AXIOMATISABILITY OF WEAKLY FLAT, FLAT AND PROJECTIVE S-ACTS

SYDNEY BULMAN-FLEMING AND VICTORIA GOULD

Abstract. This paper gives necessary and sufficient conditions on a monoid S that the class of flat left S-acts be (first order) axiomatisable. It presents also a streamlined version of the result of Stepanova that the class of projective left S-acts is axiomatisable if and only if the strongly flat left S-acts are axiomatisable and S is left perfect.

1. Introduction

Any class of algebras $\mathcal{A}$ has an associated first order language L. One can then ask whether a property P, defined for members of $\mathcal{A}$, is expressible in the language L. In other words, is there a set of sentences Π in L such that a member A of $\mathcal{A}$ has property P if and only if all the sentences of Π are true in A, that is, if and only if A is a model of Π (denoted $A \vDash \Pi$)? If Π exists we say that $\mathcal{B}$ is *axiomatisable*, where $\mathcal{B}$ is the subclass of $\mathcal{A}$ whose members have property P. Questions of axiomatisability are the first step in investigating the *model theory* of a class of algebras.

In this paper we consider the class of left S-acts over a monoid S, the subject of a recent comprehensive overview [1]. Our language is the first order language with equality L_S which has no constant or relation symbols and which has a unary function symbol λ_s for each $s \in S$: we write sx for $\lambda_s(x)$. The class of left S-acts is axiomatised by the set of sentences

$$\{(\forall x)(1x = x)\} \cup \{\mu_{s,t} : s, t \in S\}$$

where $\mu_{s,t}$ is the sentence

$$(\forall x)((st)x = s(tx)).$$

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Right S-acts have, of course, the dual definition.

Certain classes of left S-acts are axiomatisable for *any* monoid S. For example the torsion free left S-acts are axiomatised by the extra sentences

$$\{(\forall x)(\forall y)(sx = sy \Rightarrow x = y) : s \in T\}$$

where T is the set of left cancellable elements of S. Other natural classes of left S-acts are axiomatisable for some monoids and not for others. The classes we address here are the projectives and the flats. Corresponding questions for modules over a ring have been fully answered by Eklof and Sabbagh in [2]. Their work was one of the early contributions to the now flourishing area of model theory of modules. The model theory of S-acts is in its infancy, but contains analogues as well as interesting disparities with the classic work on modules.

A left S-act B is *projective* if given any diagram of S-acts and S-morphisms

$$\begin{tikzcd} & B \arrow[d, "\theta"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

where $\phi : M \to N$ is onto, there exists an S-morphism $\psi : B \to M$ such that the diagram

$$\begin{tikzcd} & B \arrow[dl, "\psi"'] \arrow[d, "\theta"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

is commutative.

Theorem 1. [3, 4] *A left S-act A is projective if and only if A is isomorphic to a coproduct of idempotent generated principal left ideals of S.*

We note that in the category of left S-acts, coproduct is simply disjoint union.

To define classes of flat left S-acts we need the notion of *tensor product* of S-acts. If A is a *right* S-act and B a left S-act then the tensor product of A and B, written $A \otimes B$, is the set $A \times B$ factored by the equivalence generated by $\{((as, b), (a, sb)) : a \in A, b \in B, s \in S\}$. For $a \in A$ and $b \in B$ we write $a \otimes b$ for the equivalence class of (a, b).

Tensor products may be approached via universal properties of balanced maps – see [1], for example, where the proof of the following standard lemma may be found.

Lemma 2. *Let A be a right S-act and B a left S-act. Then for $a, a' \in A$ and $b, b' \in B$, $a \otimes b = a' \otimes b'$ if and only if there exist $s_1, t_1, s_2, t_2, \ldots, s_m, t_m \in S$, $a_2, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$ such that*

$$\begin{array}{rclrcl} & & & b & = & s_1 b_1 \\ a s_1 & = & a_2 t_1 & t_1 b_1 & = & s_2 b_2 \\ a_2 s_2 & = & a_3 t_2 & t_2 b_2 & = & s_3 b_3 \\ & \vdots & & & \vdots & \\ a_m s_m & = & a' t_m & t_m b_m & = & b'. \end{array}$$

The sequence presented in Lemma 2 will be called a *tossing* (or scheme) $\mathcal{T}$ of length m over A and B connecting (a, b) to (a', b'). The *skeleton* of $\mathcal{T}$, $\mathcal{S} = \mathcal{S}(\mathcal{T})$, is the sequence

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m) \in S^{2m}.$$

The set of all skeletons is denoted by $\mathbb{S}$. It is easy to see that $\mathbb{S}$ consists of all sequences of elements of S of even length.

For a left S-act B, the map $- \otimes B$ is a functor from the category of right S-acts to the category of sets. It is from this functor that the various notion of *flatness* are derived.

A left S-act B is *weakly flat* if the functor $- \otimes B$ preserves embeddings of right ideals of S into S, *flat* if it preserves arbitrary embeddings of right S-acts, and *strongly flat* if it preserves pullbacks (equivalently, equalisers and pullbacks [5]). We give the reader a warning that terminology has changed over the years; in particular, left S-acts B such that $- \otimes B$ preserves equalisers and pullbacks are called weakly flat in [6] and [7], flat in [8], and pullback flat in [1].

Theorem 3. [7] *A left S-act B is strongly flat if and only if, whenever $sx = ty$ where $x, y \in B$ and $s, t \in S$, there is an element $z \in B$ and elements $s', t' \in S$ such that $x = s'z, y = t'z$ and $ss' = tt'$; moreover, if $x = y$, we may take $s' = t'$.*

The first part of the condition in the above theorem is known as condition (P) (see [1]). The classes of projective, strongly flat, flat and weakly flat left S-acts will be denoted by $\mathcal{P}, \mathcal{SF}, \mathcal{F}$ and $\mathcal{WF}$ respectively. We remark here that 'elementary' descriptions (that is, not involving arrows) of $\mathcal{F}$ and $\mathcal{WF}$, along the lines of Theorems 1 for $\mathcal{P}$ or 3 for $\mathcal{SF}$, are not available. From those results it is immediate that projective left S-acts are strongly flat. Clearly flat left S-acts are weakly flat and since embeddings are equalisers in the category of right S-acts, strongly flat left S-acts are flat. Thus

$$\mathcal{P} \subseteq \mathcal{SF} \subseteq \mathcal{F} \subseteq \mathcal{WF}.$$

Those monoids for which $\mathcal{P}$ is axiomatisable are determined by Stepanova in [9]. A monoid S is *left perfect* if $\mathcal{P} = \mathcal{SF}$.

Theorem 4. [9] *The class $\mathcal{P}$ is axiomatisable if and only if $\mathcal{SF}$ is axiomatisable and S is left perfect.*

Those monoids for which $\mathcal{SF}$ is axiomatisable are characterised in [8], so that Theorem 4 is a complete characterisation of those monoids such that $\mathcal{P}$ is axiomatisable. In Section 2 we present a simpler proof than that of [9], using Fountain's description of left perfect monoids [6].

In subsequent sections we turn our attention to flatness. Section 3 concentrates on a technical result, a consequence of which is that a left S-act B is flat if the functor $-\otimes B$ preserves embeddings of finitely generated subacts of finitely presented right acts. This enables us in Section 4 to determine those monoids S for which $\mathcal{F}$ is axiomatisable; in Section 5 we do the same for weakly flat left S-acts. The condition arrived at in Section 3 is rather strong and raises the question, if $\mathcal{F}$ is axiomatisable, does $\mathcal{F}$ consist of *all* left S-acts? In other words, is S left *absolutely flat*? In Section 6 we give a counterexample to this conjecture. We also present a monoid S for which $\mathcal{F}$ is not axiomatisable.

One of our main tools throughout will be that of an ultraproduct. If $B = \prod_{i\in I} B_i$ is a product of left S-acts B_i, $i \in I$, and Φ is an ultrafilter on I, then we denote by $\underline{b}^*$ the class of $\underline{b} = (b_i)_{i\in I}$ in the ultraproduct $U = (\prod_{i\in I} B_i)_\Phi$. We make heavy use of Łos's theorem.

Theorem 5. [10] *Let L be a first order language and $\mathcal{A}$ a class of L-structures. If $\mathcal{A}$ is axiomatisable, then $\mathcal{A}$ is closed under the formation of ultraproducts.*

We also bear in mind the following lemma.

Lemma 6. *Let Φ be a an ultrafilter on a set I. If $D \in \Phi$ and $D = D_1 \cup \ldots \cup D_n$ is a finite union of subsets of I, then $D_i \in \Phi$ for some $i \in \{1, \ldots, n\}$.*

For further details on the concepts in this Introduction, we refer the reader to [11] for general semigroup theory and to [1] for the theory of S-acts.

2. Axiomatisability of $\mathcal{P}$

This section gives an alternative proof of the result of Stepanova [9] that $\mathcal{P}$ is axiomatisable if and only if $\mathcal{SF}$ is axiomatisable and S is left perfect, that is, $\mathcal{P} = \mathcal{SF}$.

We denote by M_R and M^L the descending chain condition on principal right ideals, and the ascending chain condition on principal left ideals, respectively.

Theorem 7. [6, 12] *The following conditions are equivalent for a monoid S:*

(1) S is left perfect;

(2) S satisfies M_R and condition DL which states that each direct limit of a sequence of copies of S as a left S-act is cyclic;

(3) S satisfies M_R and the ascending chain condition A for cyclic left S-acts.

Conditions DL and A are notoriously difficult to handle. Stepanova succeeds in showing that if $\mathcal{P}$ is axiomatisable, then DL holds. In view of the following result from [8], this completes the picture for $\mathcal{P}$.

Proposition 8. [8] *Let S be a monoid such that every ultrapower of S is projective. Then:*

(1) the class $\mathcal{SF}$ is axiomatisable;

(2) S satisfies M_R;

(3) if S satisfies M^L, then S satisfies A.

We use the technique of Stepanova to show that if $\mathcal{P}$ is closed under ultraproducts, then S satisfies M^L, which together with Proposition 8 gives Theorem 4.

Lemma 9. *Let S be a monoid such that every ultraproduct of projective left S-acts is projective. Then, for any $t \in S$, there exists $n \in \mathbb{N}$ such that, for any projective left S-act P and any $a \in P$, $|(a:t)_P| \leq n$, where*

$$(a:t)_P = \{x \in P \mid tx = a\}.$$

Proof. Suppose to the contrary that $t \in S$ exists for which the condition is *not* true. That is, for each $n \in \mathbb{N}$, there exists a projective left S-act P_n and $a_n \in P_n$ such that $|(a_n : t)_{P_n}| > n$. A well-known consequence of the compactness theorem gives that for any cardinal κ, there exists a projective left S-act P_κ and $a \in P_\kappa$ such that

$$|(a:t)_{P_\kappa}| > \kappa.$$

Choosing $\kappa > |S|$ we have a contradiction, because $(a:t)_{P_\kappa}$ is contained in a connected component of P_κ which according to Theorem 1 is isomorphic to Sf for some idempotent f, since P_κ is projective. $\square$

The missing link to get Stepanova's main result is

Lemma 10. *Let S be a monoid such that every ultraproduct of projective left S-acts is projective. Then S satisfies M^L.*

Proof. Consider an ascending chain

$$Sa_1 \subseteq Sa_2 \subseteq \cdots$$

of principal left ideals of S. Let $u_2, u_3, ... \in S$ be such that $a_i = u_{i+1}a_{i+1}$, from which it follows that for any i, j with $i < j$,

$$a_i = u_{i+1}u_{i+2} \cdots u_j a_j.$$

Consider an ultrapower $U = (S^{\mathbb{N}})/\Phi$ where Φ is a non-principal ultrafilter on $\mathbb{N}$. By assumption, U is projective as a left S-act. For each $i \geq 2$ define

$$\underline{v_i}^* = (1, 1, ..., 1, u_i, u_iu_{i+1}, u_iu_{i+1}u_{i+2}, ...)_\Phi,$$

the entry u_i occurring in the ith position. Observe that

$$\underline{v_i}^* = u_i\underline{v_{i+1}}^*$$

for each i, because the corresponding vectors agree from the $(i+1)$st position on. Because all of the $\underline{v_i}^*$ lie in the same connected component of the projective left S-act U, there exist $\underline{f}^* = (f_1, f_2, ...)_\Phi$ (where $f_i \in S$) and $s_2, s_3, ... \in S$ such that

$$\underline{v_k}^* = s_k\underline{f}^*$$

for each $k \geq 2$. For each natural number k it follows that the set

$$E_k = \{j \in \mathbb{N} \mid j > k,\ u_ku_{k+1} \cdots u_j = s_kf_j\}$$

belongs to Φ. For each i we put

$$\underline{g_i}^* = f_i\underline{v_{i+1}}^* = (f_i, ..., f_i, f_iu_{i+1}, f_iu_{i+1}u_{i+2}, ...)_\Phi,$$

the entry f_iu_{i+1} occurring in the $(i+1)$st position. To complete the proof of the lemma we consider now two cases.

Case 1: there exists $i_0 \in \mathbb{N}$ such that $D := \left\{j \in \mathbb{N} \mid \underline{g_j}^* = \underline{g_{i_0}}^*\right\}$ belongs to Φ. For any $k \in D$ with $k > i_0$, pick $j \in D \cap E_k$. Then we have

$$S_j := \{m \in \mathbb{N} \mid m > j,\ f_ju_{j+1} \cdots u_m = f_{i_0}u_{i_0+1} \cdots u_m\} \in \Phi.$$

Now take any $m \in S_j$ and calculate

$$\begin{aligned} a_{k-1} &= u_k \cdots u_ju_{j+1} \cdots u_ma_m \\ &= s_kf_ju_{j+1} \cdots u_ma_m \\ &= s_kf_{i_0}u_{i_0+1} \cdots u_ma_m \\ &= s_kf_{i_0}a_{i_0}. \end{aligned}$$

In summary we have shown in this case that, for any $k \in D$ with $k > i_0$, the left ideals Sa_{k-1} and Sa_{i_0} are equal. If now $l > i_0$ is arbitrary, take any $k \in D$ with $k > l$ and note that $Sa_{i_0} \subseteq Sa_l \subseteq Sa_{k-1} = Sa_{i_0}$, and so the chain terminates, as required.

Case 2: for each $i \in \mathbb{N}$, $\left\{ j \in \mathbb{N} \mid \underline{g_i}^* = \underline{g_j}^* \right\} \notin \Phi$. In this case, for each $i \in \mathbb{N}$ put $T_i = \left\{ j \in \mathbb{N} \mid \underline{g_i}^* \neq \underline{g_j}^* \right\}$, which by assumption belongs to Φ. Now choose $j_1, j_2, ... \in \mathbb{N}$ as follows:

$$\begin{array}{rcl} j_1 & \in & E_2 \\ j_2 & \in & E_2 \cap T_{j_1},\ j_2 > j_1 \\ j_3 & \in & E_2 \cap T_{j_1} \cap T_{j_2},\ j_3 > j_2 > j_1 \\ & & \vdots \end{array}$$

For any k,

$$\begin{array}{rcl} \underline{v_2}^* & = & u_2 u_3 \cdots u_{j_k} \underline{v_{j_k+1}}^* \\ & = & s_2 f_{j_k} \underline{v_{j_k+1}}^* \\ & = & s_2 \underline{g_{j_k}}^*, \end{array}$$

and so $\underline{g_{j_k}}^* \in (\underline{v_2}^* : s_2)_U$ for each k. By definition of the sets T_i, all of the elements $\underline{g_{j_k}}^*$ are distinct. This contradicts Lemma 9. $\square$

3. The finitely presented flatness lemma

The following lemma is vital in characterising those monoids S such that $\mathcal{F}$ is axiomatisable. It is essentially an easy observation on the nature of tossings, but is worthwhile to state with care.

Let $\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m) \in \mathbb{S}$ be a skeleton. We let F^{m+1} be the free right S-act

$$xS \amalg x_2 S \amalg ... \amalg x_m S \amalg x' S$$

and let $\rho_{\mathcal{S}}$ be the congruence on F^{m+1} generated by

$$\{(xs_1, x_2 t_1), (x_2 s_2, x_3 t_2), ..., (x_{m-1} s_{m-1}, x_m t_{m-1}), (x_m s_m, x' t_m)\}.$$

We denote the $\rho_{\mathcal{S}}$-class of $w \in F^{m+1}$ by $[w]$. If B is a left S-act and $b, b_1, \ldots, b_m, b' \in B$ are such that

$$b = s_1 b_1, t_1 b_1 = s_2 b_2, \ldots, t_m b_m = b'$$

then the tossing

$$\begin{array}{rclrcl} & & & b & = & s_1b_1 \\ [x]\,s_1 & = & [x_2]\,t_1 & t_1b_1 & = & s_2b_2 \\ [x_2]\,s_2 & = & [x_3]\,t_2 & t_2b_2 & = & s_3b_3 \\ & \vdots & & & \vdots & \\ [x_{m-1}]\,s_{m-1} & = & [x_m]\,t_{m-1} & t_{m-1}b_{m-1} & = & s_mb_m \\ [x_m]\,s_m & = & [x']\,t_m & t_mb_m & = & b'. \end{array}$$

over $F^{m+1}/\rho_{\mathcal{S}}$ and B is called a *standard tossing* with skeleton $\mathcal{S}$.

Lemma 11. *The following are equivalent for a left S-act B:*

(1) B is flat;

(2) $-\otimes B$ preserves all embeddings of A in C, where A is a finitely generated subact of a finitely presented right S-act C;

(3) $-\otimes B$ preserves the embedding of $[x]\,S \cup [x']\,S$ into $F^{m+1}/\rho_{\mathcal{S}}$, for all skeletons $\mathcal{S}$;

(4) if $([x], b)$ and $([x'], b')$ are connected by a standard tossing over $F^{m+1}/\rho_{\mathcal{S}}$ and B with skeleton $\mathcal{S}$, then they are connected by a tossing over $[x]\,S \cup [x']\,S$ and B.

Proof. The only thing to show is that if (4) holds, then B is flat. Suppose therefore that B satisfies condition (4), let a, a' belong to any right S-act A, let b, b' belong to B, and assume $a \otimes b = a' \otimes b'$ in $A \otimes B$ via the tossing

$$\begin{array}{rclrcl} & & & b & = & s_1b_1 \\ as_1 & = & a_2t_1 & t_1b_1 & = & s_2b_2 \\ a_2s_2 & = & a_3t_2 & t_2b_2 & = & s_3b_3 \\ & \vdots & & & \vdots & \\ a_{m-1}s_{m-1} & = & a_mt_{m-1} & t_{m-1}b_{m-1} & = & s_mb_m \\ a_ms_m & = & a't_m & t_mb_m & = & b'. \end{array}$$

From the standard tossing

$$\begin{array}{rclrcl} & & & b & = & s_1b_1 \\ [x]\,s_1 & = & [x_2]\,t_1 & t_1b_1 & = & s_2b_2 \\ [x_2]\,s_2 & = & [x_3]\,t_2 & t_2b_2 & = & s_3b_3 \\ & \vdots & & & \vdots & \\ [x_{m-1}]\,s_{m-1} & = & [x_m]\,t_{m-1} & t_{m-1}b_{m-1} & = & s_mb_m \\ [x_m]\,s_m & = & [x']\,t_m & t_mb_m & = & b', \end{array}$$

and the given condition, we conclude that $[x] \otimes b = [x'] \otimes b'$ in $([x]\,S \cup [x']\,S) \otimes B$.

If φ denotes the S-morphism from F^{m+1} into A given by $\varphi(x) = a$, $\varphi(x_i) = a_i$ $(2 \leq i \leq m)$, and $\varphi(x') = a'$, then, because $\rho \subseteq \ker\varphi$, there exists an S-morphism $\overline{\varphi} : F^{m+1}/\rho_{\mathcal{S}} \to A$ such that $\overline{\varphi}([x]) = a$ and $\varphi([x']) = a'$. Applying $\overline{\varphi}$ to each equality in the left-hand column of any tossing corresponding to the equality $[x] \otimes b = [x'] \otimes b'$ in $([x]\, S \cup [x']\, S) \otimes B$ produces a tossing that implies $a \otimes b = a' \otimes b'$ in $(aS \cup a'S) \otimes B$, as required. ☐

4. Axiomatisability of Flat S-Acts

In this section we characterise those monoids S such that $\mathcal{F}$ is axiomatisable.

We know that if $a, a' \in A$ and $b, b' \in B$, where A is a right S-act and B a left S-act, then $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exists a tossing from (a, b) to (a', b') over A and B, with skeleton $\mathcal{S}$, say. In this case, if B is flat, then the equality $a \otimes b = a' \otimes b'$ holds also in $(aS \cup a'S) \otimes B$, and is determined by some tossing $\mathcal{T}'$ from (a, b) to (a', b') over $aS \cup a'S$ and B with skeleton $\mathcal{S}' = \mathcal{S}(\mathcal{T}')$. Such skeletons $\mathcal{S}'$ will be called *replacement skeletons for* $\mathcal{S}$.

Theorem 12. *The following conditions are equivalent for a monoid S:*

(1) the class $\mathcal{F}$ is axiomatisable;

(2) the class $\mathcal{F}$ is closed under formation of ultraproducts;

(3) for every skeleton $\mathcal{S}$ over S there exist finitely many replacement skeletons $\mathcal{S}_1, ..., \mathcal{S}_{\alpha(\mathcal{S})}$ over S such that, for any right S-act A and any flat act left S-act B, if $(a, b), (a', b') \in A \times B$ are connected by a tossing $\mathcal{T}$ over A and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, ..., \alpha(\mathcal{S})\}$.

Proof. The implication (1) implies (2) is clear from Theorem 5.

For the proof of (2) implies (3), suppose that $\mathcal{F}$ is closed under formation of ultraproducts, but that assertion (3) is false. Let J denote the family of all finite subsets of $\mathbb{S}$. Suppose $\mathcal{S}_0 = (s_1, t_1, ..., s_m, t_m) \in \mathbb{S}$ is such that, for every $f \in J$, there exist a right S-act A_f, a flat left S-act B_f, and pairs (a_f, b_f), $(a'_f, b'_f) \in A_f \times B_f$ such that (a_f, b_f) and (a'_f, b'_f) are connected over A_f and B_f by a tossing $\mathcal{T}_f$ with skeleton $\mathcal{S}_0$, but no tossing over $a_f S \cup a'_f S$ and B_f connecting (a_f, b_f) and (a'_f, b'_f) has a skeleton belonging to the set f.

For each $\mathcal{S} \in \mathbb{S}$ let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$. Each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty (because $\mathbb{S}$ is infinite), so there exists an ultrafilter Φ over J such that each $J_{\mathcal{S}}$ ($\mathcal{S} \in \mathbb{S}$) belongs to Φ. Note that $\underline{a} \otimes \underline{b} = \underline{a}' \otimes \underline{b}'$ in $A \otimes B$, where $A = \prod_{f \in J} A_f$ and $B = \prod_{f \in J} B_f$, and that this equality is determined by a tossing over A and B (the "product" of the $\mathcal{T}_f$'s) having skeleton $\mathcal{S}_0$. If follows that the equality $\underline{a} \otimes \underline{b}^* = \underline{a}' \otimes \underline{b}'^*$ holds also in $A \otimes U$, where $U = (\prod_{f \in J} B_f)_\Phi$, and is determined by a tossing over A and U with skeleton $\mathcal{S}_0$. Because U is flat, $\mathcal{S}_0$ has a replacement skeleton $\mathcal{S}(\mathcal{T}') = \mathcal{S}' = (u_1, v_1, ..., u_n, v_n)$, where $\mathcal{T}'$ is a tossing

$$\begin{array}{rclrcl} & & & \underline{b}^* & = & u_1(d_{f,1})^* \\ \underline{a}u_1 & = & (c_{f,2})v_1 & v_1(d_{f,1})^* & = & u_2(d_{f,2})^* \\ (c_{f,2})u_2 & = & (c_{f,3})v_2 & v_2(d_{f,2})^* & = & u_3(d_{f,3})^* \\ & \vdots & & & \vdots & \\ (c_{f,n})u_n & = & \underline{a}'v_n & v_n(d_{f,n})^* & = & \underline{b}'^* \end{array}$$

with each $c_{f,j} \in a_f S \cup a'_f S$ and each $d_{f,j} \in B_f$. As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$\begin{array}{rclrcl} & & & b_f & = & u_1 d_{f,1} \\ a_f u_1 & = & c_{f,2} v_1 & v_1 d_{f,1} & = & u_2 d_{f,2} \\ c_{f,2} u_2 & = & c_{f,3} v_2 & v_2 d_{f,2} & = & u_3 d_{f,3} \\ & \vdots & & & \vdots & \\ c_{f,n} u_n & = & a'_f v_n & v_n d_{f,n} & = & b'_f \end{array} \tag{1}$$

whenever $f \in D$. Now, suppose $f \in D \cap J_{\mathcal{S}'}$. Then, from the tossings (1) just considered, we see that $\mathcal{S}'$ is a replacement skeleton for skeleton $\mathcal{S}_0$, the latter being the skeleton of tossing $\mathcal{T}_f$ connecting the pairs (a_f, b_f) and (a'_f, b'_f) over A_f and B_f. But because $\mathcal{S}'$ belongs to f, this is impossible. This completes the proof that (2) implies (3).

Finally, we show that (3) implies (1). Suppose every skeleton requires only finitely many replacement skeletons, as made precise in the statement of (3) above. We aim to use this condition to construct a set of axioms for $\mathcal{F}$.

We note first, for any left S-act B and any $b, b' \in B$, a skeleton $\mathcal{S} = (s_1, t_1, ..., s_m, t_m)$ is the skeleton of a tossing from (a, b) to (a', b') over A and B for some A and some $a, a' \in A$ if and only if $\gamma_{\mathcal{S}}(b, b')$ holds in B, where $\gamma_{\mathcal{S}}$ is the formula

$$\gamma_{\mathcal{S}}(y, y') := (\exists y_1)(\exists y_2) \cdots (\exists y_m)(y = s_1 y_1 \wedge t_1 y_1 = s_2 y_2 \wedge \cdots \wedge t_m y_m = y'),$$

that is, if and only if there exist $b_1, \ldots, b_m \in B$ such that

$$\begin{array}{rcl} b & = & s_1 b_1 \\ t_1 b_1 & = & s_2 b_2 \\ & \vdots & \\ t_m b_m & = & b', \end{array} \tag{2}$$

as a glance at the proof of Lemma 11 will verify.

Let $\mathbb{S}_1$ denote the set of all elements of $\mathbb{S}$ that are *not* the skeleton of any tossing connecting two elements of $A \times B$, where A ranges over all right S-acts and B over all flat left S-acts, and let $\mathbb{S}_2 = \mathbb{S} \backslash \mathbb{S}_1$. We now introduce a sentence corresponding to each element of $\mathbb{S}$ in such a way that the resulting set of sentences axiomatises the class $\mathcal{F}$.

In the case that $\mathcal{S} \in \mathbb{S}_1$ introduce the sentence

$$\psi_{\mathcal{S}} := (\forall y)(\forall y')\neg\gamma_{\mathcal{S}}(y, y').$$

In the case that $\mathcal{S} \in \mathbb{S}_2$, we have that $\mathcal{S}$ is the skeleton of at least one tossing joining $([x], b)$ to $([x'], b')$ over $F^{m+1}/\rho_{\mathcal{S}}$ and B where B is flat, $b, b' \in B$, and F^{m+1} and $\rho_{\mathcal{S}}$ are as defined in Lemma 11. Let $\mathcal{S}_1, ..., \mathcal{S}_{\alpha(\mathcal{S})}$ be a set of replacement skeletons for $\mathcal{S}$ as provided by assertion (3). Then, for each $k \in \{1, ..., \alpha(\mathcal{S})\}$, if $\mathcal{S}_k = (u_1, v_1, ..., u_h, v_h)$, there exist a flat left S-act C, elements $c, c', c_1, ..., c_h \in C_k$, and elements $p_2, ..., p_h \in [x] S \cup [x'] S$ such that

$$\begin{array}{rclrcl} & & & c & = & u_1 c_1 \\ [x]\, u_1 & = & p_2 v_1 & v_1 c_1 & = & u_2 c_2 \\ p_2 u_2 & = & p_3 v_2 & v_2 c_2 & = & u_3 c_3 \\ & \vdots & & & \vdots & \\ p_h u_h & = & [x']\, v_h & v_h c_h & = & c'. \end{array} \tag{3}$$

For each k, we fix such a list $p_2, ..., p_h$ of elements, for future reference, and define $\varphi_{\mathcal{S}}$ to be the sentence

$$\varphi_{\mathcal{S}} := (\forall y)(\forall y')(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \cdots \vee \gamma_{\mathcal{S}_{\alpha(\mathcal{S})}}(y, y')).$$

Let

$$\Sigma = \{\psi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_1\} \cup \{\varphi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_2\}.$$

We claim that $\mathcal{F} = Mod(\Sigma)$.

Suppose first that D is any flat left S-act.

For $\mathcal{S} \in \mathbb{S}_1$, if D did not satisfy $\psi_{\mathcal{S}}$, then we would have $\gamma_{\mathcal{S}}(d, d')$ for some $d, d' \in D$, and so, by the comment at the beginning of this part of the proof, $\mathcal{S}$ is the skeleton of some tossing joining (a, d) to (a', d') over some right S-act A and flat left S-act D, contrary to the fact that $\mathcal{S} \in \mathbb{S}_1$. Therefore, $D \vDash \psi_{\mathcal{S}}$.

Now take any $\mathcal{S} \in \mathbb{S}_2$, and suppose $d, d' \in D$ are such that D satisfies $\gamma_{\mathcal{S}}(d, d')$. Then, as noted earlier, $([x], d)$ and $([x'], d')$ are joined over $F^{m+1}/\rho_{\mathcal{S}}$ and D by a tossing with skeleton $\mathcal{S}$, and therefore, by assumption, by a tossing over $[x] S \cup [x'] S$ and D with skeleton $\mathcal{S}_k$ for some $k \in \{1, ..., \alpha(\mathcal{S})\}$. It is now clear that $\gamma_{\mathcal{S}_k}(d, d')$ holds in D, as required. We have now shown that $\mathcal{F} \vDash \Sigma$.

Finally, we show that every left S-act C that satisfies Σ is flat. We need only show that condition (4) of Lemma 11 holds for C. Let $\mathcal{S} \in \mathbb{S}$ and suppose we have a tossing

$$\begin{array}{rclrcl} & & & c & = & s_1c_1 \\ [x]\,s_1 & = & [x_2]\,t_1 & t_1c_1 & = & s_2c_2 \\ & \vdots & & & \vdots & \\ [x_m]\,s_m & = & [x']\,t_m & t_mc_m & = & c' \end{array} \tag{4}$$

over $F^{m+1}/\rho_{\mathcal{S}}$ and C. If $\mathcal{S}$ belonged to $\mathbb{S}_1$, then C would satisfy the sentence $(\forall y)(\forall y')\neg\gamma_{\mathcal{S}}(y,y')$, and so $\neg\gamma_{\mathcal{S}}(c,c')$ would hold, contrary to the sequence of equalities in the right-hand column of (4). Therefore, $\mathcal{S}$ belongs to $\mathbb{S}_2$. Because C satisfies $\varphi_{\mathcal{S}}$ and because $\gamma_{\mathcal{S}}(c,c')$ holds, it follows that $\gamma_{\mathcal{S}_k}(c,c')$ holds for some $k \in \{1,...,\alpha(\mathcal{S})\}$. If $\mathcal{S}_k = (u_1,v_1,...,u_h,v_h)$, then

$$\begin{array}{rcl} c & = & u_1e_1 \\ v_1e_1 & = & u_2e_2 \\ & \vdots & \\ v_he_h & = & c' \end{array} \tag{5}$$

for certain $e_1,...,e_h \in C$. Equalities (5) and the left hand side of (3) together constitute a tossing over $[x]\,S \cup [x']\,S$ and C connecting $([x]\,,c)$ and $([x']\,,c')$, showing that C is indeed flat. The proof is now complete. $\square$

5. Axiomatisability of weakly flat S-acts

We recall from the Introduction that a left S-act B is *weakly flat* if the functor $\otimes B$ preserves all embeddings of right ideals of S into S; the class of weakly flat left S-acts is denoted by $\mathcal{WF}$. In a previous paper [13] we characterised those monoids S such that every ultrapower of S is weakly flat (as a left S-act). In view of Łos's theorem, this is a necessary condition for $\mathcal{WF}$ to be axiomatisable. The aim of this section is to improve on the result of [13] to describe those monoids for which $\mathcal{WF}$ is axiomatisable.

We proceed via a technique similar to that in the previous section, with a little more care needed in the skeletons of tossings we select.

Let S be a monoid, let $a,a' \in S$ and let $\mathcal{S} = (s_1,t_1,...,s_m,t_m)$ be a skeleton of length m. We say that the triple $(a,\mathcal{S},a')$ is *realised* if there are elements $a_2, a_3, \ldots, a_m \in S$ such that

$$\begin{array}{rcl} as_1 & = & a_2t_1 \\ a_2s_2 & = & a_3t_2 \\ & \vdots & \\ a_ms_m & = & a't_m. \end{array}$$

We let $\mathbb{T}$ denote the set of realised triples.

A realised triple $(a, \mathcal{S}, a')$ is *witnessed by* $b, b_1, \ldots, b_m, b'$ where $b, b_1, \ldots, b_m, b'$ are elements of a left S-act B if

$$\begin{array}{rcl} b & = & s_1b_1 \\ t_1b_1 & = & s_2b_2 \\ & \vdots & \\ t_mb_m & = & b'. \end{array}$$

We know that $a \otimes b = a' \otimes b'$ in $S \otimes B$ if and only if there exists a tossing from (a, b) to (a', b') in $S \otimes B$, with skeleton $\mathcal{S}$, say. In this case, if B is weakly flat, then the equality $a \otimes b$ also holds in $(aS \cup a'S) \otimes B$, and is determined by some tossing $\mathcal{T}'$ from (a, b) to (a', b') over $aS \cup a'S$ and B with skeleton $\mathcal{S}' = \mathcal{S}(\mathcal{T}')$. The triples of the form $(a, \mathcal{S}', a')$ will be called *replacement triples* for $(a, \mathcal{S}, a')$.

Theorem 13. *The following conditions are equivalent for a monoid S:*

(1) the class $\mathcal{WF}$ is axiomatisable;

(2) the class $\mathcal{WF}$ is closed under ultraproducts;

(3) for every skeleton $\mathcal{S}$ over S and $a, a' \in S$ there exist finitely many skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\beta(a,\mathcal{S},a')}$ over S, such that for any weakly flat left S-act B, if $(a, b), (a', b') \in S \times B$ are connected by a tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1 \ldots, \beta(a, \mathcal{S}, a')\}$.

Proof. As in Theorem 3, the implication (1) $\Longrightarrow$ (2) follows from Łos's theorem.

Suppose now that (2) holds but that (3) is false. Let J denote the set of all finite subsets of S and suppose that, for some skeleton $\mathcal{S}_0 = (s_1, t_1, \ldots, s_m, t_m) \in \mathbb{S}$ and $a, a' \in S$, for every $f \in J$, there is a weakly flat left S-act B_f and $b_f, b'_b \in B_f$ with the pairs $(a, b_f), (a', b'_f) \in S \times B_f$ connected by a tossing $\mathcal{T}_f$ with skeleton $\mathcal{S}_0$, but such that no tossing over $aS \cup a'S$ and B_f connecting (a, b_f) and (a', b'_f) has a skeleton belonging to the set f.

For each $\mathcal{S} \in \mathbb{S}$ let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$. Each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty (because $\mathbb{S}$ is infinite), so there exists an ultrafilter Φ over J such that each $J_{\mathcal{S}}$ $(\mathcal{S} \in \mathbb{S})$ belongs to Φ. Varying slightly on the proof of Theorem 12, notice that $a \otimes \underline{b} = a' \otimes \underline{b}'$ in $S \otimes B$ where $B = \prod_{f \in J} B_f$, and that this equality is determined by a tossing over S and B having skeleton $\mathcal{S}_0$. It follows that the equality $a \otimes \underline{b}^* = a' \otimes \underline{b}'^*$ holds also in $S \otimes U$, where $U = (\prod_{f \in J} B_f)_\Phi$, and is also determined by a tossing over S and U with skeleton $\mathcal{S}_0$.

By our assumption, U is weakly flat, so that $(a, \underline{b}^*)$ and $(a', \underline{b}'^*)$ are connected via a replacement tossing $\mathcal{T}'$ over $aS \cup a'S$ and U, say

$$
\begin{array}{rclrcl}
 & & & \underline{b}^* & = & u_1(d_{f,1})^* \\
au_1 & = & c_2v_1 & v_1(d_{f,1})^* & = & u_2(d_{f,2})^* \\
c_2u_2 & = & c_3v_2 & v_2(d_{f,2})^* & = & u_3(d_{f,3})^* \\
 & \vdots & & & \vdots & \\
c_nu_n & = & a'v_n & v_n(d_{f,n})^* & = & \underline{b}'^*.
\end{array}
$$

We put $\mathcal{S}' = \mathcal{S}(\mathcal{T}')$.

As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$
\begin{array}{rclrcl}
 & & & b_f & = & u_1d_{f,1} \\
au_1 & = & c_2v_1 & v_1d_{f,1} & = & u_2d_{f,2} \\
c_2u_2 & = & c_3v_2 & v_2d_{f,2} & = & u_3d_{f,3} \\
 & \vdots & & & \vdots & \\
c_nu_n & = & a'v_n & v_nd_{f,n} & = & b'_f
\end{array}
\tag{1}
$$

whenever $f \in D$. Now, suppose $f \in D \cap J_{\mathcal{S}'}$. Then, from the tossing (1) just considered, we see that $\mathcal{S}'$ is a replacement skeleton for skeleton $\mathcal{S}_0$, connecting (a, b_f) and (a', b'_f) over $aS \cup a'S$ and B_f. But $\mathcal{S}' \in f$, contradicting the choice of (a, b_f) and (a', b'_f). Thus (3) holds.

Finally, suppose that (3) holds. We construct a set of axioms for $\mathcal{WF}$. Again, the proof reflects that of Theorem 12, but here we need the extra care that realised triples will provide.

We let $\mathbb{T}_1$ be the set of realised triples that are not witnessed in any weakly flat left S-act B, and put $\mathbb{T}_2 = \mathbb{T} \setminus \mathbb{T}_1$. For $T = (a, \mathcal{S}, a') \in \mathbb{T}_1$ we let ψ_T be the sentence $\psi_{\mathcal{S}}$ defined in the proof of Theorem 12. If $T = (a, \mathcal{S}, a') \in \mathbb{T}_2$, then $\mathcal{S}$ is the skeleton of some scheme joining (a, b) to (a', b') over S and some weakly flat left S-act B. By our assumption (3), there is a finite list of replacement skeletons $\mathcal{S}_1, ..., \mathcal{S}_{\beta(T)}$. Then, for each $k \in \{1, ..., \beta(T)\}$, if $\mathcal{S}_k = (u_1, v_1, ..., u_h, v_h)$, there exist a weakly flat left S-act C, elements $c, c', c_1, ..., c_h \in C_k$, and elements $q_2, ..., q_h \in aS \cup a'S$ such that

$$(2)\qquad \begin{array}{rclrcl} & & & c & = & u_1c_1 \\ au_1 & = & q_2v_1 & v_1c_1 & = & u_2c_2 \\ q_2u_2 & = & q_3v_2 & v_2c_2 & = & u_3c_3 \\ & \vdots & & & \vdots & \\ q_hu_h & = & a'v_h & v_hc_h & = & c'. \end{array}$$

For each k, we fix such a list $q_2, ..., q_h$ of elements, for future reference, and let φ_T be the sentence

$$(\forall y)(\forall y')(\gamma_{\mathcal{S}}(y,y') \to \gamma_{\mathcal{S}_1}(y,y') \vee \cdots \vee \gamma_{\mathcal{S}_{\beta(T)}}(y,y')).$$

Let

$$\Pi = \{\psi_T : T \in \mathbb{T}_1\} \cup \{\varphi_T : T \in \mathbb{T}_2\}.$$

We claim that $\mathcal{WF} = \mathrm{Mod}\,(\Pi)$.

Suppose first that D is any weakly flat left S-act. Let $T \in \mathbb{T}_1$. Then $T = (a, \mathcal{S}, a')$ is a realised triple. Since T is not witnessed in *any* weakly flat left S-act, T is certainly not witnessed in D so that $D \vDash \psi_T$.

On the other hand, for $T = (a, \mathcal{S}, a') \in \mathbb{T}_2$, where $\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$, if $d, d' \in D$ are such that $\gamma_{\mathcal{S}}(d, d')$ is true, then there are elements $d_1, \ldots d_m \in D$ such that

$$\begin{array}{rcl} d & = & s_1d_1 \\ t_1d_1 & = & s_2d_2 \\ & \vdots & \\ t_md_m & = & d,' \end{array}$$

which together with the fact that T is a realised triple, gives that (a, d) is connected to (a', d') over S and D via a tossing with skeleton $\mathcal{S}$. Because D is weakly flat, (a, d) and (a', d') are connected over $aS \cup a'S$ and D, and by assumption (3), we can take the tossing to have skeleton one of $\mathcal{S}_1, \ldots, \mathcal{S}_{\beta(T)}$, say $\mathcal{S}_k$. Thus $D \vDash \gamma_{\mathcal{S}_k}(d, d')$ and it follows that $D \vDash \varphi_T$. Hence D is a model of Π.

Conversely, we show that every model of Π is weakly flat. Let $C \vDash \Pi$ and suppose that $a, a' \in S, c, c' \in C$ and we have a tossing

$$\begin{array}{rclrcl} & & & c & = & s_1c_1 \\ as_1 & = & a_2t_1 & t_1c_1 & = & s_2c_2 \\ a_2s_2 & = & a_3t_2 & t_2c_2 & = & s_3c_3 \\ & \vdots & & & \vdots & \\ a_ms_m & = & a't_m & t_mc_m & = & c' \end{array}$$

with skeleton $\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$ over S and C. Then the triple $T = (a, \mathcal{S}, a')$ is realised, so that $T \in \mathbb{T}$. Since $\gamma_{\mathcal{S}}(c, c')$ holds, C cannot be a model of $\psi_{\mathcal{S}}$. Since $C \models \Pi$ it follows that $T \in \mathbb{T}_2$. But then φ_T holds in C so that for some $k \in \{1\ldots,\beta(T)\}$ we have that $\gamma_{\mathcal{S}_k}(c,c')$ is true. Together with the equalities on the left hand side of (2) we have a tossing over $aS \cup a'S$ and C connecting (a,c) to (a',c'). Thus C is weakly flat. $\square$

6. Examples

Example 1 Theorem 12 invites the question of whether there are monoids such that $\mathcal{F}$ is axiomatisable, but which are not left absolutely flat. Indeed there are, although we do not need the results of Section 4 to show this. Let S be the monoid of natural numbers $\mathbb{N}$ under multiplication. An easy exercise in elementary number theory shows that each right S-act

$$R(a,b) = \{(u,v) \in S \times S : au = bv\}$$

is cyclic, equal to $(b',a')S$ where $b' = b/d$, $a' = a/d$, and d is the greatest common divisor of a and b. It is implicit in Theorem 3.1 of [8] that this implies the class of left S-acts satisfying condition (P) is axiomatisable. Since S is commutative and cancellative, the class of flat left S-acts coincides with the class of left S-acts satisfying condition (P) [14, 15]. Thus $\mathcal{F}$ is axiomatisable. Now if all left S-acts are weakly flat, S must be regular [16]. Since it is not, there are certainly left S-acts which are not flat.

Example 2 To see that there are monoids for which $\mathcal{F}$ is *not* axiomatisable, consider an infinite null monoid, say $S = \{0,1,2,\ldots\}$, in which all products not involving 1 are equal to 0. It follows from Proposition 5.6 of [13], that if ultrapowers of copies of S are even weakly flat as left S-acts, then $r(a) = \{(u,v) \in S : au = av\}$ is a finitely generated right congruence, for every $a \in S$. Now for any $a \in S \setminus \{0,1\}$, we have

$$r(a) = (S \setminus \{1\} \times S \setminus \{1\}) \cup \{(1,1)\}\,.$$

It is then easy to see that $r(a)$ is not finitely generated.

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Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario N2L 3C5, Canada
E-mail address: `sbulman@wlu.ca`

Department of Mathematics, University of York, Heslington, York YO1 5DD, UK
E-mail address: `varg1@york.ac.uk`