Conjugacy in epigroups

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@ the York semigroup

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Epigroups

Epigroups

A semigroup S is an *epigroup* if for any element x of S some power of x lies in a subgroup of S.

Nice classes of epigroups:

- finite semigroups;
- periodic semigroups;
- completely regular semigroups;
- completely 0-simple semigroups;
- algebraic monoids.

Some concrete examples:

- the semigroup of all matrices over a division ring;
- the infinite cyclic epigroup $C_{n,\infty}$ given by the presentation

$$\langle a, b \mid ab = ba,\ ab^2 = b,\ a^{n+1}b = a^n \rangle$$

Epigroups as unary semigroups

Let S be a semigroup:

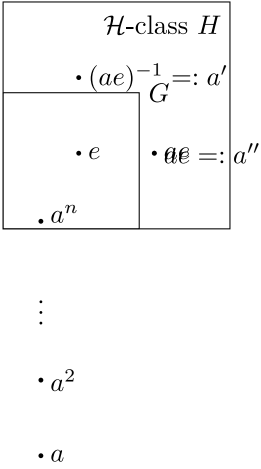
- $a \in S$ is an *epigroup element* ($a \in \mathrm{Epi}(S)$) (or a *group-bound element*) if $\exists n : a^n$ is in a subgroup of S;
- the maximum subgroup of S containing a^n is its $\mathcal{H}$-class H; with identity e;
- we define the *pseudo-inverse* a' of a by

$$a' := (ae)^{-1}$$

  the inverse of ae in the group H;
- $a \in \mathrm{Epi}(S)$ iff $\exists n\ \exists a' \in S$ such that:

$$a'aa' = a'\,, \quad aa' = a'a\,, \quad a^{n+1}a' = a^n\,;$$

  the smallest such n is the *index* of a;
- we set $a^\omega = aa'$; so $a^{\omega+1} = aa'a = a''$.
  - $\mathrm{Epi}_n(S) = \{a \in \mathrm{Epi}(S)$ with index no more than $n\}$;
  - $\mathrm{Epi}_1(S)$ are the *completely regular*;
  - S is an *epigroup* if $S = \mathrm{Epi}(S)$.



Conjugacy... in groups

G a group
$a, b \in G$ are conjugate ($a \sim b$) if:

- $\exists_{u,v \in G}\ a = uv$ and $b = vu$
- $\exists_{g \in G}\ a = g^{-1}bg$
- $\exists_{g \in G}\ ga = bg$
- $\exists_{g \in G}\ ag = gb$
- consider a representation $\rho : G \to GL_n(\mathbb{C})$;
  the character $\begin{aligned} \chi_\rho : G &\to \mathbb{C} \\ g &\mapsto \mathrm{Tr}(\rho(g)) \end{aligned}$ is a class function;

$$\text{irreducible characters} \longleftrightarrow \text{conjugacy classes}$$

Some known generalizations of conjugacy...

S a semigroup (with zero) / monoid / inverse semigroup / epigroup

$$a \sim_p b \iff \exists_{u,v \in S^1} \quad a = uv \ \wedge \ b = vu$$

$$a \sim_u b \iff \exists_{g \in U(S)} \quad g^{-1}ag = b \ \wedge \ gbg^{-1} = a$$

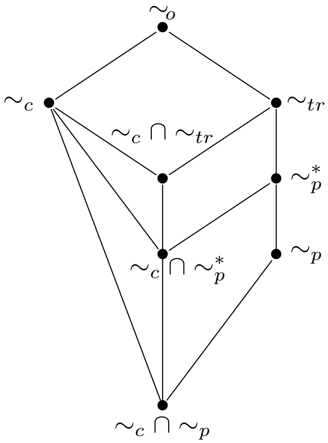
$$a \sim_i b \iff \exists_{g \in S^1} \quad g^{-1}ag = b \ \wedge \ gbg^{-1} = a$$

$$a \sim_o b \iff \exists_{g,h \in S^1} \quad ag = gb \ \wedge \ bh = ha$$

$$a \sim_c b \iff \exists_{g \in \mathbb{P}^1(a)} \ \exists_{h \in \mathbb{P}^1(b)} \quad ag = gb \ \wedge \ bh = ha$$

$$a \sim_{tr} b \iff \exists_{g,h \in S^1} \quad ghg = g \ \wedge \ hgh = h \ \wedge \ ga^{\omega+1}h = b^{\omega+1} \ \wedge \ hg = a^{\omega} \ \wedge \ gh = b^{\omega}.$$

Known inclusions between conjugacies



$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	4	4	0
1	0	0	0	0	4	4	0
2	0	0	0	0	4	4	0
3	0	0	0	0	4	4	0
4	4	4	4	4	4	4	4
5	4	4	4	4	4	4	4
6	0	0	2	3	4	5	6

`SmallSemigroup(7,542155)`

Epigroups

For $a, b \in \mathrm{Epi}(S)$, we set

$$a \sim_{tr} b \iff \exists_{g,h\in S^1}\ ghg = g,\ hgh = h,\ ha''g = b'',\ gh = aa',\ hg = bb'.$$

Theorem

Let S be a semigroup. For $a, b \in \mathrm{Epi}(S)$, the following are equivalent:

1. $a \sim_{tr} b$;
2. $\exists_{g,h\in S^1}\ ha''g = b'',\ gh = a^\omega,\ hg = b^\omega$
3. $\exists_{g,h\in S^1}\ a''g = gb'',\ gh = a^\omega,\ hg = b^\omega$;
4. $\exists_{g,h\in S^1}\ ag = gb,\ bh = ha,\ gh = a^\omega,\ hg = b^\omega$;
5. $\exists_{g,h\in S^1}\ hgh = h,\ ha''g = b'',\ gb''h = a''$;
6. $a'' \sim_p b''$.

Epigroups

Theorem

Let S be a semigroup. Then:

1. $\sim_{tr}$ is an equivalence relation on $\mathrm{Epi}(S)$;
2. for all $x \in \mathrm{Epi}(S)$, $x \sim_{tr} x''$;
3. for all $x, y \in S$ such that $xy, yx \in \mathrm{Epi}(S)$, $xy \sim_{tr} yx$;
4. $\sim_{tr}$ is the smallest equivalence relation on $\mathrm{Epi}(S)$ such that (2) and (3) hold.

Theorem

Let S be a semigroup. As relations on $\mathrm{Epi}(S)$, the following inclusions hold:

$$\sim_p \subseteq \sim_p^* \subseteq \sim_{tr} \subseteq \sim_o .$$

Completely regular semigroups and beyond ...

Completely regular as semigroup variety:

$$xx' = x'x \qquad x'xx' = x' \qquad xx'x = x$$

Corollary

Let S be a semigroup. As relations on $\mathrm{Epi}_1(S)$, we have $\sim_p = \sim_p^* = \sim_{tr}$. In particular, (as Kudryavtseva showed) $\sim_p$ is transitive on completely regular semigroups.

$\mathcal{W}$ - Another semigroup variety:

$$xx' = x'x \qquad x'xx' = x' \qquad x^3x' = x^2 \qquad (xy)'' = xy$$

Theorem

Let S be an epigroup in $\mathcal{W}$. Then $\sim_p = \sim_p^* \subset \sim_{tr}$.

Variants of CS semigroups

Theorem

Let $(S, \cdot, ')$ be a completely regular semigroup, and fix $a \in S$. Let $(S, \circ, ^*)$ be the variant of S at a, that is,

$$x \circ y = xay \qquad \text{and} \qquad x^* = (xa)'x(ax)'$$

for all $x, y \in S$. Then $(S, \circ, ^*)$ is in $\mathcal{W}$.

Corollary

The relation $\sim_p$ is transitive in every variant of a completely regular semigroup.

In general, for epigroups ...

$\sim_p$ transitive $\not\Longrightarrow$ $\sim_p$ transitive in all of the variants.

Epigroups and idempotents

Proposition

Let S be an epigroup. Then $\sim_{tr} \cap \leq$ is the identity relation on $E(S)$.

Proposition

Let S be an epigroup in which $\sim_{tr} = \sim_o$. Then $E(S)$ is an antichain.

Completely simple semigroups

- no proper ideals;
- idempotents form an antichain.

Theorem

In completely simple semigroups, we have $\sim_p = \sim_p^* = \sim_{tr} = \sim_o$.

Theorem

Let S be a regular epigroup without zero. The following are equivalent:

1. $\sim_p = \sim_o$ in S;
2. S is completely simple.

Epigroups with zero

Completely 0-simple semigroups

Rees matrix representation $\mathcal{M}^0(G; I, \Lambda; P)$:

- I and Λ are nonempty sets;
- G is a group;
- $P = (p_{\lambda j})$ is a $\Lambda \times I$ matrix with entries in $G \cup \{0\}$ such that no row or column of P consists entirely of zeros
- elements from $(I \times G \times \Lambda) \cup \{0\}$;
- multiplication is defined by $(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$ if $p_{\lambda j} \neq 0$, $(i, a, \lambda)(j, b, \mu) = 0$ if $p_{\lambda j} = 0$, and $(i, a, \lambda)0 = 0(i, a, \lambda) = 0$.

Proposition

For a completely 0-simple semigroup $\mathcal{M}^0(G; I, \Lambda; P)$, we have $\sim_c \subseteq \sim_p$. Moreover, $\sim_c = \sim_p$ if and only if the sandwich matrix P has only nonzero elements.

Epigroups with zero

Lemma

Let S be an epigroup with zero and suppose $\sim_c \subseteq \sim_{tr}$. Then $E(S)\backslash\{0\}$ is an antichain.

Theorem

Let S be a regular epigroup with zero. The following are equivalent:

1. $\sim_c \subseteq \sim_p$;
2. $\sim_c \subseteq \sim_{tr}$;
3. S is a 0-direct union of completely 0-simple semigroups.

0-direct union

A semigroup S with zero is called a 0-*direct union* of completely 0-simple semigroups if $S = \bigcup_{i \in I} S_i$, where each S_i is a completely 0-simple semigroup and $S_i \cap S_j = S_i S_j = \{0\}$ if $i \neq j$

The conjugacies $\sim_o$ and $\sim_c$ in epigroups

Recall $\sim_o$

$$a \sim_o b \iff \exists_{g,h \in S^1}\ ag = gb \ \wedge\ bh = ha$$

Theorem

Let S be an epigroup and suppose $a \sim_o b$ for some $a, b \in S$. Then there exist mutually inverse $g, h \in S^1$ such that $ag = gb$ and $bh = ha$.

Recall $\sim_c$

$$a \sim_c b \iff \exists_{g \in \mathbb{P}^1(a)} \exists_{h \in \mathbb{P}^1(b)}\ ag = gb \ \wedge\ bh = ha$$

Theorem

Let S be an epigroup with zero in $\mathcal{W}$ and suppose $a \sim_c b$ for some $a, b \in S$. Then there exist mutually inverse $g \in \mathbb{P}^1(a)$, $h \in \mathbb{P}^1(b)$ such that $ag = gb$ and $bh = ha$.

Conjugacy in symmetric inverse semigroups - $\mathcal{I}(X)$

X countable set

Basic partial injective transformations on X:

- *cycle* - $\delta = (x_0\, x_1 \ldots x_{k-1})$, with $x_i\delta = x_{i+1}$ for all $0 \leq i < k-1$, and $x_{k-1}\delta = x_0$.
- *chain* - $\theta = [x_0\, x_1 \ldots x_k]$, with $x_i\theta = x_{i+1}$ for all $0 \leq i \leq k-1$.
- *double ray* - $\omega = \langle \ldots x_{-1}\, x_0\, x_1 \ldots \rangle$, with $x_i\omega = x_{i+1}$ for all i.
- *right ray* - $\upsilon = [x_0\, x_1\, x_2 \ldots \rangle$, with $x_i\upsilon = x_{i+1}$ for all $i \geq 0$.
- *left ray* - $\lambda = \langle \ldots x_2\, x_1\, x_0]$, with $x_i\lambda = x_{i-1}$ for all $i > 0$.

An element $\beta \in \mathcal{I}(X)$ has a unique *cycle-chain-ray decomposition*:

$\beta = (2\;4) \sqcup [6\;8\;10] \sqcup \langle \ldots -6\; -4\; -2\; -1\; -3\; -5\; \ldots \rangle \sqcup [1\;5\;9\;13\; \ldots \rangle \sqcup \langle \ldots 15\;11\;7\;3]$

The *cycle-chain-ray type* of

$$\alpha = (2\,6\,8) \sqcup [1\,3] \sqcup [4\,5\,9]$$

(has the form $(*\,*\,*)[*\,*][*\,*\,*]$)
is the sequence of cardinalities

$$\langle 0,0,1,0,\ldots;0,1,1,0,\ldots;0,0,0\rangle.$$

The *cycle-chain-ray type* of

$$\beta = (2\,4)\sqcup[6\,8\,10]\sqcup\langle\ldots{-6}\,{-4}\,{-2}\,{-1}\,{-3}\,{-5}\ \ldots\rangle\sqcup[1\,5\,9\,13\ \ldots\rangle\sqcup\langle\ldots 15\,11\,7\,3]$$

(has the form $(*\,*)[*\,*\,*]\langle\ldots *\,*\,*\ \ldots\rangle[*\,*\,*\ \ldots\rangle\langle\ldots *\,*\,*]$)
is the sequence of cardinalities

$$\langle 0,1,0,\ldots;0,0,1,0,\ldots;1,1,1\rangle.$$

$$\langle |\Delta^1_\alpha|,|\Delta^2_\alpha|,|\Delta^3_\alpha|,\ldots;|\Theta^1_\alpha|,|\Theta^2_\alpha|,|\Theta^3_\alpha|,\ldots;|\Omega_\alpha|,|\Upsilon_\alpha|,|\Lambda_\alpha|\rangle$$

Conjugacy in symmetric inverse semigroups - $\mathcal{I}(X)$

Theorem

Suppose that X is countable. Let $\alpha, \beta \in \mathcal{I}(X)$. Then $\alpha \sim_c \beta$ if and only if the following conditions are satisfied:

(a) $|\Delta_\alpha^k| = |\Delta_\beta^k|$ for every $k \in \mathbb{Z}_+$, $|\Omega_\alpha| = |\Omega_\beta|$, and $|\Lambda_\alpha| = |\Lambda_\beta|$;

(b) if Ω_α is finite, then $|\Upsilon_\alpha| = |\Upsilon_\beta|$; and

(c) if Λ_α is finite, then

 (i) $k_\alpha = k_\beta$ ($k_\alpha = \sup\{k \in \mathbb{Z}_+ : \Theta_\alpha^k \neq \emptyset\}$); and

 (ii) if $k_\alpha \in \mathbb{Z}_+$, then $m_\alpha = m_\beta$ ($m_\alpha = \max\{m \in \{1, 2, \ldots, k_\alpha\} : |\Theta_\alpha^m| = \aleph_0\}$) and for every $k \in \{m_\alpha + 1, \ldots, k_\alpha\}$, $|\Theta_\alpha^k| = |\Theta_\beta^k|$.

$$\langle |\Delta_\alpha^1|, |\Delta_\alpha^2|, |\Delta_\alpha^3|, \ldots; |\Theta_\alpha^1|, |\Theta_\alpha^2|, |\Theta_\alpha^3|, \ldots; |\Omega_\alpha|, |\Upsilon_\alpha|, |\Lambda_\alpha| \rangle$$

$$\langle |\Delta_\beta^1|, |\Delta_\beta^2|, |\Delta_\beta^3|, \ldots; |\Theta_\beta^1|, |\Theta_\beta^2|, |\Theta_\beta^3|, \ldots; |\Omega_\beta|, |\Upsilon_\beta|, |\Lambda_\beta| \rangle$$

Conjugacy in symmetric inverse semigroups - $\mathcal{I}(X)$

Corollary

Suppose that X is finite and $\alpha, \beta \in \mathcal{I}(X)$. Then the following are equivalent:

- $\alpha \sim_c \beta$;
- α and β have the same cycle-chain type;
- there exists a permutation σ on the set X such that $\alpha = \sigma^{-1}\beta\sigma$.

Theorem (Ganyushkin, Kormysheva, 1993)

Suppose that X is finite and $\alpha, \beta \in \mathcal{I}(X)$. Then $\alpha \sim_p^* \beta$ if and only if α and β have the same cycle type.

Conjugacy in symmetric inverse semigroups - $\mathcal{I}(X)$

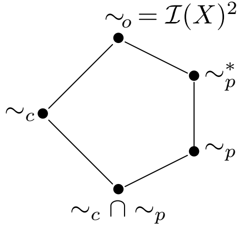
Proposition

Suppose that X is finite with $|X| \geq 2$. Then $\sim_c \subset \sim_p$ in $\mathcal{I}(X)$.

Proposition

Suppose that X is countably infinite. Then, with respect to inclusion, $\sim_p$ and $\sim_c$ are not comparable in $\mathcal{I}(X)$.

For a countably infinite X:



Epigroup elements of $\mathcal{I}(X)$

Lemma

Let $\alpha \in \mathcal{I}(X)$. Then α is an epigroup element if and only if $\Omega_\alpha = \Upsilon_\alpha = \Lambda_\alpha = \emptyset$ and there is a positive integer n such that $\Theta_\alpha^k = \emptyset$ for all $k > n$.

Lemma

Let $\alpha \in \mathrm{Epi}(\mathcal{I}(X))$. Then α and α'' have the same cycle type.

Theorem

Let X be a countable set. Then for all $\alpha, \beta \in \mathrm{Epi}(\mathcal{I}(X))$, $\alpha \sim_{tr} \beta$ if and only if α and β have the same cycle type.

When is conjugacy the identity relation?

Fact:

In a group $\sim$ is the identity if and only if G is commutative.

Theorem

Let S be a semigroup. Then, $\sim_p$ is the identity relation in S if and only if S is commutative.

Theorem

Let S be an epigroup. Then, $\sim_{tr}$ is the identity relation in S if and only if S is a commutative completely regular epigroup.

Theorem

Let S be a semigroup. Then:

1. if S is commutative, then $\sim_o$ is the minimum cancellative congruence on S;
2. $\sim_o$ is the identity relation in S if and only if S is commutative and cancellative.

When is conjugacy the identity relation?

Corollary

Let S be a commutative and cancellative semigroup. Then $\sim_p$, $\sim_o$, and $\sim_c$ all coincide, and are equal to the identity relation.

Corollary

Let S be an epigroup. Then $\sim_p$, $\sim_o$, $\sim_{tr}$ and $\sim_c$ all coincide and are equal to the identity relation if and only if S is a commutative group.

When is conjugacy the universal relation?

Theorem

Let S be an epigroup. The following are equivalent:

1. $\sim_{tr}$ is the universal relation;
2. $E(S)$ is an antichain and for all $x \in S$, $x'' = x^\omega$;
3. for all $x, y \in S$, $x'yx' = x'$;
4. for all $x, y \in S$, $x^\omega y x^\omega = x^\omega$;
5. for all $x \in S$, $e \in E(S)$, $exe = e$.

Theorem

Let S be a semigroup.

1. If S is a rectangular band, then $\sim_p$ is the universal relation.
2. If $\sim_p$ is the universal relation in S, then S is simple. If, in addition, S contains an idempotent, then S is a rectangular band.

Corollary

In a finite semigroup (or more generally, an epigroup) S, $\sim_p$ is the universal relation if and only if S is a rectangular band.

Some research problems:

For inverse semigroups we can define the conjugacy notion

$$a \sim_i b \Leftrightarrow \exists g \in S^1, \quad g^{-1}ag = b \ \wedge \ gbg^{-1} = a.$$

This can be naturally generalized for epigroups setting

$$a \sim b \Leftrightarrow \exists g \in S^1, \quad g'ag = b \ \wedge \ gbg' = a.$$

In general, $\sim$ is not transitive so we should consider $\sim^*$.

Left restriction

In $P(X)$ (the semigroup of partial transformations on any nonempty set X) which can be regarded as a left restriction semigroup with respect to the set of partial identities $E = \{id_Y : Y \subseteq X\}$ we have

$$\alpha \sim_c \beta \Leftrightarrow \exists \phi, \psi \in S^1 : \alpha\phi = \phi\beta \wedge \beta\psi = \psi\alpha \wedge (\alpha\phi)^+ = \alpha^+ \wedge (\beta\psi)^+ = \beta^+.$$

Thanks for your attention!