graph 3-colourability*.





Combinatorially, one looks for a homomorphism from $\mathcal{C}_5$ to $\mathcal{K}_3$.
Logically, one asks if $\mathcal{K}_3 \models \Phi$.

$$\begin{aligned}\Phi := \exists v_1, v_2, v_3, v_4, v_5 \quad & E(v_1, v_2) \wedge E(v_2, v_1) \wedge E(v_2, v_3) \wedge E(v_3, v_2) \\ & E(v_3, v_4) \wedge E(v_4, v_3) \wedge E(v_4, v_5) \\ & E(v_5, v_4) \wedge E(v_5, v_1) \wedge E(v_!, v_5).\end{aligned}$$

Quantified Constraint Satisfaction

The *quantified constraint satisfaction problem* QCSP($\mathcal{B}$) has

- Input: a sentence ϕ of $\{\forall, \exists, \wedge, =\}$-FO.
- Question: does $\mathcal{B} \models \phi$?

It is the CSP with $\forall$ returned.

"The QCSP might be thought of as the dissolute younger brother of its better-studied restriction, the CSP. . . . CSPs are ubiquitous in CS . . . , while QCSPs can not nearly claim to be so important in applications."

useful QCSPs	classified?
relativised ($\forall x \in X, \exists y \in Y$)	$\surd$
Boolean (QBF or QSAT)	$\surd$

". . . what is left of the true non-Boolean QCSP is a problem I believe to be mostly of interest to theorists."

First-order structures

Relational structures:

$$\mathcal{B} := (B; R_1, R_2, \ldots)$$

Functional structures:

$$\mathbb{B} := (D; f_1, f_2, \ldots)$$

$$\text{functional structures} \quad = \quad \text{algebras.}$$

What is the interplay between relational and functional structures?

$$\text{Model Theory} \quad = \quad \text{Logic} \quad + \quad \text{Universal Algebra}$$

All our structures are finite-domain.

Interplay

Let R be an m-ary relation on $\mathcal{B}$. We say that a k-ary operation $f : B^k \to B$ *preserves* R (or R is *invariant*) under f if:

$$
\begin{array}{llll}
f, & f, & \ldots, & f \\
(x_{11}, & x_{12}, & \ldots, & x_{1m}) \in R \\
(x_{21}, & x_{22}, & \ldots, & x_{2m}) \in R \\
\vdots & \vdots & & \vdots \\
(x_{k1}, & x_{k2}, & \ldots, & x_{km}) \in R \\
\hline
(y_1, & y_2, & \ldots, & y_m) \in R
\end{array}
$$

where each $y_i = f(x_{1i}, x_{2i}, \ldots, x_{ki})$.

- operations that preserve each of the relations of $\mathcal{B}$ are $\mathrm{Pol}(\mathcal{B})$
- relations invariant under each operation of $\mathbb{B}$ are $\mathrm{Inv}(\mathbb{B})$.

one-side of a Galois Correspondence

Let $\mathcal{B}$ and $\mathbb{B}$ be over the same finite domain B.

$$\mathrm{Inv}(\mathrm{Pol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\{\exists,\wedge,=\}}$$
$$\mathrm{Inv}(\mathrm{surPol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\{\forall,\exists,\wedge,=\}}$$

Idempotent operations are surjective! The algebraic definition for QCSP($\mathbb{B}$) has

- Input: a sentence ϕ of $\{\forall, \exists, \wedge\}$-FO with some relations $\mathcal{B} \in \mathrm{Inv}(\mathbb{B})$.
- Question: does $\mathcal{B} \models \phi$?

What if $\mathrm{Inv}(\mathbb{B})$ is infinite?

Infinite languages on a finite domain

Each relation R can be given as a list of tuples, but this is far too lengthy! How about a Boolean formula ϕ in atoms

- $v = v'$ and $v = c$,

where c is a domain element. The problem is that recognising, e.g., non-emptiness of the relation can be NP-hard! Following others, e.g. [Bodirsky & Dalmau 2006] we will ask for

- ϕ in DNF,

However, our main result will be a separation NP versus co-NP-hard, so this is not a big deal!

Infinite languages on a finite domain

Example 1.

$$\begin{array}{llll} \{ & (1,2), & (2,1), & \qquad (x \neq y \vee x = 1) \\ & (2,3), & (3,2), & \\ & (1,3), & (3,1), & \\ & (1,1) & \qquad \} & \end{array}$$

Example 2.

$$\begin{array}{lllll} \{ & (1,0,0), & (0,1,0), & (0,0,1), & \qquad (x \neq y \vee y \neq z) \\ & (1,1,0), & (1,0,1), & (1,1,0), & \} \end{array}$$

Back to PGP

Call an algebra $\mathbb{B}$ *k*-PGP-switchable if $\mathbb{B}^m$ is generated from the set of m-tuples of the form

- $(x_1, \ldots, x_1, x_2, \ldots x_2, \ldots, \ldots, x_{k'}, \ldots, x_{k'})$ for some $k' \leq k$.

switchability were originally introduced in connection with the QCSP by Hubie Chen!

Theorem (Chen 2008)

If $\mathbb{A}$ is switchable then QCSP($\mathbb{A}$) is in NP.

Theorem (LICS 2015)

$\mathbb{A}$ is PGP-switchable iff it is switchable.

A number of algebraists worked on the PGP-EGP dichotomy conjecture.

Conjecture

Let $\mathbb{B}$ be a finite idempotent algebra, then either $\mathbb{B}$ has PGP or it has EGP.

In 2015, Dmitriy Zhuk solved it.

Theorem (Zhuk 2015)

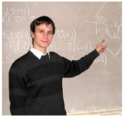
Let $\mathbb{B}$ be a finite algebra, then either $\mathbb{B}$ is PGP-switchable or it has EGP.

In order to prove this result, Zhuk assumes $\mathbb{B}$ is not PGP-switchable and finds the existence of a certain class of relations in $\mathrm{Inv}(\mathbb{B})$.

Church of Switchability

B. M.

switchability



- H. Chen: *Quantified constraint satisfaction and the polynomially generated powers property*. ICALP 2008.
- D. Zhuk: *The Size of Generating Sets of Powers*. Arxiv 2015.

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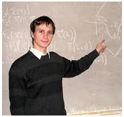  

- H. Chen: *Quantified constraint satisfaction and the polynomially generated powers property*. ICALP 2008.
- D. Zhuk: *The Size of Generating Sets of Powers*. Arxiv 2015.
- C. Carvalho, F. Madelaine, B. M.: *From Complexity to Algebra and Back: Digraph Classes, Collapsibility, and the PGP*. LICS 2015.

Notes & Queries

Henceforth, let $\mathbb{A}$ be an idempotent algebra on a finite domain A.

Conjecture (Chen Conjecture 2012)

Let $\mathcal{B}$ be a finite relational structure expanded with all constants. If Pol($\mathcal{B}$) has PGP, then QCSP($\mathcal{B}$) is in NP; otherwise QCSP($\mathcal{B}$) is Pspace-complete.

Theorem (Revised Chen Conjecture)

If $\mathrm{Inv}(\mathbb{A})$ satisfies PGP, then QCSP($\mathrm{Inv}(\mathbb{A})$) is in NP. Otherwise, if $\mathrm{Inv}(\mathbb{A})$ satisfies EGP, then QCSP($\mathrm{Inv}(\mathbb{A})$) is co-NP-hard.

Conjecture (Alternative Chen Conjecture)

If $\mathrm{Inv}(\mathbb{A})$ satisfies PGP, then for every finite reduct $\mathcal{B} \subseteq \mathrm{Inv}(\mathbb{A})$, QCSP($\mathcal{B}$) is in NP. Otherwise, there exists a finite reduct $\mathcal{B} \subseteq \mathrm{Inv}(\mathbb{A})$ so that QCSP($\mathcal{B}$) is co-NP-hard.

Notes & Queries

Henceforth, let $\mathbb{A}$ be an idempotent algebra on a finite domain A.

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Let $\mathcal{B}$ be a finite relational structure expanded with all constants. If Pol($\mathcal{B}$) has PGP, then QCSP($\mathcal{B}$) is in NP; otherwise QCSP($\mathcal{B}$) is Pspace-complete.

Theorem (Revised Chen Conjecture)

Either QCSP($\mathrm{Inv}(\mathbb{A})$) is co-NP-hard or QCSP($\mathrm{Inv}(\mathbb{A})$) has the same complexity as CSP($\mathrm{Inv}(\mathbb{A})$).

Conjecture (Alternative Chen Conjecture False)

If $\mathrm{Inv}(\mathbb{A})$ satisfies PGP, then for every finite reduct $\mathcal{B} \subseteq \mathrm{Inv}(\mathbb{A})$, QCSP($\mathcal{B}$) is in NP. Otherwise, there exists a finite reduct $\mathcal{B} \subseteq \mathrm{Inv}(\mathbb{A})$ so that QCSP($\mathcal{B}$) is co-NP-hard.

Tractability

We know from Zhuk 2015 that

$$\mathrm{PGP} \longrightarrow \text{PGP-switchability}$$

and from [LICS 2015]

$$\text{PGP-switchability} \longrightarrow \text{switchability}$$

whereupon Chen 2008 gives

$$\text{switchability} \longrightarrow \text{QCSP tractability.}$$

Henceforth, α, β be strict subsets of A so that $\alpha \cup \beta = A$.

Theorem (Zhuk 2015)

Algebra $\mathbb{A}$ (idempotent) has EGP iff exists such α, β with

$$\sigma_k(x_1, y_1, \ldots, x_k, y_k) := \rho(x_1, y_1) \vee \ldots \vee \rho(x_k, y_k),$$

where $\rho(x, y) = (\alpha \times \alpha) \cup (\beta \times \beta)$, is in $\mathrm{Inv}(\mathbb{A})$, for each $k \in \mathbb{N}$.

We prefer the relation $\tau_k(x_1, y_1, z_1 \ldots, x_k, y_k, z_k)$ defined by

$$\tau_k(x_1, y_1, z_1 \ldots, x_k, y_k, z_k) := \rho'(x_1, y_1, z_1) \vee \ldots \vee \rho'(x_k, y_k, z_k),$$

where $\rho'(x, y, z) = (\alpha \times \alpha \times \alpha) \cup (\beta \times \beta \times \beta)$.

Corollary

Algebra $\mathbb{A}$ (idempotent) has EGP iff exists such α, β with $\tau_k(x_1, y_1, z_1 \ldots, x_k, y_k, z_k)$ in $\mathrm{Inv}(\mathbb{A})$, for each $k \in \mathbb{N}$.

co-NP-hardness

Theorem

If $\mathrm{Inv}(\mathbb{A})$ *satisfies EGP, then QCSP*$(\mathrm{Inv}(\mathbb{A}))$ *is co-NP-hard.*

Proof.

Reduce from the complement of (monotone) 3-*not-all-equal-sat*.

$$\exists x_1^1, x_1^2, x_1^3, \ldots, \ldots, x_m^1, x_m^2, x_m^3 \ \mathrm{NAE}(x_1^1, x_1^2, x_1^3) \wedge \ldots \wedge \mathrm{NAE}(x_m^1, x_m^2, x_m^3)$$

becomes

$$\forall x_1^1, x_1^2, x_1^3, \ldots, \ldots, x_m^1, x_m^2, x_m^3 \ \rho'(x_1^1, x_1^2, x_1^3) \vee \ldots \vee \rho'(x_m^1, x_m^2, x_m^3)$$

where we note that $\tau_m(x_1, y_1, z_1 \ldots, x_m, y_m, z_m) :=$

$$\rho'(x_1, y_1, z_1) \vee \ldots \vee \rho'(x_m, y_m, z_m)$$

has a DNF representation that is polynomially-sized in m.

Recall, α, β be strict subsets of A so that $\alpha \cup \beta = A$. Now ask further that $\alpha \cap \beta \neq \emptyset$.

Corollary

$QCSP(A; \{\tau_n : n \in \mathbb{N}\}, \{a : a \in A\})$ is co-NP-hard.

In fact,

Proposition

$QCSP(A; \{\tau_n : n \in \mathbb{N}\}, \{a : a \in A\})$ is in co-NP.

Proof.

Roughly speaking, evaluate all existential variables to something in $\alpha \cap \beta$. $\square$

Proposition

For every finite reduct $\mathcal{B}$ of $(A; \{\tau_n : n \in \mathbb{N}\}, \{a : a \in A\})$, $QCSP(\mathcal{B})$ is in NL.

Conjecture

Let $\mathbb{A}$ *be an algebra. Either*

- *QCSP*$(\mathrm{Inv}(\mathbb{A}))$ *is in NP, or*
- *QCSP*$(\mathrm{Inv}(\mathbb{A}))$ *is co-NP-complete, or*
- *QCSP*$(\mathrm{Inv}(\mathbb{A}))$ *is Pspace-complete.*

Or even

Conjecture

Let $\mathbb{A}$ *be an algebra on a 3-element domain. Either*

- *QCSP*$(\mathrm{Inv}(\mathbb{A}))$ *is in NP, or*
- *QCSP*$(\mathrm{Inv}(\mathbb{A}))$ *is co-NP-complete, or*
- *QCSP*$(\mathrm{Inv}(\mathbb{A}))$ *is Pspace-complete.*

3-element vignette

The closest we can do is

Theorem

Let $\mathbb{A}$ be an algebra on a 3-element domain. Either

- Π_k*-CSP*$(\mathrm{Inv}(\mathbb{A}))$ *is in NP, for all* k*; or*
- Π_k*-CSP*$(\mathrm{Inv}(\mathbb{A}))$ *is co-NP-complete, for all* k*; or*
- Π_k*-CSP*$(\mathrm{Inv}(\mathbb{A}))$ *is* Π_2^{P}*-hard, for some* k*.*