Zero-divisor graphs of MV-algebras

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Definition [1, 2]

An MV-algebra is an algebra \((A, \oplus, *, 0)\) of type \((2, 1, 0)\) satisfying the following axioms: for all \(x, y \in A\),

1. \((M1)\) \((A, \oplus, 0)\) is a commutative monoid,
2. \((M2)\) \(x^{**} = x\),
3. \((M3)\) \(x \oplus 0^* = 0^*\),
4. \((M4)\) \((x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x\).

MV-algebras are the algebraic counterpart of Lukasiewicz logic, a many-valued logic with infinitely many values. They have been introduced by C.C.Chang [1](C. C. Chang, Algebraic analysis of many-valued logic, Trans Am Math Soc., 88 (1958) 467-490) to prove the completeness of a certain axiom system.

The class of MV-algebras is an equation class, so it forms a variety.
On every MV-algebra $A$, define the constant 1 and the operation $\odot$ as follows:

$$1 = 0^* \quad \text{and} \quad x \odot y = (x^* \oplus y^*)^*.$$ 

Then for all $x, y \in A$, the following well-known properties hold:

- $(A, \odot, *, 1)$ is an MV-algebra,
- $*$ is an isomorphism between $(A, \oplus, *, 0)$ and $(A, \odot, *, 1)$,
- $1^* = 0$,
- $x \oplus y = (x^* \odot y^*)^*$,
- $x \oplus 1 = 1$,
- $x \oplus x^* = 1$,
- $x \odot x^* = 0$. 

Example 1

Equip the real unit interval \([0, 1]\) with the operations

\[
x \oplus y = \min\{1, x + y\} \quad \text{and} \quad x^* = 1 - x.
\]

Then \([0, 1] = ([0, 1], \oplus, *, 0)\) is an MV-algebra and

\[
x \circ y = \max\{0, x + y - 1\}.
\]

The rational numbers in \([0, 1]\) and for each \(n \geq 2\), the \(n\)-element set

\[
L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\},
\]

yield examples of subalgebras of \([0, 1]\).

The MV-algebra \([0, 1]\) is important because

- it generates the variety of all MV-algebras, and
- Chang Completeness Theorem says that an equation holds in \([0, 1]\) if and only if it holds in every MV-algebra.
Example 2

For any Boolean algebra \((A, \lor, \land, -, 0, 1)\), the structure \((A, \lor, -, 0)\) is an MV-algebra, where \(\lor\), \(\neg\) and 0 denote, respectively, the join, the complement and the smallest element in \(A\).

- Boolean algebras form a subvariety of the variety of MV-algebras. They are precisely the MV-algebras satisfying the additional equation \(x \oplus x = x\).
- MV-algebras are the non-idempotent generation of Boolean algebra.
The natural order

For any MV-algebra $A$ and $x, y \in A$, define

$$x \leq y \iff x^* \oplus y = 1.$$ 

It is well-known that $\leq$ is a partial order on $A$, called the natural order of $A$.

The natural order determines a structure of bounded distributive lattice on $A$, with $0$ and $1$ are respectively the bottom and the top element, and

$$x \lor y = (x \odot y^*) \oplus y \text{ and } x \land y = x \odot (x^* \oplus y).$$

- **Note:** the axiom $(M4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ ensures that $x \lor y = y \lor x$. 
The relation among MV-algebras and some other algebras

It is known that MV-algebras are categorically equivalent to the following algebras:

- Abelian l-groups with strong unit [2];
- Bounded commutative BCK-algebras [3];
- Bounded DRI -semigroups satisfying the identity \(1 - (1 - x) = x\) [4];
- Bézout domains with non-zero unit-radical.

Thus one can study MV-algebras from these aspects.
**Definition of a semiring**

A semiring is an algebra \((S, +, \cdot, 0, 1)\) of type \((2, 2, 0, 0)\) satisfying the following axioms:

1. \((S, +, 0)\) is a commutative monoid,
2. \((S, \cdot, 1)\) is a monoid,
3. \(\cdot\) distributes over \(+\) from either side,
4. \(0 \cdot x = 0 = x \cdot 0\) for all \(x \in S\).

A semiring \(S\) is called commutative if so is the multiplication; additively idempotent if it satisfies the equation \(x + x = x\).
For an additively idempotent semiring $S$, there exists a natural order given by

$$s \leq t \iff s + t = t$$

with $s, t \in S$.

**Definition**

An MV-semiring is a commutative, additively idempotent semiring $(S, +, \cdot, 0, 1)$ for which there exists a map $\ast : S \to S$, called the negation, satisfying the following conditions: for all $a, b \in S$,

(i) $a \cdot b = 0$ if and only if $a \leq b^\ast$,

(ii) $a + b = (a^\ast \cdot (a^\ast \cdot b)^\ast)^\ast$. 
The next proposition gives a bridge between MV-algebras and MV-semirings.

**Proposition [5]**

For any MV-algebra \((A, \oplus, *, 0)\), both the semiring reducts \(A^{\lor\odot} = (A, \lor, \odot, 0, 1)\) and \(A^{\land\oplus} = (A, \land, \oplus, 1, 0)\) are MV-semirings. Conversely, if \((S, +, \cdot, 0, 1)\) is an MV-semiring, with negation \(*\), the structure \((S, \oplus, *, 0)\) with

\[
a \oplus b = (a^* \cdot b^*)^* \quad \text{for all} \quad a, b \in S
\]

is an MV-algebra.
Let \((S, \cdot, 0)\) be a commutative semigroup with zero. Recall that an element \(a\) of \(S\) is called a zero divisor if \(a \cdot b = 0\) for some non-zero element \(b\) of \(S\).

The zero-divisor graph \(\Gamma(S)\) of \(S\) is the simple graph

- whose vertex set \(V(\Gamma(S))\) is the set of all non-zero zero-divisors of \(S\), and
- two distinct such elements \(a, b\) form an edge precisely when \(a \cdot b = 0\).

The concept of the zero-divisor graph is extended to many algebras, such as, noncommutative semigroups, semirings, posets, etc. Now we extend it to MV-algebras.
For an MV-algebra $A$, since $(A, \odot, 0)$ is a commutative semigroup with zero, we define the zero-divisor graph of $(A, \odot, 0)$ to be the zero-divisor graph of MV-algebra $A$, and we denote it by $\Gamma(A)$, that is to say, $\Gamma(A)$ is the simple graph

- whose vertex set $V(\Gamma(A))$ is the set of all non-zero zero-divisors of $A$, and
- two distinct such elements $a, b$ form an edge precisely when $a \odot b = 0$.

**Note**

For an MV-algebra $A$, the zero-divisor graph of $A$ is the same as the zero-divisor graph of MV-semiring $A^{\vee \odot} = (A, \lor, \odot, 0, 1)$. 

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Zero-divisor graphs of MV-algebras
Proposition

\[ V(\Gamma(A)) = A \setminus \{0, 1\} \] for any MV-algebra \( A \).

Proof.

Let \( A \) be an MV-algebra. It is obvious that \( V(\Gamma(A)) \subseteq A \setminus \{0, 1\} \); conversely, for any \( a \in A \setminus \{0, 1\} \), we have \( a^* \in A \setminus \{0, 1\} \) and \( a \odot a^* = 0 \), so \( a \in V(\Gamma(A)) \).
The zero-divisor graph of an MV-algebra

The graphs $\Gamma(L_{11})$ is as follows:
Theorem

Let $A$ be an MV-algebra and $\Gamma(A)$ is not null. Then $\Gamma(A)$ is connected and $\text{diam}(\Gamma(A)) \leq 3$.

Theorem

Let $A$ be an MV-algebra and $\Gamma(A)$ is not null. Then the following statements are equivalent:

(i) $\text{diam}(\Gamma(A)) = 1$,

(ii) $\Gamma(A) \cong K_2$,

(iii) $|A| = 4$,

(iv) $A \cong L_4$ or $A \cong B_4$, where $B_4$ is the 4-element Boolean algebra.
Theorem

Let $A$ be an MV-algebra, $|A| = n \geq 5$ and $\Gamma(A) \cong \Gamma(L_n)$ implies $A \cong L_n$.

Proposition

Let $A$ be an MV-chain. Then the following statements are true:

(i) $\text{diam}(\Gamma(A)) \leq 2$;

(ii) $\text{diam}(\Gamma(A)) = 2$ if and only if $|A| \geq 5$. 
Proposition

Let $A$, $B$ and $C$ be MV-algebras such that $A \cong B \times C$. Then we have

$$(|B| \geq 3 \text{ or } |C| \geq 3) \iff \text{diam}(\Gamma(A)) = 3.$$


4 J. Rachunek, Olomouc, MV-ALGEBRAS ARE CATEGORICALLY EQUIVALENT TO A CLASS OF \( DRI_{1(i)} \)-SEMIGROUPS, Mathematica Bohemica, 1998(4), 437-441.

Thank you for your attention!