Maximal left ideals in Banach algebras

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Algebras

Throughout, an algebra is linear and associative and over the complex field \( \mathbb{C} \).

A left ideal in \( A \) is a linear subspace \( I \) of \( A \) such that \( ax \in I \) when \( a \in A \) and \( x \in I \); a left ideal \( M \) is maximal if \( M \neq A \) and \( I = M \) or \( I = A \) when \( I \) is a left ideal in \( A \) with \( I \supset M \).

A proper left ideal \( I \) in an algebra \( A \) is modular if there exists \( u \in A \) with \( a - au \in I \) (\( a \in A \)).

Let \( I \) be a left ideal in an algebra \( A \) with such a \( u \). By Zorn’s lemma, the family of left ideals \( J \) in \( A \) with \( J \supset I \) and \( u \notin J \) has a maximal member, say \( M \). Clearly \( M \) is a maximal left ideal in \( A \).

The radical, \( \text{rad} A \), of \( A \) is the intersection of the maximal modular left ideals, with \( \text{rad} A = A \) if there are no such (and then \( A \) is radical). It is an ideal in \( A \). The algebra is semi-simple if \( \text{rad} A = \{0\} \).
Banach Algebras

These are Banach spaces \((A, \| \cdot \|)\) which are also algebras such that

\[
\|ab\| \leq \|a\| \|b\| \quad (a, b \in A).
\]

**Example** For a Banach space \(E\), \(B(E)\) is the Banach algebra of all bounded, linear operators on \(E\).

**Example** Let \(S\) be a semigroup. The point mass at \(s \in S\) is \(\delta_s\). The Banach space \(\ell^1(S)\) consists of the functions \(f : S \to \mathbb{C}\) such that

\[
\|f\|_1 = \sum \{|f(s)| : s \in S\} < \infty.
\]

There is a Banach algebra product \(*\) called convolution such that \(\delta_s * \delta_t = \delta_{st} \quad (s, t \in S)\). Then \((\ell^1(S), *)\) is a Banach algebra that is the **semigroup algebra** on \(S\).
More definitions

Let $I$ be a closed ideal in a Banach algebra $A$. Then $A/I$ is also a Banach algebra. The algebra $A/\text{rad } A$ is a semi-simple Banach algebra.

A maximal left ideal in a Banach algebra is either closed or dense.

A Banach algebra is a **Banach $\ast$-algebra** if there is an involution $\ast$ on $A$ such that

$$\|a^\ast\| = \|a\| \quad (a \in A).$$

For example, $C^\ast$-algebras are Banach $\ast$-algebras.
Matrices

For $n \in \mathbb{N}$, we denote by $\mathbb{M}_n$ the algebra of $n \times n$ matrices over $\mathbb{C}$. The algebras $\mathbb{M}_n$ are simple, i.e., no proper, non-zero ideals.

Let $A$ be an algebra. Then $\mathbb{M}_n(A)$ is the algebra of all $n \times n$ matrices with coefficients in $A$. In the case where $A$ is a Banach algebra, $\mathbb{M}_n(A)$ is also a Banach algebra with respect to the norm given by

$$\| (a_{i,j}) \| = \sum_{i,j=1}^{n} \| a_{i,j} \| \quad ((a_{i,j}) \in \mathbb{M}_n(A)).$$

Suppose that $A$ is a Banach $\ast$-algebra. Then $\mathbb{M}_n(A)$ is also a Banach $\ast$-algebra with respect to the involution given by the transpose map $(a_{i,j}) \mapsto (a_{j,i}^\ast)$.
Maximal modular left ideals

Let $A$ be a Banach algebra. The following basic result is in all books on Banach algebras.

**Theorem** Every maximal modular left ideal $M$ in $A$ is closed (and $A/M$ is a simple Banach left $A$-module).

**Proof** Take $u \in A$ as in the definition. Assume that there is $a \in M$ with $\|a - u\| < 1$. Then there is $b \in A$ with $a - u + b + b(a - u) = 0$, and so $u = a + ba + (b - bu) \in M$, a contradiction. So $M$ is not dense, and hence it is closed. $\square$
Codimension of maximal modular left ideals in Banach algebras

What is the codimension of such an ideal $M$?

Suppose that $A$ is commutative. Then $A/M$ is a field, and $A/M = \mathbb{C}$ by Gel’fand–Mazur, so $M$ is the kernel of a continuous character and has codimension 1.

Suppose that $A$ is non-commutative. For example, take $A = \mathcal{B}(E)$ for a Banach space $E$, and take $x \in E$ with $x \neq 0$. Then

$$M = \{T \in \mathcal{B}(E) : Tx = 0\}$$

is a closed, singly-generated maximal left ideal. When $E$ has dimension $n \in \mathbb{N}$, $M$ has codimension $n$; when $E$ has infinite dimension, $M$ has infinite codimension.
Detour to Fréchet algebras

A Fréchet algebra has a countable series of semi-norms, rather than one norm.

Let $A$ be a commutative, unital Fréchet algebra. Then each closed maximal ideal is the kernel of a continuous character, but it is a formidable open question, called Michael’s problem, whether all characters on each commutative Fréchet algebra are continuous.
An example

Let $O(\mathbb{C})$ denote the space of entire functions on $\mathbb{C}$, a Fréchet algebra with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}$.

Then each maximal ideal $M$ of codimension 1 in $O(\mathbb{C})$ is closed, and there exists $z \in \mathbb{C}$ such that

$$M = M_z := \{ f \in O(\mathbb{C}) : f(z) = 0 \}.$$ 

Let $I$ be the set of functions $f \in O(\mathbb{C})$ such that $f(n) = 0$ for each sufficiently large $n \in \mathbb{N}$. Clearly $I$ is an ideal in $O(\mathbb{C})$, $I$ is dense in $O(\mathbb{C})$, and $I$ is contained in a maximal ideal, say $M$. Then $M$ is dense in $O(\mathbb{C})$, but $M$ is not of the form $M_z$. The quotient $A/M$ is a ‘very large field’ of infinite dimension. (For large fields, see a book with W. H. Woodin.)
Maximal left ideals that are not modular

**Example** Let $E$ be an infinite-dimensional Banach space. Then $E$ has a dense subspace $F$ that has codimension 1 in $E$; it is the kernel of a discontinuous linear functional. The space $E$ is a commutative Banach algebra with respect to the zero product, and $F$ is a maximal (left) ideal in this algebra such that $F$ is not closed, and $F$ is obviously not modular.

This suggested:

**Conjecture** Let $A$ be a Banach algebra. Then every maximal left ideal in $A$ is either closed or of codimension 1.

We shall give a counter-example.
The following are little calculations.

Here \( A \) is any algebra, \( A^{[2]} = \{ab : a, b \in A\} \), and \( A^2 = \text{lin} \, A^{[2]} \). The algebra \( A \) factors if \( A = A^{[2]} \) and factors weakly if \( A = A^2 \) (not the same).

**Fact 1** Suppose that \( A^2 \nsubseteq A \). Then \( A \) contains a maximal left ideal that is an ideal in \( A \) and that contains \( A^2 \). Each maximal left ideal that contains \( A^2 \) has codimension 1 in \( A \).

Just take \( M \) to be a subspace of codimension 1 in \( A \) such that \( A^2 \subset M \). \( \square \)
Fact 2 Suppose that $M$ is a maximal left ideal in $A$ and $b \in A$, and set $J_b = \{a \in A : ab \in M\}$. Then either $J_b = A$ or $J_b$ is a maximal modular left ideal in $A$.

Either $Ab \subset M$, and hence $J_b = A$, or $A/M$ is a simple left $A$-module, and $J_b = (b + M)^\perp$ is a maximal modular left ideal. □

Fact 3 $A$ has no maximal left ideals iff $A$ is a radical algebra and $A^2 = A$.

Suppose that $A$ has no maximal left ideals. Then $A$ is radical, and $A^2 = A$ by Fact 1.

For the converse, assume that $M$ is a maximal left ideal, and take $b \in A$. By Fact 2, $J_b = A$, and so $Ab \subset M$, whence $A^2 \subset M \neq A$, a contradiction. □
A simple, radical algebra

A simple, radical algebra was constructed by Paul Cohn in 1967. Since a simple algebra $A$ is such that $A^2 = A$, it follows from Fact 3 that this algebra has no maximal left or maximal right ideal. However, it does have a maximal ideal, namely $\{0\}$.

A topologically simple Banach algebra $A$ is one in which the only closed ideals are $\{0\}$ and $A$. Is there a commutative, radical Banach algebra that is topologically simple?

Maybe this is the hardest question in Banach algebra theory.
An example

A Banach algebra $A$ with a bounded approximate identity is such that $A = A^{[2]}$; this follows from Cohen’s factorization theorem.

Let $\mathcal{V}$ be the Volterra algebra. This is the Banach space $L^1([0,1])$ with truncated convolution multiplication:

$$(f \ast g)(t) = \int_0^t f(t - s)g(s) \, ds \quad (t \in [0,1])$$

for $f, g \in \mathcal{V}$. This is a radical Banach algebra with a BAI, and so $\mathcal{V}^{[2]} = \mathcal{V}$. Thus there are no maximal ideals in $\mathcal{V}$ (and so the conjecture holds vacuously for $\mathcal{V}$).
Some positive results

**Theorem** Let $A$ be a Banach algebra with maximal left ideal $M$. Suppose that $A^2 \not\subset M$ and $M$ is also a right ideal. Then $M$ is closed.

**Proof** Set $J_A = \{ a \in A : aA \subset M \}$. By Fact 2, $J_A$ is a closed left ideal. Since $A^2 \not\subset M$, it is not true that $J_A = A$. Since $M$ is a right ideal, $M \subset J_A$. So $M = J_A$ is closed. □

**Corollary** Let $A$ be a commutative Banach algebra with a maximal ideal $M$. Then $M$ has codimension 1. Either $A/M = \mathbb{C}$ and $M$ is closed, or $A^2 \subset M$. □

Thus the conjecture holds in the commutative case.
Null sequences factoring

Let $A$ be a Banach algebra. A null sequence $(a_n)$ factors if there is a null sequence $(b_n)$ in $A$ and $a \in A$ with $a_n = b_na$ ($n \in \mathbb{N}$). This holds when $A$ has a BAI (but is more general).

**Theorem** Let $A$ be a Banach algebra in which null sequences factor. Then every maximal left ideal $M$ in $A$ is closed.

**Proof** Take $a \in A$ and $(a_n)$ in $M$ with $a_n \to a$. There is a null sequence $(b_n)$ and $b \in A$ with $a - a_n = b_nb$ and $a = b_0b$. Again set 

$$J = J_b = \{x \in A : xb \in M\}.$$

By Fact 2, $J$ is closed. Now we have $(b_0 - b_n)b = a_n \in M$, so $b_0 = \lim(b_0 - b_n) \in J$, whence $a \in M$. So $M$ is closed. \qed
Applications

Corollary Every maximal left ideal in each $C^*$-algebra is closed. □

Let $E$ be a Banach space. Then $A(E)$ and $K(E)$ are the Banach algebras of approximable and compact operators, respectively. Suppose that $E$ has certain approximation properties. Then null sequences in $A(E)$ and $K(E)$ factor, and so every maximal left ideal is closed.

What are they? Are they all modular? What happens if $E$ does not have the ‘certain approximation properties’?
An example - algebraic preliminary

**Definition** Let $A$ be an algebra with a character $\varphi$. Then $M_\varphi$ is the kernel of $\varphi$ and

$$J_\varphi = \text{lin} \{ab - \varphi(a)b : a, b \in A\}.$$  

Then $J_\varphi$ is a right ideal and $M_\varphi A \subset J_\varphi \subset M_\varphi$.

Suppose that there is an idempotent $u$ in $A \setminus M_\varphi$. Then

$$J_\varphi = M_\varphi^2 + M_\varphi u + (1 - u)M_\varphi.$$  

**Fact** Take a non-zero linear functional $\lambda$ on $A$ with $\lambda | J_\varphi = 0$, and set $M = \ker \lambda$. Then $M$ is a maximal left ideal in $A$ of codimension 1 and $A^2 \not\subseteq M$.

This is easily checked.
A Banach algebra

**Theorem** Let $A$ be a Banach algebra with a character $\varphi$, and suppose that $J_\varphi$ is not closed. Then there is a dense maximal left ideal $M$ of codimension 1 in $A$ with $A^2 \not\subset M$.

**Proof** Take a linear functional $\lambda$ with $\lambda | J_\varphi = 0$ and $\lambda | \overline{J_\varphi} \neq 0$, and set $M = \ker \lambda$. □

A starting point

We suppose that we have a Banach algebra $(I, \| \cdot \|_I)$ with $I^2 \subsetneq \overline{I^2} = I$, and we take $B = I^\#$ to be the unitization of $I$, so that $B$ is a unital Banach algebra, with identity $e_B$, say, and $I$ is a maximal ideal in $B$. 
**A construction**

From our starting point, consider the Banach algebra $\mathcal{B} = \mathbb{M}_2(B)$, so that $\mathcal{B}$ is also a unital Banach algebra. Set $\mathcal{I} = \mathbb{M}_2(I)$. Then $\mathcal{I}$ is a closed ideal in $\mathcal{B}$ (of codimension 4).

Consider the elements

$$
P = \begin{pmatrix} e_B & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & e_B \end{pmatrix}
$$

in $\mathcal{B}$. Then $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$, and $P + Q$ is the identity of $\mathcal{B}$.

Next, consider the subset $\mathcal{A} = \mathcal{I} + \mathbb{C}P$ in $\mathcal{B}$. Symbolically, $\mathcal{A}$ has the form

$$
\mathcal{A} = \begin{pmatrix} B & I \\ I & I \end{pmatrix}.
$$

Then $\mathcal{A}$ is a closed subalgebra of $\mathcal{B}$, and $\mathcal{I}$ is a maximal ideal in $\mathcal{A}$ of codimension 1; the quotient map $\varphi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is a character on $\mathcal{A}$. 
We define $M_\varphi$ and $J_\varphi$ (in relation to $\mathfrak{A}$ and the character $\varphi$) as above. Then $\mathcal{I} = M_\varphi$ and

$$J_\varphi = \mathcal{J}^2 + \mathcal{J}P + Q\mathcal{J} \subset P\mathcal{J}^2Q + P\mathcal{J}P + Q\mathcal{J} \subset \mathcal{I},$$

and so $\mathcal{J}^2 \subset J_\varphi \subset \mathcal{I} = M_\varphi$. Also

$$\mathcal{I} = (P + Q)\mathcal{J}(P + Q) = P\mathcal{J}P + P\mathcal{J}Q + Q\mathcal{J}.$$

We claim that $\mathcal{J}^2$ is dense in $M_\varphi$. Indeed, given $\varepsilon > 0$ and $x \in I$, there exist $n \in \mathbb{N}$ and $u_1, \ldots, u_n, v_1, \ldots, v_n \in I$ with $\left\|x - \sum_{i=1}^{n} u_i v_i\right\|_I < \varepsilon$. It follows that

$$\left\|\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^{n} \begin{pmatrix} u_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_i & 0 \\ 0 & 0 \end{pmatrix}\right\|_I$$

$$= \left\|\begin{pmatrix} x - \sum_{i=1}^{n} u_i v_i & 0 \\ 0 & 0 \end{pmatrix}\right\|_I < \varepsilon,$$

with similar calculations in the other positions. The claim follows. Hence $\overline{J_\varphi} = M_\varphi$.  

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A construction, continued further

We also claim that $J_\varphi \neq M_\varphi$. Assume towards a contradiction that $J_\varphi = M_\varphi$. Then

$$\mathcal{I} = P\mathcal{I}P + P\mathcal{I}Q + Q\mathcal{I} = P\mathcal{I}^2Q + P\mathcal{I}P + Q\mathcal{I}.$$ 

Since $\mathcal{I} = P\mathcal{I}P \oplus P\mathcal{I}Q \oplus Q\mathcal{I}$, this implies that $P\mathcal{I}Q = P\mathcal{I}^2Q$. However, take $x \in I \setminus I^2$, and consider the element

$$x = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathcal{I}.$$ 

Since $PxQ = x$, we see that $x \in P\mathcal{I}Q$. But every element of $P\mathcal{I}^2Q$ has the form

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix},$$

where $u \in I^2$, and so $x \not\in P\mathcal{I}^2Q$, the required contradiction. Thus the claim holds.

So far we have:

**Theorem** The Banach algebra $\mathcal{A}$ contains a dense maximal left ideal $\mathcal{M}$ with $\mathcal{A}^2 \not\subset \mathcal{M}$ such that $\mathcal{M}$ has codimension 1 in $\mathcal{A}$. \(\square\)
Another algebraic calculation

Proposition Let $A$ be an algebra containing a maximal left ideal $M$ of codimension 1 such that $A^2 \not\subset M$, and take $n \in \mathbb{N}$. Then the matrices $(a_{i,j})$ in $\mathbb{M}_n(A)$ such that $a_{i,1} \in M$ ($i \in \mathbb{N}_n$) form a maximal left ideal in $\mathbb{M}_n(A)$ of codimension $n$.

Proof The matrices that we are considering have the form

$$
\mathcal{M} = \begin{pmatrix}
M & A & \cdots & A \\
M & A & \cdots & A \\
\vdots & \vdots & \ddots & \vdots \\
M & A & \cdots & A 
\end{pmatrix}.
$$

It is clear that $\mathcal{M}$ is a left ideal of codimension $n$ in $\mathbb{M}_n(A)$. Consider a left ideal $\mathcal{J}$ in $\mathbb{M}_n(A)$ with $\mathcal{J} \supsetneq \mathcal{M}$. Since $A^2 \not\subset M$, there exist $a,b \in A$ with $ab \not\in M$, and so $b \not\in M$ and this implies that $\mathbb{C}ab + M = \mathbb{C}b + M = A$. A little multiplication shows that $\mathcal{J} = \mathbb{M}_n(A)$, and so $\mathcal{M}$ is maximal. \qed
Conclusion

We combine the above results to exhibit our main example (assuming that we can reach the starting point).

**Theorem** Let \( n \in \mathbb{N} \). Then there is a Banach algebra \( \mathcal{A} \) with a dense maximal left ideal \( \mathcal{M} \) with codimension \( n \) in \( \mathcal{A} \). We can arrange that \( \mathcal{A} \) be semi-simple and a Banach \( \ast \)-algebra. \( \square \)

**Challenge** Modify the above to find a Banach algebra with a dense maximal left ideal of infinite codimension. Maybe a semigroup algebra of the form \( \ell^1(S) \)?
An equivalence

The existence of such a Banach algebra is equivalent to the existence of a Banach algebra $A$ that has a discontinuous left $A$-module homomorphism into an infinite-dimensional, simple Banach left $A$-module, an ‘automatic continuity’ question.

See a book of mine on ‘automatic continuity’.
A small modification

Replace \( \mathfrak{A} \) and \( \mathcal{I} \) by

\[
\mathfrak{A} = \begin{pmatrix} B & I \\ B & I \end{pmatrix} \quad \text{and} \quad \mathcal{I} = \begin{pmatrix} I & I \\ B & I \end{pmatrix},
\]

respectively. Then nearly the same calculation works, and the bonus is that we get \( \mathfrak{A}^2 = \mathfrak{A} \), and hence \( \mathcal{A}^2 = \mathcal{A} \), so that \( \mathcal{A} \) factors weakly. Indeed, take

\[
x = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in \mathfrak{A},
\]

where \( x_{1,1}, x_{2,1} \in B \) and \( x_{1,2}, x_{2,2} \in I \). Then

\[
x = Px + \begin{pmatrix} 0 & 0 \\ e_B & 0 \end{pmatrix} \begin{pmatrix} x_{2,1} & x_{2,2} \\ 0 & 0 \end{pmatrix} \in \mathfrak{A}^2.
\]

However I do not know the answer to the following:

Let \( \mathcal{A} \) be a Banach algebra that factors. Is it true that every maximal left ideal in \( \mathcal{A} \) is closed?
Commutative starting points

Recall that we require Banach algebras $I$ such that $I^2$ is dense in $I$ and $I^2 \subsetneq I$.

1) Let $I = (\ell^p, \| \cdot \|_p)$, where $1 \leq p < \infty$, taken with the coordinatewise product, so that $I$ is a commutative, semi-simple Banach algebra. The final algebra $\mathcal{A}$ is semi-simple.

2) Take $R$ to be the commutative Banach algebra $C([0, 1])$ with the above truncated convolution multiplication. Here $R$ has an approximate identity, so $R^{[2]}$ is dense in $R$, but $R^2 \subsetneq R$. This example is radical. So the final algebra $\mathcal{A}$ has a large radical.
Non-commutative starting points

3) Let $H$ be an infinite-dimensional Hilbert space, and take $I$ to be the non-commutative Banach algebra of all Hilbert–Schmidt operators on $H$, with the standard norm on $I$. Then $I^2 = I^{[2]}$ is the space of trace-class operators. Here $I$ is a semi-simple algebra and a Banach $\ast$-algebra, and we can show that he corresponding algebra $\mathcal{A}$ has the same properties.

4) Let $E$ be an infinite-dimensional Banach space, and let $I = \mathcal{N}(E)$, the nuclear operators on $E$, so that $I$ is a non-commutative Banach algebra with respect to the nuclear norm. Then $I^{[2]}$ is dense in $I$ and $I^2$ has infinite codimension in $I$.  \[ \square \]

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Finitely-generated maximal left ideals

A left ideal $I$ in a unital algebra $A$ is finitely-generated if there exist $a_1, \ldots, a_n \in A$ such that $I = Aa_1 + Aa_2 + \cdots + Aa_n$.

Theorem, Sinclair–Tullo, 1974 Let $A$ be a unital Banach algebra. Suppose that all closed left ideals are finitely-generated. Then $A$ is finite dimensional.

Conjecture, D-Zelazko, 2012 Let $A$ be a unital Banach algebra. Suppose that all maximal left ideals are finitely-generated. Then $A$ is finite dimensional.

Theorem True when $A$ is commutative, and for various other examples.

Theorem, D-Kania-Kochanek-Koszmider-Laustsen, 2013 Consider $\mathcal{B}(E)$. Then the conjecture holds for very many different classes of Banach spaces $E$. No counter-example is known.
Semigroup algebras

Theorem, Jared White, 2017 Consider $\ell^1(S)$ for a monoid $S$, or its weighted version $\ell^1(S,\omega)$. Then the conjecture holds for many different classes of semigroup $S$, including all groups.

For a semigroup algebra $\ell^1(S)$, set

$$\ell^1_0(S) = \left\{ f : \sum_{s \in S} f(s) = 0 \right\}$$

Then $\ell^1_0(S)$ is a maximal ideal, called the augmentation ideal.

Theorem, Jared White, 2017 Let $S$ be a monoid. Then $\ell^1_0(S)$ is finitely generated (as a left ideal) iff $S$ is ‘pseudo-finite’.

Could infinite, pseudo-finite semigroups give counters to the DZ conjecture? One needs all maximal left ideals to be finitely generated.
Counter-examples?

White There are rather trivial infinite, pseudo-finite semigroups. But these do not give counter-examples to the main conjecture.

Example, VG, et al There is a non-trivial infinite, pseudo-finite semigroup.

Question Does this give a counter-example to the DZ conjecture? What are the maximal left ideals for this example?