Some semigroup theory on some endomorphism monoids of relational first-order structures

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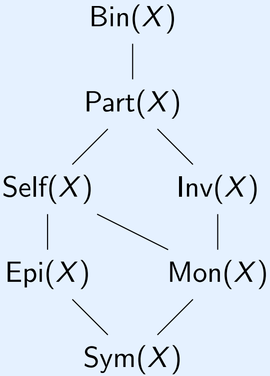
Famous semigroups

Let X be a countable set.



If X is finite, then $\mathrm{Epi}(X) = \mathrm{Mon}(X) = \mathrm{Sym}(X)$.

Relational structures

Let M be a set. A **relational signature** σ contains a set $\{\bar{R}_i \ : \ i \in I\}$ of relation symbols indexed by some countable set I, where each $\bar{R}_i$ of these symbols is assigned an **arity** $n_i \in \mathbb{N}$. In addition, σ contains variables ranging over M and logical connectives (including equality), quantifiers and punctuation.

A σ-**structure** $\mathcal{M}$ is a set M (called the **domain**) together with subsets $R_i^{\mathcal{M}} \subseteq M^{n_i}$ **interpreting** each relation $\bar{R}_i \in \sigma$. If $\bar{x}$ is an n_i-tuple of M, say that $\bar{R}_i(\bar{x})$ **holds** in $\mathcal{M}$ if and only if $\bar{x} \in R_i^{\mathcal{M}}$. (We often drop the bar on $\bar{R}_i \in \sigma$.)

A function $\alpha : \mathcal{M} \to \mathcal{N}$ between two σ-structures is a **homomorphism** if for all $\bar{x} \in R_i^{\mathcal{M}}$ then $\bar{x}\alpha \in R_i^{\mathcal{N}}$. A function $\beta : \mathcal{M} \to \mathcal{N}$ is an **embedding** when $\bar{x} \in R_i^{\mathcal{M}}$ if and only if $\bar{x}\beta \in R_i^{\mathcal{N}}$.

Distinguishing relational structures

Example 1

Let σ be the signature containing a single binary relation R. Then a σ-structure $\mathcal{M}$ could be a graph, a digraph, a poset... (not very useful!)

What differentiates any two of these structures is the rules that they satisfy. A σ-**sentence** is a string of characters from σ with no unquantified variables. Say that a σ-structure **models** a σ-sentence ϕ (or set of σ-sentences) if the sentence is true in $\mathcal{M}$.

Example 2

Let σ be as in Example 1. Suppose that

$$\phi = (\forall x, y)(\neg R(x,x) \wedge (R(x,y) \to R(y,x))).$$

Any model of ϕ is a simple, undirected graph.

Key difference between sets and relational structures

Let X be a countable set. A bog-standard transformation $f : X \to X$ could do one of the following two things:

- send two points $x, y \in X$ to the same point $xf = yf$ in X.
- omit a point from the image; there may exist $x \in X$ such that $x \notin Xf$.

The properties of the function are dependent on these; for instance, if there is no such pair $x \neq y$ where $xf = yf$, then f is injective.

As every endomorphism $\alpha : \mathcal{M} \to \mathcal{M}$ is a function, this behaviour can also be observed in this setting. However, there is also the possibility of a further type of behaviour:

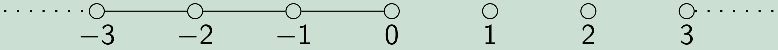
- non-relations of $\mathcal{M}$ changed to relations; so $\bar{x} \notin R_i^{\mathcal{M}}$ but $\bar{x}\alpha \in R_i^{\mathcal{M}}$.

This difference expands the range of potential monoids associated to $\mathcal{M}$.

Example behaviour

Example 3

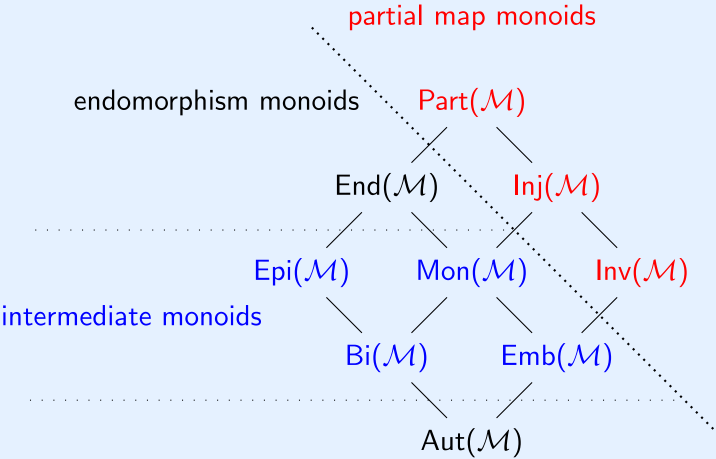
Let $\mathcal{M}$ be a graph with vertex set $\mathbb{Z}$ and adjacencies $i \sim j$ if and only if $i \leq 0$ and $j = i - 1$.



The function $\alpha : \mathcal{M} \to \mathcal{M}$ defined by $i\alpha = i - 2$ for all $i \in \mathbb{Z}$ is an endomorphism, as every edge is preserved. Furthermore, the two nonedges $(0, 1)$ and $(1, 2)$ are changed to edges.

α is a bijection; but the inverse function $\alpha^{-1} : \mathcal{M} \to \mathcal{M}$ is **not** an endomorphism, as the edge $(-1, 0)$ becomes the nonedge $(1, 2)$.

Picture for relational structures



If $\mathcal{M}$ finite, then $\text{Mon}(\mathcal{M}) = \text{Epi}(\mathcal{M}) = \text{Bi}(\mathcal{M}) = \text{Emb}(\mathcal{M}) = \text{Aut}(\mathcal{M})$.
This is boring, so assume $\mathcal{M}$ is countably infinite.

Examples (1/3)

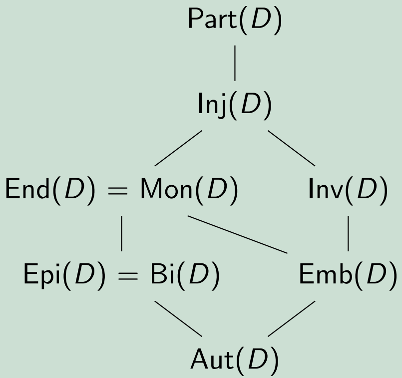
Example 4

(1) Let $\mathcal{M} = (\mathbb{Q}, <)$. Then every partial homomorphism preserves non-relations (as there are none) so is a partial isomorphism. Similarly, any endomorphism is an embedding. So the picture looks like this:

$$\begin{array}{c} \mathsf{Part}(\mathcal{M}) = \mathsf{Inj}(\mathcal{M}) = \mathsf{Inv}(\mathcal{M}) \\ | \\ \mathsf{End}(\mathcal{M}) = \mathsf{Mon}(\mathcal{M}) = \mathsf{Emb}(\mathcal{M}) \\ | \\ \mathsf{Epi}(\mathcal{M}) = \mathsf{Bi}(\mathcal{M}) = \mathsf{Aut}(\mathcal{M}) \end{array}$$

Example 5

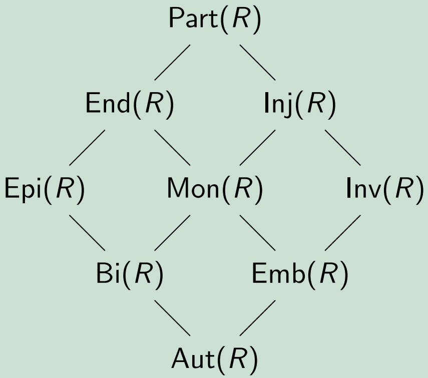
(2) Let $\mathcal{M} = D$, the generic digraph without 2-cycles. Then every endomorphism must be injective as to avoid the creation of 2-cycles. Partial homomorphisms need not be injective, however.



Examples (3/3)

Example 6

(3) Let $\mathcal{M} = R$, the random graph. There exist examples that distinguish each of the aforementioned monoids from each other; so for R, the picture looks like this:



Initial case: Mon(X)

Mon(X) is right-cancellative.

Definition 7

Let $\alpha \in \text{Mon}(X)$. Define the **defect** of α to be the set $D(\alpha) = X \smallsetminus X\alpha$. Write $d(\alpha) = |D(\alpha)|$, and say that $d(\alpha) = \infty$ if $X \smallsetminus X\alpha$ is infinite.

Here, $D(\alpha\beta) = D(\beta) \cup D(\alpha)\beta$ and this is a disjoint union. Following this, Mon(X) is not regular, and its only idempotent is the identity. The set $I_k = \{\alpha \in \text{Mon}(X) : d(\alpha) \geq k\}$ is an ideal of X.

Proposition 8

Let $\alpha, \beta \in \text{Mon}(X)$. Then:

(1) $\alpha \mathscr{L} \beta$ if and only if $D(\alpha) = D(\beta)$;

(2) $\alpha \mathscr{R} \beta$ if and only if $d(\alpha) = d(\beta)$, and hence $\mathscr{L} = \mathscr{H}$;

(3) $\alpha \mathscr{J} \beta$ if and only if $d(\alpha) = d(\beta)$, and hence $\mathscr{R} = \mathscr{D} = \mathscr{J}$.

First pieces of semigroup theory of $\mathrm{Bi}(\mathcal{M})$

Every bimorphism of $\mathcal{M}$ is a permutation of the domain M. So $\mathrm{Bi}(\mathcal{M}) \leq \mathrm{Sym}(M)$ and therefore it is a group-embeddable monoid. This means that we get lots of nice semigroup-theoretic properties for free...

Lemma 9

- $\mathrm{Bi}(\mathcal{M})$ is a cancellative monoid.
- The only idempotent of $\mathrm{Bi}(\mathcal{M})$ is the identity element e.
- $\mathrm{Bi}(\mathcal{M})$ is regular if and only if $\mathrm{Bi}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$. The only regular elements of $\mathrm{Bi}(\mathcal{M})$ are automorphisms.

In addition, the group of units of $\mathrm{Bi}(\mathcal{M})$ is $\mathrm{Aut}(\mathcal{M})$.

Green's relations

Because these maps are bijective...

Proposition 10

Let $\alpha, \beta \in \mathrm{Bi}(\mathcal{M})$.

- If $\gamma, \delta \in \mathrm{Bi}(\mathcal{M})$ are such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$, the maps γ and δ are automorphisms.
- If $\gamma, \delta \in \mathrm{Bi}(\mathcal{M})$ are such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$, the maps γ and δ are automorphisms.
- Suppose that $\alpha \mathscr{J} \beta$, and α and β add in finitely many relations. For all $\gamma, \delta, \epsilon, \zeta \in \mathrm{Bi}(\mathcal{M})$ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$, the maps $\gamma, \delta, \epsilon, \zeta$ are automorphisms.

So $\mathscr{L}, \mathscr{R}$ and $\mathscr{J}$ depend on how and where α, β add in relations.

Adding in relations

Understanding how relations are added by bimorphisms is crucial to the study of $\text{Bi}(\mathcal{M})$.

Definition 11

- Define a σ-structure $\mathcal{A}(\alpha)$ with domain M and relations

$$\bar{a} \in R_i^{\mathcal{A}(\alpha)} \text{ if and only if } \bar{a} \notin R_i^{\mathcal{M}} \text{ and } \bar{a}\alpha \in R_i^{\mathcal{M}}$$

  for all $i \in I$. We say that $\mathcal{A}(\alpha)$ is the **additional structure** of α.

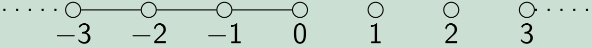
- Define the **support** of α to be the set

$$S(\alpha) = \{x \in M : x \in \bar{a} \text{ and } \bar{a} \in R_i^{\mathcal{A}(\alpha)} \text{ for some } i \in I\}.$$

- Define the **support structure** of α to be the σ-structure $\mathcal{M}[S(\alpha)]$ induced on $S(\alpha)$ with relations from $\mathcal{M}$.

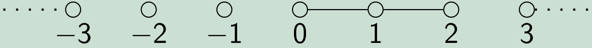
Example 12

Let $\mathcal{M}$ be the graph from Example 3.



Set $\alpha \in \text{Bi}(\mathcal{M})$ to be the function $i\alpha = i - 2$ for all $i \in \mathbb{Z}$. Then $\mathcal{A}(\alpha)$ is the graph on $\mathbb{Z}$ with the only two adjacencies given by $0 \sim 1$ and $1 \sim 2$, $S(\alpha)$ is the set $\{0, 1, 2\}$, and $\mathcal{M}[S(\alpha)]$ is the null graph induced by $\mathcal{M}$ on the vertex set $S(\alpha)$.

$\mathcal{A}(\alpha)$



$\mathcal{M}[S(\alpha)]$

Composition of bimorphisms

Let α, β be bimorphisms. How are relations added by $\alpha\beta$?

Lemma 13

Suppose that $\alpha, \beta \in \mathrm{Bi}(\mathcal{M})$ and $R_i \in \sigma$. Define the set

$$R_i^{\mathcal{A}(\beta)}\alpha^{-1} = \left\{\bar{x} \in M^{n_i} : \bar{x}\alpha \in R_i^{\mathcal{A}(\beta)}\right\}.$$

Then

$$R_i^{\mathcal{A}(\alpha\beta)} = R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}$$

and this is a disjoint union.

Here $R_i^{\mathcal{A}(\alpha)}$ is the set of relations added by α, and $R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ is the set of relations added by β once α has been applied.

Ideals

Bimorphisms do not delete relations; so this lemma gives a basis for an ideal structure.

Define $e_i(\alpha) = |R_i^{\mathcal{A}(\alpha)}|$, writing $e_i(\alpha) = \infty$ if $R_i^{\mathcal{A}(\alpha)}$ is infinite.

Corollary 14

Let $\alpha, \beta \in \mathrm{Bi}(\mathcal{M})$.

- If both $e_i(\alpha)$ and $e_i(\beta)$ are finite then $e_i(\alpha\beta) = e_i(\alpha) + e_i(\beta)$.
- $e_i(\alpha\beta) = \infty$ if and only if at least one of $R_i^{\mathcal{A}(\alpha)}$ or $R_i^{\mathcal{A}(\beta)}$ is infinite.
- Let $k \in \mathbb{N} \cup \{\infty\}$. Define $I(i,k) := \{\alpha \in \mathrm{Bi}(\mathcal{M}) \ : \ e_i(\alpha) \geq k\}$. Then, if non-empty, $I(i,k)$ is an ideal of $\mathrm{Bi}(\mathcal{M})$.

Green's relations

Theorem 15

Let $\alpha, \beta \in \mathrm{Bi}(\mathcal{M})$.

- $\alpha \mathscr{L} \beta$ if and only if $S(\alpha)\alpha = S(\beta)\beta$ and the bimorphism $\alpha\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$.
- $\alpha \mathscr{R} \beta$ if and only if $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$.
- $\alpha \mathscr{D} \beta$ if and only if there exists a bimorphism η such that: $\eta\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$, $\alpha^{-1}\eta$ induces an isomorphism from $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$, and $S(\beta)\beta = S(\eta)\eta$.
- If α is such that $e_i(\alpha)$ is finite for all $i \in I$, then $D_\alpha = J_\alpha$.

Open problem

Characterise Green's $\mathscr{J}$-relation in $\mathrm{Bi}(\mathcal{M})$.

Embeddings

Embeddings are monomorphisms α of $\mathcal{M}$ such that $\mathcal{M}\alpha \cong \mathcal{M}$. Need to measure how things are left out!

Definition 16

Let $\alpha \in \mathrm{Emb}(\mathcal{M})$.

- Define the **defect** of α to be the set $O(\alpha) = M \setminus M\alpha$, and write $o(\alpha) = |O(\alpha)|$.
- Define the **omitted structure** of α to be the set $\mathcal{O}(\alpha) = \mathcal{M}[O(\alpha)]$.

Lemma 17

Let $\alpha, \beta \in \mathrm{Emb}(\mathcal{M})$. Then $O(\alpha\beta) = O(\beta) \cup O(\alpha)\beta$ and this is disjoint.

Initial semigroup theory

$\text{Emb}(\mathcal{M}) \leq \text{Mon}(M)$, and so is right cancellative.

Lemma 18

- The only idempotent element in $\text{Emb}(\mathcal{M})$ is the identity.
- If $\text{Emb}(\mathcal{M}) \neq \text{Aut}(\mathcal{M})$, then $\text{Emb}(\mathcal{M})$ is not regular.
- The set $J_k = \{\epsilon \in \text{Emb}(\mathcal{M}) \ : \ o(\epsilon) \geq k\}$, if non-empty, is an ideal of $\text{Emb}(\mathcal{M})$ for $k \in \mathbb{N} \cup \{\infty\}$.

Green's relations (1/2)

Due to right-cancellativity, we have:

Lemma 19

- If $\gamma, \delta \in \text{Emb}(\mathcal{M})$ are such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$, the maps γ and δ are automorphisms.
- Suppose that $\alpha \mathscr{J} \beta$, and α and β omit finitely many vertices. For all $\gamma, \delta, \epsilon, \zeta \in \text{Emb}(\mathcal{M})$ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$, the maps $\gamma, \delta, \epsilon, \zeta$ are automorphisms.

In addition:

Proposition 20

$\mathscr{R} = \mathscr{D}$ in $\text{Emb}(\mathcal{M})$.

Green's relations (2/2)

Theorem 21

- Let $\alpha, \beta \in \mathsf{Emb}(\mathcal{M})$. Then $\alpha \mathscr{L} \beta$ if and only if $O(\alpha) = O(\beta)$.
- If there exists an isomorphism between $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ that extends to an automorphism of $\mathcal{M}$, then $\alpha \mathscr{R} \beta$. If $o(\alpha), o(\beta)$ are finite, then the converse is true.
- If $o(\alpha) < \infty$, then $L_\alpha = H_\alpha$ and $R_\alpha = D_\alpha = J_\alpha$.

Compare and contrast with Green's relations in $\mathsf{Mon}(M)$! Here's a teaser:

Open problem

Characterise Green's $\mathscr{J}$-relation in $\mathsf{Emb}(\mathcal{M})$.

Monomorphisms

A general monomorphism of $\mathcal{M}$ may change a non-relation to a relation and leave out vertices. So we need to measure how things are left out **and** added in. Thankfully, machinery for this already exists; we can extend concepts used for bimorphisms and embeddings.

Lemma 22

- The only idempotent element in $\mathrm{Mon}(\mathcal{M})$ is the identity.
- If $\mathrm{Mon}(\mathcal{M}) \neq \mathrm{Aut}(\mathcal{M})$, then $\mathrm{Mon}(\mathcal{M})$ is not regular.
- The sets $I(i,k) := \{\alpha \in \mathrm{Bi}(\mathcal{M}) \ : \ e_i(\alpha) \geq k\}$ and $J_k = \{\epsilon \in \mathrm{Mons}(\mathcal{M}) \ : \ o(\epsilon) \geq k\}$, if non-empty, are ideal of $\mathrm{Emb}(\mathcal{M})$ for $i \in I$ and $k \in \mathbb{N} \cup \{\infty\}$.

Green's relations (1/2)

Proposition 23

- If $\gamma, \delta \in \mathrm{Mon}(\mathcal{M})$ are such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$, the maps γ and δ are automorphisms.
- Suppose that $\alpha \mathscr{J} \beta$, and α and β omit finitely many vertices. For all $\gamma, \delta, \epsilon, \zeta \in \mathrm{Mon}(\mathcal{M})$ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$, the maps $\gamma, \delta, \epsilon, \zeta$ are automorphisms.

Green's relations (2/2)

Theorem 24

- Let $\alpha, \beta \in \mathrm{Mon}(\mathcal{M})$. Then $\alpha \mathscr{L} \beta$ if and only if $O(\alpha) = O(\beta)$, $S(\alpha)\alpha = S(\beta)\beta$ and $\mathcal{M}[S(\alpha)] \cong \mathcal{M}[S(\beta)]$ via the isomorphism induced by $\alpha\beta^{-1}$.
- Let $\alpha, \beta \in \mathrm{Mon}(\mathcal{M})$. Then $\alpha \mathscr{R} \beta$ if and only if there exists a monomorphism $f : \mathcal{O}(\alpha) \to \mathcal{O}(\beta)$ that extends to a monomorphism η of $\mathcal{M}$ such that $\eta|_{\mathcal{M}\alpha} = \alpha^{-1}\beta : \mathcal{M}\alpha \to \mathcal{M}\beta$; and there exists a monomorphism $g : \mathcal{O}(\beta) \to \mathcal{O}(\alpha)$ that extends to a monomorphism θ of $\mathcal{M}$ such that $\theta|_{\mathcal{M}\beta} = \beta^{-1}\alpha : \mathcal{M}\beta \to \mathcal{M}\alpha$.

Open problem

Characterise Green's $\mathscr{D}$ and $\mathscr{J}$-relation in $\mathrm{Emb}(\mathcal{M})$. In particular, does $\mathscr{R} = \mathscr{D}$ in $\mathrm{Mon}(\mathcal{M})$?

A reminder

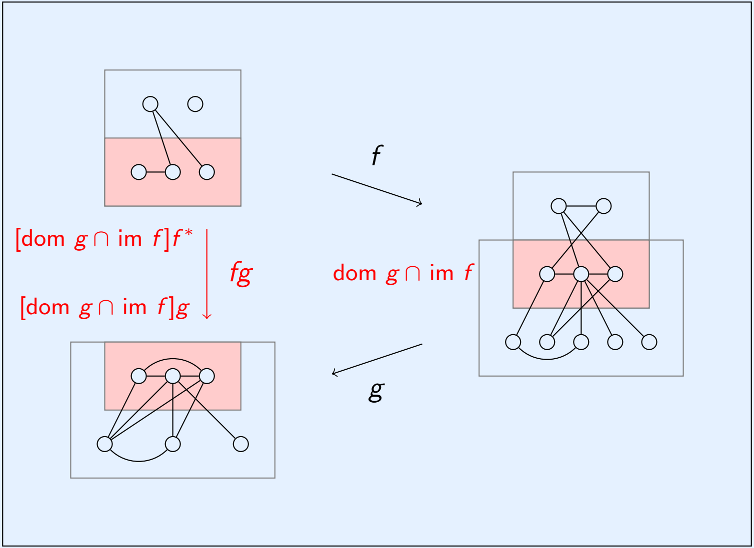
For a countably infinite relational first-order structure $\mathcal{M}$:

- $\mathrm{Inv}(\mathcal{M})$ is the symmetric inverse monoid of $\mathcal{M}$; the monoid of all isomorphisms between substructures of $\mathcal{M}$.
- $\mathrm{Part}(\mathcal{M})$ is the partial homomorphism monoid of $\mathcal{M}$.
- $\mathrm{Inj}(\mathcal{M})$ is the partial monomorphism monoid of $\mathcal{M}$.

Fun aside!

Much like $\mathrm{Bi}(\mathcal{M}) \subseteq \mathrm{Sym}(M)$ is a group-embeddable monoid that isn't a group, $\mathrm{Inj}(\mathcal{M}) \subseteq \mathrm{Inv}(M)$ is a inverse semigroup-embeddable monoid that isn't an inverse semigroup.

Example 25: Composition in Part(R)



Where to begin...

Lemma 26

Let $\epsilon \in \mathrm{Inj}(\mathcal{M})$. Then ϵ is an idempotent if and only if ϵ is the identity map on some substructure of $\mathcal{M}$.

This is decidedly **not** true in $\mathrm{Part}(\mathcal{M})$!

But Lemma 26 does give us this!

Corollary 27

If $\mathcal{M}$ is countably infinite, then $|\mathrm{Inv}(\mathcal{M})| = |\mathrm{Inj}(\mathcal{M})| = |\mathrm{Part}(\mathcal{M})| = 2^{\aleph_0}$.

Now onto Green's relations...

Easy case: $\mathrm{Inv}(\mathcal{M})$

Every partial isomorphism α of $\mathcal{M}$ has an inverse α^{-1}. Therefore, $\mathrm{Inv}(\mathcal{M})$ is an inverse semigroup, and hence is regular; therefore, it inherits Green's relations from $\mathrm{Inv}(M)$:

Corollary 28

Suppose $\alpha, \beta \in \mathrm{Inv}(\mathcal{M})$. Then:

- $\alpha \mathscr{L} \beta$ if and only if im $\alpha =$ im β;
- $\alpha \mathscr{R} \beta$ if and only if dom $\alpha =$ dom β;
- $\alpha \mathscr{D} \beta$ if and only if $\mathcal{M}[\text{im } \alpha] \cong \mathcal{M}[\text{im } \beta]$, and;
- $\mathscr{D} = \mathscr{J}$.

Harder case: $\mathsf{Inj}(\mathcal{M})$

$\mathsf{Inj}(\mathcal{M})$ is decidedly not regular. As each partial monomorphism can be assumed to be bijective (wlog), we can use the same machinery for partial monomorphisms as we could for bimorphisms (with the caveat that we restrict to the appropriate domain). Here's the main technical result:

Lemma 29

Suppose that $\alpha, \beta \in \mathsf{Inj}(\mathcal{M})$. Then

$$R_i^{\mathcal{A}(\alpha\beta)} = (R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}) \cap (\mathrm{dom}\ \alpha\beta)^n$$

and the first term of the intersection is a disjoint union.

Lemma 13 is a direct consequence of this result.

Harder case: $\mathrm{Inj}(\mathcal{M})$

Theorem 30

Suppose $\alpha, \beta \in \mathrm{Inj}(\mathcal{M})$. Then:

- $\alpha \mathscr{L} \beta$ in $\mathrm{Inj}(\mathcal{M})$ if and only if im $\alpha =$ imβ, and the resulting map $\alpha\beta^{-1}$ is an isomorphism sending $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$, and $S(\alpha)\alpha = S(\beta)\beta$.
- $\alpha \mathscr{R} \beta$ if and only if dom $\alpha =$ dom β and $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$.
- $\alpha \mathscr{D} \beta$ if and only if there exists a partial monomorphism η such that
  - dom $\alpha =$ dom η and im $\eta =$ im β;
  - $\eta\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$, and $\alpha^{-1}\eta$ induces an isomorphism from $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$, and;
  - $S(\beta)\beta = S(\eta)\eta$.

Open question

$\mathscr{J}$?

Hardest case: Part($\mathcal{M}$)

???? (Note: progress has been made in some areas.)

Questions (?)

- Investigate semigroup theory of Epi($\mathcal{M}$) in more detail. Studies in this direction should presumably extend work of Bi($\mathcal{M}$) (due to the example of D) could influence work on Part($\mathcal{M}$).
- Links with constraint satisfaction?
- Is there a structural analogue for the binary relation monoid?
- $\mathscr{J}$.