ABUNDANT AND AMPLE STRAIGHT LEFT ORDERS

VICTORIA GOULD

ABSTRACT. A subsemigroup S of a semigroup Q is a *straight left order* in Q if every $q \in Q$ can be written as $a^\sharp b$ for some $a, b \in S$ with $a\,\mathcal{R}\,b$ in Q and if, in addition, every element of S that is *square cancellable* lies in a subgroup of Q. Here $a^\sharp$ denotes the group inverse of a in some (hence any) subgroup of Q. If S is a straight left order in Q, then Q is necessarily regular; the idea is that Q has a better understood structure than that of S.

Necessary and sufficient conditions exist on a semigroup S for S to be a straight left order. The technique is to consider a pair $\mathcal{P}$ of *preorders* on S. If such a pair satisfies conditions mimicking those satisfied by $(\leq_{\mathcal{L}}, \leq_{\mathcal{R}})$ on a regular semigroup, and if certain subsemigroups of S are *right reversible*, then S is a straight left order.

The conditions required for $\mathcal{P}$ to satisfy are somewhat lengthy. In this paper we aim to circumvent some of these by specialising in two ways. First we consider only *fully stratified* left orders, that is, the case where $\mathcal{P} = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ (certainly the most natural choice for $\mathcal{P}$) and the other is to insist that S be *abundant*, that is, every $\mathcal{R}^*$-class and every $\mathcal{L}^*$-class of S contains an idempotent.

Our results may be used to show that the monoid of endomorphisms of a hereditary basis algebra of finite rank is a fully stratified straight left order.

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1. INTRODUCTION

The relation $\leq_{\mathcal{L}^*}$ is defined on a semigroup S by the rule that $a \leq_{\mathcal{L}^*} b$ if and only if for all $x, y \in S^1$,

$$bx = by \text{ implies } ax = ay.$$

Certainly $\leq_{\mathcal{L}^*}$ is a right compatible preorder, with associated equivalence $\mathcal{L}^*$. It is well known (see, for example [2, 10],) that $a \leq_{\mathcal{L}^*} b$ $(a\,\mathcal{L}^*\,b)$ if and only if $a \leq_{\mathcal{L}} b$ $(\,a\,\mathcal{L}\,b)$ in some oversemigroup of S, where $\leq_{\mathcal{L}}$ is the preorder associated with Green's relation $\mathcal{L}$. The relations $\leq_{\mathcal{R}^*}$ and $\mathcal{R}^*$ are defined dually; we put $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$. A semigroup S is *abundant* if every $\mathcal{R}^*$-class and every $\mathcal{L}^*$-class contains an idempotent. Many results for regular semigroups have analogues for abundant semigroups: the general pattern is that the role of a

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group (or group action) in the regular case is replaced by that of a cancellative monoid (see, for example [2, 5]). In this paper we show that certain abundant semigroups are special subsemigroups of regular semigroups; to be precise, they are fully stratified straight left orders.

The concepts of left order in ring theory and in group theory have proved fruitful in the investigation of non-regular rings and cancellative semigroups. Here we are concerned with the analogous notion for semigroups, introduced by Fountain and Petrich in [7]. Their aim was to develop concepts that reflect the equal importance of *all* subgroups of a semigroup, not only the group of units, which of course may not even exist. The role of non-zero divisors in the notion of left order in the ring case is replaced by that of square cancellable elements. An element a of a semigroup S is *square cancellable* if $a\,\mathcal{H}^*\,a^2$. In view of Green's theorem and the comments at the beginning of this section, being square cancellable is a strong necessary condition for lying in a subgroup of an oversemigroup. We insist that all such elements lie in subgroups of any semigroup of left quotients.

A subsemigroup S of a semigroup Q is a *left order* in Q and Q is a *semigroup of left quotients* of S if every element of Q can be written as $a^\sharp b$ for some $a, b \in S$ and if, in addition, every square cancellable element of S lies in a subgroup of Q. Here $a^\sharp$ denotes the group inverse of a in a subgroup of Q; use of the notation $a^\sharp$ implies that a lies in a subgroup of Q. *Right orders* and *semigroups of right quotients* are defined dually; if S is both a left order and a right order in Q then S is an *order* in Q and Q is a *semigroup of quotients* of S. If S is a left order in Q and every element of Q can be written as $a^\sharp b$ where $a, b \in S$ and $a\,\mathcal{R}\,b$ in Q, we say that S is *straight* in Q; all the common examples of left orders are straight.

In the case where S is a subsemigroup of a group Q, our concept of a semigroup of left quotients becomes the classical one of a group of left quotients. A theorem of Ore and Dubreil [1] says that a semigroup S has a group of left quotients if and only if it is right reversible, that is, for all $a, b \in S$, $Sa \cap Sb \neq \emptyset$, and cancellative.

For arbitrary semigroups, such a neat characterisation of left orders is impossible. Left orders are determined by right reversible, cancellative subsemigroups *and also* by pairs of preorders $\mathcal{P}$. The role of $\mathcal{P}$ is to determine the $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ structure of the semigroup of left quotients.

In [8] we give necessary and conditions on a pair of preorders $\mathcal{P}$ on a semigroup S such that S is a left order in a semigroup Q and Q *induces* $\mathcal{P}$, that is,

$$\mathcal{P} = (\leq_{\mathcal{L}} \cap (S \times S), \leq_{\mathcal{R}} \cap (S \times S))$$

(where here, as elsewhere, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are relations on Q). The conditions split into two parts; one guarantees that $\mathcal{P}$ 'looks like' $(\leq_{\mathcal{L}}, \leq_{\mathcal{R}})$ on a regular semigroup; the other says that certain subsemigroups of S are right reversible and also ensures that square cancellable elements lie in subgroups of any semigroup of left quotients. Details are given in Section 2.

If a pair $\mathcal{P}$ of preorders on a semigroup S satisfies the first set of conditions, we say it is an *embeddable $*$-pair*. We aim to simplify the definition of an embeddable $*$-pair in the case where S is abundant and $\mathcal{P} = \mathcal{P}^* = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$. If S is a left order in Q and Q induces $\mathcal{P}^*$ we say that S is *fully stratified* in Q.

The layout of the paper is as follows. In Section 2 we give the required definitions and prove some preliminary facts for abundant semigroups. In Section 3 we specialise the characterisation of straight left orders given in [8] to the fully stratified case. It is then an easy matter in the final section to combine our earlier results to characterise abundant fully stratified straight left orders. Further simplification is possible if we insist that S is IC-abundant or bountiful. These conditions ensure that the idempotents of an abundant semigroup have some of the properties of those of a regular semigroup. These results are used in [6] to show that the endomorphism monoid of a hereditary basis algebra of finite rank is a fully stratified straight left order in a regular monoid. The best example of a hereditary basis algebra is a free module over a principal ideal domain. Finally we specialise further to left orders that are *ample* semigroups. An ample semigroup is a bountiful semigroup in which the idempotents commute. We show that an ample semigroup is a fully stratified straight left order if and only if $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$, for any $a, b, c \in S$ with a square cancellable, $b, c \leq_{\mathcal{R}^*} a$ and $ab \, \mathcal{R}^* \, ac$, we have $b \, \mathcal{R}^* \, c$, and the $\mathcal{H}^*$-class of any square cancellable element is right reversible. Moreover, any semigroup Q in which S is such an order must be inverse and contain S as a full subsemigroup.

2. Preliminaries

In this section we give some details of classes of abundant semigroups and recall the characterisation of straight left orders given in [8].

We assume a familiarity with the basic notions of semigroup theory, as in [1] and [9], in particular, with Green's relations. We bear in mind the fact that Green's relations $\mathcal{L}$ and $\mathcal{R}$ are the equivalence relations associated with the preorders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ respectively, where for elements p and q of a semigroup Q,

$$p \leq_{\mathcal{L}} q \text{ if and only if } Q^1 p \subseteq Q^1 q$$

and dually,

$$p \leq_{\mathcal{R}} q \text{ if and only if } pQ^1 \subseteq qQ^1.$$

Notice that the relations $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are respectively right and left compatible with multiplication.

To a certain extent, the relations $\leq_{\mathcal{L}^*}, \mathcal{L}^*, \leq_{\mathcal{R}^*}, \mathcal{R}^*$ behave as their corresponding unstarred versions. Notice that $\leq_{\mathcal{L}} \, \subseteq \, \leq_{\mathcal{L}^*}$ and that for regular elements the two relations coincide. Corresponding statements are true for $\mathcal{L}^*$ and $\mathcal{L}$, etc. For any elements a, b of a semigroup S we have that $ab \leq_{\mathcal{L}^*} b$ and if e is idempotent and $a \leq_{\mathcal{L}^*} e$, then $ae = a$.

If S is a left order in Q, then it is easy to see that any element of Q can be written as $a^\sharp b$ where $a, b \in S$ and $a \leq_{\mathcal{R}} b$ in Q. If we can insist that $a \, \mathcal{R} \, b$ in Q, then S is a *straight* left order in Q. In this case, S intersects every $\mathcal{H}$-class of

Q and inherits more of its structure. Many left orders are naturally straight, for example, this is always the case if $\mathcal{H}$ is a congruence on Q. Straight right orders are defined dually.

The main result of [8] is a complete characterisation of those semigroups that are straight left orders. The approach is via embeddable *-pairs, which are pairs of preorders whose properties reflect those of $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ on a regular semigroup and which for convenience and reference we now detail.

An ordered pair $\mathcal{P} = (\leq_l, \leq_r)$ of preorders on a semigroup S is a $*$-*pair* if $\leq_l$ is right compatible with multiplication, $\leq_r$ is left compatible, $\leq_l \subseteq \ \leq_{\mathcal{L}^*}$ and $\leq_r \subseteq \ \leq_{\mathcal{R}^*}$. Clearly $\mathcal{P}^* = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ is a $*$-pair for any semigroup S. From the comments at the beginning of the Introduction we have the following.

Corollary 2.1. *Let S be a subsemigroup of Q. Then*

$$(\leq_{\mathcal{L}^Q} \cap (S \times S), \leq_{\mathcal{R}^Q} \cap (S \times S))$$

*is a $*$-pair for S, the $*$-*pair for S induced by Q.

In particular the above applies when S is a left order in Q; moreover the induced $*$-pair satisfies a number of further conditions.

For a $*$-pair $\mathcal{P} = (\leq_l, \leq_r)$, we denote by $\mathcal{L}'$ and $\mathcal{R}'$ the equivalence relations associated with $\leq_l$ and $\leq_r$ respectively. Notice that $\mathcal{L}'$ is a right congruence contained in $\mathcal{L}^*$ and $\mathcal{R}'$ is a left congruence contained in $\mathcal{R}^*$; $\mathcal{H}'$ denotes the intersection of $\mathcal{L}'$ and $\mathcal{R}'$ and we put

$$\mathcal{G}(S) = \{a \in S : a\,\mathcal{H}'\,a^2\} \text{ and } \mathcal{S}(S) = \{a \in S : a\,\mathcal{H}^*\,a^2\},$$

that is, $\mathcal{S}(S)$ is the set of square cancellable elements of S. Clearly $\mathcal{G}(S) \subseteq \mathcal{S}(S)$.

Definition 2.2. *Let $\mathcal{P} = (\leq_\ell, \leq_r)$ be a *-pair for a semigroup S. Then $\mathcal{P}$ is an* embeddable *$*$-pair if it satisfies the following conditions and the duals (Eii)(r),..., (Evii)(r) of (Eii)(l),..., (Evii)(l) respectively.*

(Ei) $\mathcal{L}' \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{L}'$.
(Eii)(l) For all $b, c \in S$, $b \leq_l c$ if and only if $b\,\mathcal{L}'\,dc$ for some $d \in S$.
(Eiii)(l) Every $\mathcal{L}'$-class contains an element from $\mathcal{G}(S)$.
(Ev)(l) For all $a \in \mathcal{G}(S)$ and $b \in S$, if $b \leq_l a$, then $ba\,\mathcal{R}'\,b$.
(Evi)(l) For all $a \in \mathcal{G}(S)$ and $b, c \in S$, if $b, c \leq_l a$ and $ba = ca$, then $b = c$.
(Evii)(l) For all $a \in \mathcal{G}(S)$ and $b, c \in S$, if $b, c \leq_l a$ and $ba\,\mathcal{L}'\,ca$, then $b\,\mathcal{L}'\,c$.

If $\mathcal{P}$ satisfies conditions (Ex)(l) and (Ex)(r), we say that $\mathcal{P}$ satisfies condition (Ex).

If S is a straight left order in Q, then Q is perforce regular and S intersects every $\mathcal{H}$-class of Q, so that Lemma 4.2 of [8] gives that Q induces an embeddable $*$-pair. Embeddable $*$-pairs are crucial in determining the existence and uniqueness of semigroups of straight left quotients.

Theorem 2.3. [8] *Let S be a semigroup having an embeddable $*$-pair $\mathcal{P}$. Then S is a straight left order in a semigroup Q inducing $\mathcal{P}$ if and only if $\mathcal{P}$ satisfies conditions*

(Gi) $\mathcal{S}(S) = \mathcal{G}(S)$
and
(Gii) If $a \in \mathcal{S}(S)$ *then* H'_a *is right reversible.*

We aim to simplify the above theorem to the case where S is abundant and fully stratified in Q.

Definition 2.4. *Let S be a left order in Q. Then S is* stratified *in Q if $\mathcal{R} \cap (S \times S) = \mathcal{R}^*$ and $\mathcal{L} \cap (S \times S) = \mathcal{L}^*$ and* fully stratified *in Q if Q induces $\mathcal{P}^* = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$. Clearly a fully stratified left order is stratified, but the converse need not be true.*

We recall from [2] that a semigroup S is *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class contains an idempotent. Since $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$ on a regular semigroup, regular semigroups are certainly abundant. Being abundant is not, in general, enough to produce pleasing analogues of structure theorems for regular semigroups. Usually one needs the extra condition on idempotents of being idempotent connected.

If a is an element of an abundant semigroup S, then a^* denotes an idempotent in the $\mathcal{L}^*$-class of a and a^+ an idempotent in the $\mathcal{R}^*$-class of a. If $E(S)$ is a semilattice, then a^* and a^+ are *uniquely* determined by a; this need not be true otherwise. For an abundant semigroup S we let B be the semiband generated by $E(S)$ and for an idempotent e we denote by $\langle e \rangle$ the subsemigroup of eBe generated by the idempotents of eBe.

Definition 2.5. *An abundant semigroup S is idempotent connected (IC) if for each element a of S and for any a^* and a^+ there is a bijection $\alpha : \langle a^+ \rangle \to \langle a^* \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^+ \rangle$.*

The bijection α in the above definition is necessarily an isomorphism [4], referred to as a *connecting isomorphism*. It follows in particular that $a^+\alpha = a^*$.

Definition 2.6. *An IC-abundant semigroup S such that $E(S)$ is a band is called* bountiful [12]*. If moreover $E(S)$ is a semilattice, S is* ample *(formerly,* type A [3].

The more familiar definition of an ample semigroup is an abundant semigroup S, with semilattice of idempotents, satisfying the ample identities

$$ae = (ae)^+a, ea = a(ea)^*$$

for all $a \in S$ and $e \in E(S)$. That this concept coincides with that in Definition 2.6 is shown in Proposition 3.1 of [4]. Bountiful and ample semigroups are the $*$-analogues of orthodox and inverse semigroups, respectively.

The first results of this paper will enable us to simplify the definition of embeddable $*$-pair in the case where S is an IC-abundant semigroup.

Proposition 2.7. *Let S be an IC-abundant semigroup. Then S satisfies condition E(ii) with respect to $\mathcal{P}^* = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$.*

Proof. Let $b, c \in S$. Certainly if $b \,\mathcal{L}^*\, dc$ for some $d \in S$, then $b \leq_{\mathcal{L}^*} c$.

Conversely, suppose that $b \leq_{\mathcal{L}^*} c$. Since S is abundant there are idempotents b^*, c^* in S with

$$b^* \,\mathcal{L}^*\, b \leq_{\mathcal{L}^*} c \,\mathcal{L}^*\, c^*.$$

Hence $b^* \leq_{\mathcal{L}} c^*$ so that $c^*b^* \in E(S)$, indeed $c^*b^* \in \langle c^* \rangle$. Now

$$b = b(c^*b^*) \leq_{\mathcal{L}^*} c(c^*b^*) \leq_{\mathcal{L}^*} b$$

so that $b \,\mathcal{L}^*\, c(c^*b^*) = (c^*b^*)\alpha^{-1}c$ for some connecting isomorphism $\alpha : \langle c^* \rangle \to \langle c^+ \rangle$. Thus E(ii)(l) holds for $\mathcal{P}^* = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$; dually for E(ii)(r). □

For our next proposition, we need a technical lemma. Note that in our 'egg-box' pictures the rows denote $\mathcal{R}^*$-classes and the columns $\mathcal{L}^*$-classes.

Lemma 2.8. *Let S be a semigroup, $a \in \mathcal{S}(S)$ and suppose that $f \,\mathcal{L}^*\, a \,\mathcal{R}^* e$ where $e, f \in E(S)$. Then we have the following egg-box diagram:*

a	ae		e
fa	fe		
f			

Further, if $fe \in E(S)$, then for any $u \in H^*_{fa}$ and $v \in H^*_{ae}$, we have $vu \in H^*_a$.

Proof. The structure of the table is a consequence of the fact that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ a left congruence. For the final part bear in mind that an idempotent is a left (right) identity for its $\mathcal{R}^*$-class ($\mathcal{L}^*$-class). □

We say that a semigroup S *satisfies condition* (*) if every square cancellable element has an idempotent in its $\mathcal{H}^*$-class. This condition will be crucial for the results of the final section.

Proposition 2.9. *Let S be a bountiful semigroup. Then S satisfies condition (*).*

Proof. Let $a \in \mathcal{S}(S)$ and let e, f be idempotents with $f \,\mathcal{L}^*\, a \,\mathcal{R}^* e$. Since S is bountiful there is a connecting isomorphism $\alpha : \langle e \rangle \to \langle f \rangle$. By the last statement of Lemma 2.8, $a \,\mathcal{H}^*\, aefa$.

Using the fact that $efe \in \langle e \rangle$ and the definition of a connecting isomorphism, we have that

$$aefa = a(efe)a = a^2(efe)\alpha.$$

Thus

$$f \,\mathcal{L}^*\, aefa \leq_{\mathcal{L}^*} (efe)\alpha \leq_{\mathcal{L}^*} f$$

so that $f \,\mathcal{L}^*\, (efe)\alpha$ and it follows that $f(efe)\alpha = f$. But $(efe)\alpha \leq_{\mathcal{R}} f$ so that $e\alpha = f = (efe)\alpha$ whence $e = efe$ since α is a bijection.

We have shown that $e \,\mathcal{R}\, ef$. Thus $a \,\mathcal{R}^*\, ef$; dually, $a \,\mathcal{L}^*\, ef$ and the result follows. □

Condition (*) implies two of the conditions necessary for $\mathcal{P}^*$ to be an embeddable $*$-pair.

Proposition 2.10. *Let S be a semigroup. Then*
(a) if S satisfies (*), condition (Evi) holds for $\mathcal{P}^*$;
(b) if $\mathcal{P}^*$ satisfies condition (Evi)(l), then $\mathcal{P}^*$ satisfies condition (Ev)(l);
(c) if S is abundant and $E(S)$ is a band, then if $\mathcal{P}^*$ satisfies condition (Ev), S satisfies $(*)$.

Proof. Suppose that S satisfies (*), that $a \in \mathcal{S}(S)$ and $b, c \in S$ with $b, c \leq_{\mathcal{L}^*} a$ and $ba = ca$. Then $a\,\mathcal{H}^*\,e$ for some $e \in E(S)$ so that

$$b = be = ce = c.$$

For part (b), suppose that $\mathcal{P}^*$ has condition (Evi)(l), that $a \in \mathcal{S}(S)$ and $b \in S$ with $b \leq_{\mathcal{L}^*} a$. By the remarks at the beginning of this section, certainly $ba \leq_{\mathcal{R}^*} b$. Suppose now that $x, y \in S^1$ and $xba = yba$. Then $xb, yb \leq_{\mathcal{L}^*} b \leq_{\mathcal{L}^*} a$ so that by (Evi)(l), $xb = yb$. Thus $b\,\mathcal{R}^*\,ba$ as required.

Finally, assume that S is abundant, $E(S)$ is a band, and that $\mathcal{P}^*$ has (Ev). Let $a \in S$ be square cancellable and choose idempotents e, f as in Lemma 2.8. Now e is certainly square cancellable so that by (Ev)(r), $ae\,\mathcal{L}^*e$. This gives that $ae\,\mathcal{H}^*\,e$ and dually, $fa\,\mathcal{H}^*\,f$. The final part of Lemma 2.8 yields $ef\,\mathcal{H}^*\,a$. □

3. Fully stratified orders

We specialise Theorem 2.3 to the case where the induced $*$-pair is $\mathcal{P}^* = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$, that is, the left order is fully stratified. We make further progress in the next section where we insist additionally that the left order is abundant. First, a technical lemma.

Lemma 3.1. *Let S be a semigroup. If $\mathcal{P}^*$ satisfies conditions*

$$(Ei), (Eii), (Eiii), (Evi) \text{ and } (Gii),$$

then $\mathcal{P}^*$ satisfies condition

$$(Evii)(l).$$

Proof. By Proposition 2.10, $\mathcal{P}^*$ satisfies (Ev). Now Lemma 4.11 of [8] gives that (Evii)(l) holds.

□

Proposition 3.2. *Let S be a semigroup. Then S is a fully stratified straight left order if and only if $\mathcal{P}^*$ satifies conditions*

$$(Ei), (Eii), (Eiii), (Evi), (Evii)(r) and (Gii).$$

Proof. Suppose that $\mathcal{P}^*$ satifies the given conditions. By Proposition 2.10, (Ev) holds for $\mathcal{P}^*$. In view of Lemma 3.1, $\mathcal{P}^*$ satisfies (Evii). Clearly (Gi) always holds for $\mathcal{P}^*$. The result follows from Theorem 2.3. □

An order S in Q is *straight* in Q if it is straight as a left order and as a right order.

Lemma 3.3. *Let S be a fully stratified straight left order in Q and a fully stratified straight right order in P. Then S is a fully stratified straight order in Q (and P).*

Proof. As commented in Section 2, S intersects every $\mathcal{H}$-class of both Q and P. From Corollary 3.14 of [11], S is an order in Q. The fact that S is straight as a right order in Q follows from Proposition 3.1 of [8]. $\square$

Corollary 3.4. *Let S be a semigroup. Then S is a fully stratified straight order if and only if $\mathcal{P}^*$ satisfies conditions*

$$(Ei), (Eii), (Eiii), (Evi), (Gii)$$

and the dual

$$(Gii)'$$

of (Gii).

Proof. Suppose that $\mathcal{P}^*$ satisfies the given conditions. By Lemma 3.1 and its dual, (Evii) holds for $\mathcal{P}^*$. Proposition 3.2 and its dual, together with Lemma 3.3, now give the result. $\square$

We end this section with a remark concerning $\mathcal{P}^*$, which will be used in conjunction with Proposition 2.7.

Lemma 3.5. *Let S be a straight left order in a semigroup Q. Suppose that $\mathcal{P}^*$ satisfies condition (Eii). Then S is stratified in Q if and only if S is fully stratified in Q.*

Proof. If S is stratified in Q and $a, b \in S$ are such that $b \leq_{\mathcal{L}^*} c$ in S, then by (Eii) for $\mathcal{P}^*$, $b \,\mathcal{L}^*\, dc$ for some $d \in S$. But then $b \,\mathcal{L}\, dc \leq_{\mathcal{L}} c$ in Q. $\square$

4. Abundant orders

We put together the technical results of the two previous sections to characterise classes of abundant semigroups that are fully stratified straight (left) orders.

We remark first that if S is abundant, then since idempotents are certainly square cancellable, condition (Eiii) always holds for $\mathcal{P}^*$. Proposition 3.2 and Corollary 3.4 can then be immediately simplified when applied to abundant semigroups by removing condition (Eiii) from their statements.

For IC-abundant semigroups we can do a little more. The following result is used in [6] to show that the endomorphism monoid of a hereditary basis algebra of finite rank is a straight left order.

Proposition 4.1. *Let S be an idempotent connected abundant semigroup. Then*

(a) S is a fully stratified straight left order if and only if $\mathcal{P}^*$ satisfies conditions

$$(Ei), (Evi), (Evii)(r) \text{ and } (Gii);$$

(b) S is a fully stratified straight order if and only if $\mathcal{P}^*$ satisfies conditions

$$(Ei), (Evi), (Gii) \text{ and } (Gii)'.$$

Proof. From Proposition 2.7, $\mathcal{P}^*$ satifies (Eii). □

Finally we may deduce the characterisation result for ample straight left orders stated at the end of the Introduction. For clarity we state the conditions in full. Part (b) is probably the closest possible analogue for straight orders in inverse semigroups of the theorem of Ore and Dubreil characterising orders in groups.

Theorem 4.2. *Let S be a bountiful semigroup. Then*

(a) S is a fully stratified straight left order if and only if $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$, for all $a \in \mathcal{S}(S)$ and $b, c \in S$ with $b, c \leq_{\mathcal{R}^*} a$,

$$ab\,\mathcal{R}^*\,ac \text{ implies } b\,\mathcal{R}^*\,c,$$

and the $\mathcal{H}^*$-class of each square cancellable element is right reversible;

(b) S is a fully stratified straight order if and only if $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$ and the $\mathcal{H}^*$-class of each square cancellable element is right and left reversible.

Further, for any semigroup Q, S is a stratified straight left order in Q if and only if S is fully stratified in Q. Moreover, any such Q is orthodox, inverse if S is ample, and contains S as a full subsemigroup.

Proof. From Propositions 2.9 and 2.10, S satisfies condition (Evi). Proposition 2.7 together with Lemma 3.5 gives that if S is a straight left order in Q, then S is fully stratified if and only if it is stratified.

If S is a fully stratified straight left order in Q, then we know that Q is regular and S intersects every $\mathcal{H}$-class of Q. Let $e \in E(Q)$; picking $a \in H_e \cap S$ we have that a is square cancellable. By Proposition 2.9, there is an idempotent f of S in the $\mathcal{H}^*$-class of a. But S is fully stratified in Q, so that $e\,\mathcal{H}\,f$ in Q, whence $e = f$. Clearly then Q is orthodox and inverse if S is ample. □

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Department of Mathematics, University of York, Heslington, York YO10 5DD, UK
E-mail address: varg1@york.ac.uk