Zero-divisor graphs of MV-algebras

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Contents

Definition [1, 2]

An MV-algebra is an algebra $(A, \oplus, *, 0)$ of type $(2, 1, 0)$ satisfying the following axioms: for all $x, y \in A$,

($M1$) $(A, \oplus, 0)$ is a commutative monoid,

($M2$) $x^{**} = x$,

($M3$) $x \oplus 0^* = 0^*$,

($M4$) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

MV-algebras are the algebraic counterpart of Lukasiewicz logic, a many-valued logic with infinitely many values. They have been introduced by C.C.Chang [1](*C. C. Chang, Algebraic analysis of many-valued logic, Trans Am Math Soc., 88 (1958) 467-490*) to prove the completeness of a certain axiom system.

The class of MV-algebras is an equation class, so it forms a varity.

MV-algebras

On every MV-algebra A, define the constant 1 and the operation $\odot$ as follows:

$$1 = 0^* \quad \text{and} \quad x \odot y = (x^* \oplus y^*)^*.$$

Then for all $x, y \in A$, the following well-known properties hold:

- $(A, \odot, *, 1)$ is an MV-algebra,
- $*$ is an isomorphism between $(A, \oplus, *, 0)$ and $(A, \odot, *, 1)$,
- $1^* = 0$,
- $x \oplus y = (x^* \odot y^*)^*$,
- $x \oplus 1 = 1$,
- $x \oplus x^* = 1$,
- $x \odot x^* = 0$.

Example 1

Equip the real unit interval $[0,1]$ with the operations

$$x \oplus y = min\{1, x+y\} \text{ and } x^* = 1-x.$$

Then $[0,1] = ([0,1], \oplus, *, 0)$ is an MV-algebra and $x \odot y = max\{0, x+y-1\}$.
The rational numbers in $[0,1]$ and for each $n \geq 2$, the n-element set

$$L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\},$$

yield examples of subalgebras of $[0,1]$.

The MV-algebra $[0,1]$ is important because

- it generates the variety of all MV-algebras, and
- Chang Completeness Theorem says that an equation holds in $[0,1]$ if and only if it holds in every MV-algebra.

MV-algebras

Example 2

For any Boolean algebra $(A, \vee, \wedge, -, 0, 1)$, the structure $(A, \vee, -, 0)$ is an MV-algebra, where $\vee, -$ and 0 denote, respectively, the join, the complement and the smallest element in A.

- Boolean algebras form a subvariety of the variety of MV-algebras. They are precisely the MV-algebras satisfying the additional equation $x \oplus x = x$.
- MV-algebras are the non-idempotent generation of Boolean algebra.

The natural order

For any MV-algebra A and $x, y \in A$, define

$$x \leq y \Longleftrightarrow x^* \oplus y = 1.$$

It is well-known that $\leq$ is a partial order on A, called the natural order of A.

The natural order determines a structure of bounded distributive lattice on A, with 0 and 1 are respectively the bottom and the top element, and

$$x \vee y = (x \odot y^*) \oplus y \text{ and } x \wedge y = x \odot (x^* \oplus y).$$

- Note: the axiom $(M4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ ensures that $x \vee y = y \vee x$.

The relation among MV-algebras and some other algebras

It is known that MV-algebras are categorically equivalent to the following algebras:

- Abelian l-groups with strong unit [2];
- Bounded commutative BCK-algebras [3];
- Bounded DRl -semigroups satisfying the identity $1-(1-x)=x$ [4];
- Bézout domains with non-zero unit-radical.

Thus one can study MV-algebras from these aspects.

MV-semirings

Definition of a semiring

A semiring is an algebra $(S, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the following axioms:

$(S1)$ $(S, +, 0)$ is a commutative monoid,

$(S2)$ $(S, \cdot, 1)$ is a monoid,

$(S3)$ $\cdot$ distributes over $+$ from either side,

$(S4)$ $0 \cdot x = 0 = x \cdot 0$ for all $x \in S$.

A semiring S is called commutative if so is the multiplication; additively idempotent if it satisfies the equation $x + x = x$.

MV-semirings

For an additively idempotent semiring S, there exists a natural order given by

$$s \leq t \Longleftrightarrow s + t = t$$

with $s, t \in S$.

Definition

An MV-semiring is a commutative, additively idempotent semiring $(S, +, \cdot, 0, 1)$ for which there exists a map $* : S \longrightarrow S$, called the negation, satisfying the following conditions: for all $a, b \in S$,

(*i*) $a \cdot b = 0$ if and only if $a \leq b^*$,

(*ii*) $a + b = (a^* \cdot (a^* \cdot b)^*)^*$.

MV-semirings

The next proposition gives a bridge between MV-algebras and MV-semirings.

Proposition [5]

For any MV-algebra $(A, \oplus, *, 0)$, both the semiring reducts $A^{\vee\odot} = (A, \vee, \odot, 0, 1)$ and $A^{\wedge\oplus} = (A, \wedge, \oplus, 1, 0)$ are MV-semirings. Conversely, if $(S, +, \cdot, 0, 1)$ is an MV-semiring, with negation $*$, the structure $(S, \oplus, *, 0)$ with

$$a \oplus b = (a^* \cdot b^*)^* \quad \text{for all} \quad a, b \in S$$

is an MV-algebra.

The zero-divisor graph of an MV-algebra

The zero-divisor graph of a semigroup [5]

Let $(S, \cdot, 0)$ be a commutative semigroup with zero. Recall that an element a of S is called a zero divisor if $a \cdot b = 0$ for some non-zero element b of S.

The zero-divisor graph $\Gamma(S)$ of S is the simple graph

- whose vertex set $V(\Gamma(S))$ is the set of all non-zero zero-divisors of S, and
- two distinct such elements a, b form an edge precisely when $a \cdot b = 0$.

The concept of the zero-divisor graph is extended to many algebras, such as, noncommutative semigroups, semirings, posets , etc. Now we extend it to MV-algebras.

The zero-divisor graph of an MV-algebra

For an MV-algebra A, since $(A, \odot, 0)$ is a commutative semigroup with zero, we define the zero-divisor graph of $(A, \odot, 0)$ to be the zero-divisor graph of MV-algebra A, and we denote it by $\Gamma(A)$, that is to say, $\Gamma(A)$ is the simple graph

- whose vertex set $V(\Gamma(A))$ is the set of all non-zero zero-divisors of A, and
- two distinct such elements a, b form an edge precisely when $a \odot b = 0$.

Note

For an MV-algebra A, the zero-divisor graph of A is the same as the zero-divisor graph of MV-semiring $A^{\vee\odot} = (A, \vee, \odot, 0, 1)$.

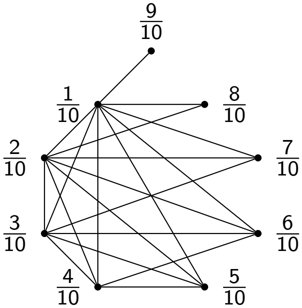
Proposition

$V(\Gamma(A)) = A \setminus \{0, 1\}$ for any MV-algebra A.

Proof.

Let A be an MV-algebra. It is obvious that $V(\Gamma(A)) \subseteq A \setminus \{0, 1\}$; conversely, for any $a \in A \setminus \{0, 1\}$, we have $a^* \in A \setminus \{0, 1\}$ and $a \odot a^* = 0$, so $a \in V(\Gamma(A))$. □

The graphs $\Gamma(L_{11})$ is as follows:



Theorem

Let A be an MV-algebra and $\Gamma(A)$ *is not null. Then* $\Gamma(A)$ *is connected and* $diam(\Gamma(A)) \leq 3$.

Theorem

Let A be an MV-algebra and $\Gamma(A)$ *is not null. Then the following statements are equivalent:*

(i) $diam(\Gamma(A)) = 1$,

(ii) $\Gamma(A) \cong K_2$,

(iii) $|A| = 4$,

(iv) $A \cong L_4$ *or* $A \cong B_4$, *where* B_4 *is the 4-element Boolean algebra.*

Theorem

Let A be an MV-algebra, $|A| = n \geq 5$ and $\Gamma(A) \cong \Gamma(L_n))$ implies $A \cong L_n$.

Proposition

Let A be an MV-chain. Then the following statements are true:

(i) $diam(\Gamma(A)) \leq 2$;

(ii) $diam(\Gamma(A)) = 2$ if and only if $|A| \geq 5$.

Proposition

Let A, B and C be MV-algebras such that $A \cong B \times C$. Then we have

$$(|B| \geq 3 \text{ or } |C| \geq 3) \Longleftrightarrow diam(\Gamma(A)) = 3.$$

References

1 C. C. Chang, Algebraic analysis of many-valued logic, Trans Am Math Soc., 88 (1958) 467-490.

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5 Frank R. DeMeyer, Thomas McKenzie, and Kim Schneider, The zero-divisor graph of a commutative semigroup, Semigroup Forum 65(2) (2002) 206-214.

Thanks

Thank you for your attention!