COHERENT MONOIDS

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Abstract

This paper is concerned with a new notion of *coherency* for monoids. A monoid S is *right coherent* if the first order theory of right S-sets is coherent; this is equivalent to the property that every finitely generated S-subset of every finitely presented right S-set is finitely presented. If every finitely generated right ideal of S is finitely presented we say that S is *weakly right coherent.* As for the corresponding situation for modules over a ring, we show that our notion of coherency is related to products of flat left S-sets, although there are some marked differences in behaviour from the case for rings. Further, we relate our work to *ultraproducts* of flat left S-sets and so to the question of axiomatisability of certain classes of left S-sets.

We show that a monoid S is weakly right coherent if and only if the right annihilator congruence of every element is finitely generated and the intersection of any two finitely generated right ideals is finitely generated. A similar result describes right coherent monoids. We use these descriptions to recognise several classes of (weakly) right coherent monoids. In particular we show that any free monoid is weakly right (and left) coherent and any free commutative monoid is right (and left) coherent. [1]

1 Introduction

There are several obvious candidates for the definition of a right coherent monoid. In [22] Wheeler defines a coherent theory (in a first order language). The theory of right S–sets over a monoid S is coherent in Wheeler's sense if and only if every finitely generated S–subset of every finitely presented right

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S–set is finitely presented; we will say that such monoids are *right coherent*. If every finitely generated right ideal of a monoid S is finitely presented we say that S is *weakly right coherent*. These notions of coherency appeared in [12] in connection with the axiomatisability of certain classes of right S–sets.

Right coherent rings have been investigated by a number of authors. We recall that a ring R (with unity) is *right coherent* if every finitely generated R–submodule of every finitely presented right R–module is finitely presented, that is, the theory of right R–modules is coherent in Wheeler's sense. This is equivalent [5,8] to the condition that every finitely generated right ideal of R is finitely presented. Chase also proved in [5] that a ring R is right coherent if and only if every product of flat left R–modules is flat, moreover he described such rings internally as those for which the right annihilator ideals of elements are finitely generated and the intersection of any two finitely generated right ideals is finitely generated. Taking inspiration from the results of Chase, Bulman–Fleming and McDowell define a monoid to be right coherent if every product of flat left S–sets is flat. In Section 5 we point out some connections between our approach and theirs.

Following Section 2, where we give some preliminaries, we investigate in Section 3 some connections between noetherian properties of a monoid and coherency. If every right congruence on S is finitely generated then every finitely generated right S–set is finitely presented, a result due to Normak [18]. Certainly then S is right coherent. However it is not true that if every right ideal of S is finitely generated then S is weakly right coherent. But analogously to the result of Chase, one can show that a monoid S is weakly right coherent if and only if the right annihilator congruence of every element is finitely generated and the intersection of any two finitely generated right ideals is finitely generated. A similar result describes right coherent monoids.

We note that any right abundant monoid has the property that the right annihilator of any element is finitely generated. This enables us to recognise immediately several examples of weakly right coherent monoids. Further, we use our description of right coherent monoids to show that, for example, semilattices of groups are right coherent.

Section 4 is concerned with the free commutative monoid $\mathcal{FC}_X^*$ and the free monoid $\mathcal{F}_X^*$ on a given set X. Using results from Section 3 we show that $\mathcal{FC}_X^*$ is right (and left) coherent and $\mathcal{F}_X^*$ is weakly right (and left) coherent.

The final section considers some relations between weak right coherency, products and ultraproducts of flat left S–sets, and the work of Bulman–Fleming and McDowell [4]. In view of Łos's theorem this is also connected with the question of determining, for which monoids S is the class of flat left S–sets (first order) axiomatisable ? For completeness we also describe those monoids for which every product of strongly flat left S–sets is strongly flat.

2 Preliminaries

We assume the reader has a basic knowledge of semigroup theory, including a familiarity with the tensor product $A \otimes B$ of a right S–set A and a left S–set B. As a reference for the background required here we recommend [7,13], the notation and terminology of which we follow as far as possible; one exception being that S–sets are referred to in [13] (and elsewhere) as S–systems. We remark that when making a definition with an obvious left–right dual, we omit for the sake of brevity the statement of the dual.

We recall from the introduction that a monoid S is *right coherent* (*weakly right coherent*) if every finitely generated S–subset of every finitely presented right S–set (S) is finitely presented.

Proposition 2.1 *[12, Theorem 6] A monoid S is right coherent if and only if every finitely generated S–subset of every right S–set of the form S/ρ , where ρ is a finitely generated right congruence, is finitely presented.*

It is a consequence of a general result concerning finite presentations (see [22], p326) that a monoid S is right coherent (weakly right coherent) if and only if the kernel of every S–homomorphism $\phi : F^n \to S/\rho(\phi : F^n \to S)$ is finitely generated. Here F^n is the free right S–set on n generators, $n \in N$, and ρ is a finitely generated right congruence on S. Usual notation for F^n is $F^n = \dot{\bigcup}\{x_i S : 1 \leq i \leq n\}$, where $\dot{\bigcup}$ denotes disjoint union.

It is useful notationally to consider the categories of left (right) S–sets and S–homomorphisms : these we denote by S–$\underline{Ens}$ and $\underline{Ens}$–S, respectively. Let S be a monoid and $B \in S$–$\underline{Ens}$. Then B is *strongly flat* if the functor $- \otimes B$ preserves equaliser and pullback diagrams in $\underline{Ens}$–S. Strongly flat S–sets have a nice characterisation in terms of elements.

Proposition 2.2 *[21] A left S–set B is strongly flat if and only if for all $s, t \in S$ and $a, b \in B$, if $sa = tb$ then there are $s', t' \in S$ and $c \in B$ with $ss' = tt', a = s'c$ and $b = t'c$; moreover if $a = b$, then we can take $s' = t'$.*

Note that in [21], strong flatness is referred to as weak flatness. The notion of strongly flat is rather restrictive in the sense that all left S–sets are strongly flat if and only if S is trivial [16].

The left S–set B is *flat* if the functor $- \otimes B$ preserves monomorphisms (which are here injections) in $\underline{Ens}$–S. If $-\otimes B$ preserves embeddings of right ideals of S into S (considered as a right S–set), then B is called *weakly flat.* Since monomorphisms are equalisers in $\underline{Ens}$–S, every strongly flat left S–set is flat; clearly every flat left S–set is weakly flat.

A monoid S is *left absolutely flat* if every left S–set is flat. Such monoids have been widely investigated, in particular by S.Bulman–Fleming and K.McDowell. They form an extensive class – for example it is shown in [1] that every inverse monoid is left and right absolutely flat. Thus every non–trivial inverse monoid S has a left S–set which is flat but not strongly flat. Further, by the results of [2] and [3], given any right normal band S which does not have constant structure mappings then there is a weakly flat S^1–set which is not flat.

For $A \in \underline{Ens}$–S and $a, b \in A$ we define

$$R(a,b) = \{(u,v) \in S \times S : au = bv\}, r(a,b) = \{u \in S : au = bu\},$$

$$r(a) = R(a,a).$$

Then $R(a,b) = \emptyset$ or is an S–subset of the right S–set $S \times S$, $r(a,b) = \emptyset$ or is a right ideal of S and $r(a)$ is a right congruence on S, the *right annihilator congruence* of a.

If $A \in \underline{Ens}$–S and $H \subseteq A \times A$ then we denote by $\rho(H)$ the smallest congruence on A containing H. As a special case of the Malcev Lemma we have the following.

Lemma 2.3 *Given $A \in \underline{Ens}$–S , $H \subseteq A \times A$ and $a, b \in A$, then $a\rho(H)b$ if and only if $a = b$ or there is a sequence*

$$a = c_1t_1, d_1t_1 = c_2t_2, ..., d_{\ell-1}t_{\ell-1} = c_\ell t_\ell, d_\ell t_\ell = b,$$

where for $i \in \{1, ..., \ell\}$, $t_i \in S$ and $(c_i, d_i) \in H \cup H^{-1}$.

A sequence as in Lemma 2.3 will be referred to as a *$\rho(H)$–sequence of length n*. For any $A \in \underline{Ens}$–S and congruence ρ on A, we denote by $[a]$ the ρ–equivalence class of an element a of A.

Finally in this section we quote a result from [1] which determines when two elements in a tensor product of S–sets are equal.

Lemma 2.4 *Let S be a monoid, $A \in \underline{Ens}$–S, $B \in S$–$\underline{Ens}$. Then $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exist $a_1, ..., a_n \in A$, $b_2, ..., b_n \in B, s_1, ..., s_n, t_1, ..., t_n \in S$ such that*

$$\begin{array}{ll} a = a_1s_1, & \\ a_1t_1 = a_2s_2, & s_1b = t_1b_2 \\ a_2t_2 = a_3s_3, & s_2b_2 = t_2b_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ a_nt_n = a', & s_nb_n = t_nb'. \end{array}$$

A system of equalities as in Lemma 2.4 is called a *scheme over A and B of length n.*

3 Noetherian conditions and coherency

If a ring R is right noetherian, that is, every right ideal is finitely generated, then it is well known (see, for example [20]) that every finitely generated right R–module is finitely presented, so that certainly R is right coherent. Of course for rings a right congruence is determined by a right ideal, which is not true for monoids in general. We shall say that a monoid S is *right noetherian* (*weakly right noetherian*) if every right congruence on S (right ideal of S) is finitely generated. By considering Rees right congruences it is easy to see that every right noetherian monoid is weakly right noetherian.

Thus two questions naturally arise. First, is every right noetherian monoid right coherent? And second, is every weakly right noetherian monoid weakly right coherent? The first is answered in the affirmative by Normak, who shows in [18] that every finitely generated right S–set over a right noetherian monoid S is finitely presented. The second is answered negatively by the following example, for which I am grateful to J.B. Fountain.

Example 3.1 .

Let G be an abelian group which is not finitely generated. Let $x, 0$ be symbols not in G. Put $S_0 = G, S_1 = \{xg : g \in G\}, S_2 = \{x^2g : g \in G\}, S_3 = \{x^3\}, S_4 = \{0\}$ (here x^2, x^3 are merely symbols). Put $S = \bigcup\{S_i : 0 \leq i \leq 4\}$ and define a multiplication on S extending that of $G = S_0$ as follows :

$g(xh) = x(gh) = (xh)g$
$g(x^2h) = x^2(gh) = (x^2h)g$
$gx^3 = x^3 = x^3g$
$(xg)(xh) = x^2(gh) = (xh)(xg)$
$(xg)(x^2h) = x^3 = (x^2h)(xg)$
$(xg)x^3 = 0 = x^3(xg)$
$(x^2g)(x^2h) = 0 = (x^2h)(x^2g)$
$(x^2g)x^3 = 0 = x^3(x^2g)$
$x^3x^3 = 0 = 0s = s0$ for all $s \in S$.

For ease of notation we identify x, x^2 with $x1_G, x^21_G$ respectively. It is not difficult to see that S is a monoid which is weakly right noetherian. Indeed, every right ideal is principal. However, we claim that the right ideal x^2S is not finitely presented so that S is not weakly right coherent . Since $x^2S \cong S/r(x^2)$, by remarks in the previous section it is enough to show that $r(x^2)$ is not finitely generated.

Suppose therefore that k is a natural number and

$$H = \{(\alpha_1, \beta_1), ..., (\alpha_k, \beta_k)\}$$

is a symmetric set of generators for $r(x^2)$. Let $g \in G$,where $g \neq 1_G$. Then $(xg, x) \in r(x^2)$ and as $xg \neq x$, xg and x are connected by a $\rho(H)$–sequence

$$xg = c_1t_1, d_1t_1 = c_2t_2, ..., d_nt_n = x$$

which we may assume is of minimal length. From the construction of S it follows that each pair (c_i, d_i) is of the form (xh_i, xk_i) where $h_i, k_i \in G$ and each $t_i \in G, 1 \leq i \leq n$. One thus obtains that $g = h_1 k_1^{-1} ... h_n k_n^{-1}$. It is now an easy consequence that G is finitely generated. From this contradiction we have that S is not weakly right coherent.

On the positive side we have the following.

Proposition 3.2 *Let A be a right S–set. Then the following conditions are equivalent:*
(i) the kernel of every S–homomorphism from a finitely generated free right S–set to A is finitely generated;
(ii) for all $a, b \in A$, $r(a)$ is finitely generated and $aS \cap bS = \emptyset$ or is finitely generated.

Proof (i)⇒(ii) Let $a, b \in A$. Define $\phi : S \to A$ by $s\phi = as$. Then $ker\phi = r(a)$ and is finitely generated by assumption.

Consider now $\psi : F = xS \cup yS \to A$ where $x\psi = a$ and $y\psi = b$. Again by assumption, $ker\psi =< H >$ for some finite subset H of $F \times F$, where we may suppose that H is symmetric. Let

$$C = \bigcup \{asS : (xs, yt) \in H \text{ for some } t \in S\};$$

clearly C is empty or is a finitely generated S–subset of A. We claim that $C = aS \cap bS$.

Given $(xs, yt) \in H$ then $as = bt \in aS \cap bS$ so that $asS \subseteq aS \cap bS$. Thus $C \subseteq aS \cap bS$. Conversely, suppose that $au = bv \in aS \cap bS$, where $u, v \in S$. Then $(xu, yv) \in ker\psi$ and as $xu \neq yv$ there is a $\rho(H)$–sequence

$$xu = c_1 t_1, d_1 t_1 = c_2 t_2, ..., d_\ell t_\ell = yv.$$

Let n be the least number such that $(c_n, d_n) = (xh_n, yk_n)$. Then for $1 \leq i < n, (c_i, d_i) = (xh_i, xk_i)$ and

$$au = ah_1 t_1 = ak_1 t_1 = ah_2 t_2 = ... = ah_n t_n$$

so that $au \in C$. It follows that $aS \cap bS = C$ as required.

(ii)⇒(i) Let $\phi : x_1 S \cup ... \cup x_n S \to A$ where $x_i \phi = a_i$. For $i \in \{1, ..., n\}$ we have that $x_i s ker\phi x_i t$ if and only if $a_i s = a_i t$ if and only if $(s, t) \in r(a_i)$ and by assumption $r(a_i)$ has a finite set of generators H_i. Thus $x_i H_i := \{(x_i h, x_i k) : (h, k) \in H_i\} \subseteq ker\phi$ and $x_i s ker\phi x_i t$ if and only if $x_i s < x_i H_i > x_i t$.

For $i, j \in \{1, ..., n\}$ where $i \neq j$, let $C_{ij} = a_i S \cap a_j S$. If $C_{ij} \neq \emptyset$ then $C_{ij} = \bigcup \{b_{ij}^k S : 1 \leq k \leq p(ij)\}$ for some $p(ij) \in N$ and for each $k \in \{1, ..., p(ij)\}, b_{ij}^k =$

$a_iu_{ij}^k = a_jv_{ij}^k$. Let $K_{ij} = \{(x_iu_{ij}^k, x_jv_{ij}^k) : 1 \leq k \leq p(ij)\}$, with the convention that if $C_{ij} = \emptyset$ then $K_{ij} = \emptyset$. Certainly $\nu = < \bigcup\{x_iH_i : 1 \leq i \leq n\} \cup \bigcup\{K_{ij} : 1 \leq i, j \leq n, i \neq j\} > \subseteq ker\phi$.

Conversely, suppose that $(x_is, x_it) \in ker\phi$ for some $i \in \{1, ..., n\}$. As remarked above, $x_is < x_iH_i > x_it$ so that certainly $x_is\nu x_it$. If $(x_is, x_jt) \in ker\phi$ where i and j are distinct elements of $\{1, ..., n\}$, then $a_is = a_jt \in C_{ij}$ so that

$$a_is = a_jt = a_iu_{ij}^kr = a_jv_{ij}^kr$$

for some $r \in S$. But then $(x_is, x_iu_{ij}^kr), (x_jt, x_jv_{ij}^kr) \in ker\phi$ so that

$$x_is\nu x_iu_{ij}^kr\nu x_jv_{ij}^kr\nu x_jt.$$

Hence $\nu = ker\phi$ and $ker\phi$ is finitely generated.

Corollary 3.3 *The following are equivalent for a monoid S:*
(i)S is weakly right coherent;
(ii)for all $a, b \in S, r(a)$ is finitely generated and $aS \cap bS = \emptyset$ or is a finitely generated right ideal;
(iii)for all $a, b \in S, r(a)$ is finitely generated and the intersection of any two finitely generated right ideals of S is empty or is finitely generated.

Corollary 3.4 *The following are equivalent for a monoid S:*
(i)S is right coherent;
(ii)for any finitely generated right congruence ρ on S and any $a, b \in S/\rho$, $r(a)$ is finitely generated and $aS \cap bS = \emptyset$ or is finitely generated;
(iii)for any finitely generated right congruence ρ on S then for any $a \in S/\rho$, $r(a)$ is finitely generated and the intersection of any two finitely generated $S-$subsets of S/ρ is empty or is finitely generated.

Corollary 3.3 provides us with many examples of weakly right coherent monoids. We recall that the relation $\mathcal{L}^*$ is defined on a monoid S by the rule that $a\mathcal{L}^*b$ if for all $x, y \in S$,

$$ax = ay \text{ if and only if } bx = by.$$

A monoid is *right abundant* if every $\mathcal{L}^*-$class contains an idempotent. Such monoids are also called *right principally projective* or *right* PP, since they are precisely those monoids in which every principal right ideal is projective. These monoids were first considered in [14], and have subsequently been studied by Fountain in [9,10]. They also occur in the work of Bulman-Fleming and McDowell (see [3] and [4]). If S is a regular monoid then $\mathcal{L}^* = \mathcal{L}$ so that certainly S is right abundant. Obviously left cancellative monoids are also right abundant.

Lemma 3.5 *If S is right abundant then $r(s)$ is finitely generated for all $s \in S$.*

Proof If S is right abundant and $s \in S$ then $r(s) =< (1, e) >$ where $s\mathcal{L}^*e$ and $e \in E(S)$.

Corollary 3.6 *Let S be a monoid. Any of the following conditions is sufficient for S to be weakly right coherent:*
(i)S is right abundant and for all $a, b \in S$, if $aS \cap bS \neq \emptyset$ then $aS \cap bS$ is finitely generated;
(ii)S is right abundant and weakly right noetherian;
(iii)$S = T^1$ where T is completely (0–) simple;
(iv)S is inverse;
(v)S is the multiplicative semigroup of a regular ring.

Proof Everything is clear except perhaps (v). The fact that the intersection of two principal right ideals in a regular ring is again principal is well known. For a reference, we quote [17] together with [5]. On p.68 the former states that in a regular ring, every finitely generated right ideal is principal. Thus if I is a finitely generated right ideal of a regular ring R, then $I = aR$ for some element a in R. As $r(a)$ is finitely generated it follows that I is finitely presented ('finitely related' in the terminology of [5]). Then Theorem 2.2 of [5] gives that the intersection of two principal right ideals is finitely generated and hence principal.

We remark that in view of the comments following Proposition 5.4, parts (iii),(iv) and (v) of Corollary 3.6 also follow from Theorem 7 of [4].

Our next aim is to use Corollary 3.4 to obtain examples of right coherent monoids.

Lemma 3.7 *Let S be a monoid such that given any $s \in S$ and finite subset X of S, there is an $e \in E(S)$ such that $s\mathcal{R}e$ and e commutes with all $x \in X$. Then for every finitely generated right congruence ρ on S and every $a \in S/\rho$, $r(a)$ is finitely generated.*

Proof Let ρ be finitely generated by $H = \{(u_1, v_1), ..., (u_n, v_n)\}$. Put $X = \{u_1, v_1, u_2, ..., v_n\}$ and suppose that $a = [s] \in S/\rho$. By hypothesis we may choose $e \in E(S)$ with $s\mathcal{R}e$ such that e commutes with every element of X. Pick $s' \in S$ such that $s = ss's$ and $e = ss'$, so that $(1, s's) \in r(a)$. Further, for any $i \in \{1, ..., n\}, ss'u_i = u_iss'\rho v_iss' = ss'v_i$, so that $(s'u_i, s'v_i) \in r(a)$. Thus if $\nu =< \{(1, s's), (s'u_i, s'v_i) : 1 \leq i \leq n\} >$ then $\nu \subseteq r(a)$.

Conversely, if $(p, q) \in r(a)$, then either $sp = sq$ so that $p\nu s'sp = s'sq\nu q$, or there is a $\rho(H)$–sequence

$$sp = c_1t_1, d_1t_1 = c_2t_2, ..., d_\ell t_\ell = sq.$$

In the latter case $p\nu s'sp = s'c_1t_1\nu s'd_1t_1...\nu s'd_\ell t_\ell = s'sq\nu q$. Thus $r(a) = \nu$ and so is finitely generated.

Corollary 3.8 *Every inverse monoid with central idempotents is (right) coherent.*

Proof In view of Corollary 3.4 and Lemma 3.7 it is enough to show that for any finitely generated right congruence ρ of S and $a, b \in S/\rho$, $aS \cap bS$ is finitely generated.

Given ρ, a, b as above,we suppose that $a = [s]$ and $b = [t]$. Let s', t' be the inverses of s, t respectively. We claim that $aS \cap bS = cS$ where $c = [s'st't]$. For using the fact that S is a semilattice of groups, $c = [s'st't] = [ss't't] = [s]s't't = as't't$ and similarly, $c = bt's's$, so that $c \in aS \cap bS$ and $cS \subseteq aS \cap bS$.

Conversely, if $d = ah = bk$, then

$$d = [s]h = [t]k = [tt't]k = [t]kt't$$

so that

$$d = [s]ht't = [ss's]ht't = [s'st't]sh = csh \in cS.$$

Thus the claim and hence the lemma hold.

4 Free monoids and coherency

For a given non–empty set X we denote by $\mathcal{F}_X$, ($\mathcal{FC}_X$) the free semigroup (free commutative semigroup) on X. The free monoid (free commutative monoid) on X we write as $\mathcal{F}_X^*$ ($\mathcal{FC}_X^*$). If $w \in \mathcal{F}_X^*$ or $\mathcal{FC}_X^*$ the *content* of w, $c(w)$,is defined by $c(w) = \{x \in X : x \text{ occurs in } w\}$.

Proposition 4.1 *The free monoid on any non–empty set is weakly right coherent.*

Proof Let $S = \mathcal{F}_X^*$ where $X \neq \emptyset$. Since S is cancellative, S is certainly right abundant. It is clear that the intersection of any two principal right ideals is empty or principal, and so by Corollary 3.3, S is weakly right coherent.

The proof of the next lemma is routine and so is omitted.

Lemma 4.2 *Let T be a semigroup. If T is right noetherian, then so is T^1.*

Redei's Theorem [7,19] states that $\mathcal{FC}_X$ is (right) noetherian for any finite, non–empty set X. Thus in this case, certainly $\mathcal{FC}_X^*$ is (right) coherent. We make use of Redei's Theorem to show that for *any* non–empty set X, $\mathcal{FC}_X^*$ is (right) coherent.

Theorem 4.3 *For any non–empty set X, $S = \mathcal{FC}^*_X$ is a right (and left) coherent monoid.*

Proof We show that conditions (ii) of Corollary 3.4 hold. Our convention is that $\mathcal{FC}^*_\emptyset$ is the trivial monoid. Let H be a finite subset of $S \times S$ and put $\rho = \rho(H)$. Let K be any finite subset of X containing U, where U consists of all letters occurring in any u where $(u,v) \in H \cup H^{-1}$ for some $v \in S$.

We show first that if $(s,t) \in S$ and $s\rho t$, then $(s,t) = (x,y)z$ for some $x, y \in \mathcal{FC}^*_K$ and $z \in \mathcal{FC}^*_{X\setminus K}$ such that $x\rho_K(H)y$ where $\rho_K(H)$ is the congruence on $\mathcal{FC}^*_K$ generated by H. For if $s\rho t$, then either $s = t$ so that $(s,t) = (x,x)z$ where $x \in \mathcal{FC}^*_K, z \in \mathcal{FC}^*_{X\setminus K}$, or there is a $\rho(H)$–sequence

$$s = c_1t_1, d_1t_1 = c_2t_2, ..., d_\ell t_\ell = t.$$

Now writing $s = xz$, where $c(x) \subseteq K, c(z) \subseteq X \setminus K$, we must have for each $i \in \{1, ..., \ell\}$ that $t_i = t'_i z, t'_i \in S$, and so $t'_i \in \mathcal{FC}^*_K, i \in \{1, ..., \ell\}$. Thus $t = yz$ where $y \in \mathcal{FC}^*_K, x\rho_K(H)y$ and $(s,t) = (x,y)z$ as required.

Let $a \in S$ and define K_1 to be the subset of X consisting of U together with all letters occurring in a. Then $r([a]) \cap \mathcal{FC}^{*2}_{K_1}$ is a (right) congruence on $\mathcal{FC}^*_{K_1}$ and so by Redei's theorem has a finite set M of generators *as a congruence on* $\mathcal{FC}^*_{K_1}$. But if $(s,t) \in r([a])$ then $as\rho at$ so that $(as, at) = (u,v)w$ for some $(u,v) \in \rho_{K_1}(H)$ and $w \in \mathcal{FC}^*_{X\setminus K_1}$. Since $c(a) \subseteq K_1$ we must have that $u = au', v = av'$ for some $u', v' \in \mathcal{FC}^*_{K_1}$ and so $(s,t) = (u',v')w$. Further, $au' = u\rho v = av'$ so that $(u',v') \in r([a]) \cap \mathcal{FC}^{*2}_{K_1}$. It follows that M generates $r([a])$ as a congruence on S.

Consider now elements $a, b \in S$ where $I = [a]S \cap [b]S \neq \emptyset$. Let K_2 be the subset of X consisting of U together with all letters occurring in a or b. Since $I \neq \emptyset$ there are elements $u, v \in S$ with $au\rho bv$. But then $(au, bv) = (x,y)z$ for some $x, y \in \mathcal{FC}^*_{K_2}, z \in \mathcal{FC}^*_{X\setminus K_2}$ such that $x\rho_{K_2}(H)y$. Thus $x = ax', y = by'$ for some $x', y' \in \mathcal{FC}^*_{K_2}$ and $[x]' \in J := [a]'S' \cap [b]'S'$, where for $w \in S' := \mathcal{FC}^*_{K_2}, [w]'$ denotes the $\rho_{K_2}(H)$–equivalence class of w. In particular, $J \neq \emptyset$ and as S' is certainly right coherent, J is finitely generated as an S'–subset of $S'/\rho_{K_2}(H)$. It is easy to see that if $N' = \{[r_i]' : 1 \leq i \leq n\}$ generates J, then $N = \{[r_i] : 1 \leq i \leq n\}$ generates I, thus completing the proof.

5 Ultraproducts, axiomatisability and coherency

For a monoid S we denote by L^ℓ_S the first order language with equality, which has no constant or relation symbols and which has a unary function symbol λ_s for each $s \in S$. We write sx for $x\lambda_s$ and we regard left S–sets as L^ℓ_S–structures in the obvious way. If $s, t \in S$ we denote by $\psi_{s,t}$ the sentence of L^ℓ_S

$$(\forall x)((st)x = s(tx)).$$

Then an L_S^ℓ–structure is a left S–set if and only if it is a model of $\sum_S^\ell = \{(\forall x)(1x = x)\} \cup \{\psi_{s,t} : s,t \in S\}$.

It seems natural to consider the conditions a monoid S must satisfy in order that a given class of left S–sets be axiomatisable, that is, is precisely the class of models satisfying a given set of sentences of L_S^ℓ. Here we find coherency notions appearing. Note that $\sum_S^\ell$ axiomatises the class of all left S–sets.

A major tool in investigating axiomatisability is Łos's theorem.

Theorem 5.1 *(Łos,[6]) Let L be a first order language and let $\mathcal{C}$ be a class of L-structures. If $\mathcal{C}$ is axiomatisable, then $\mathcal{C}$ is closed under ultraproducts.*

In [12] we show that the class of existentially closed left S–sets, or the class of absolutely pure left S–sets, is axiomatisable if and only if S is left coherent, and the class of weakly f–injective left S–sets is axiomatisable if and only if S is weakly left coherent.

Theorem 3.1 of [11] states that the class of strongly flat left S–sets is axiomatisable if and only if for each $(a,b) \in S \times S, R(a,b) = \emptyset$ or is finitely generated (as a right S–subset of $S \times S$) and $r(a,b) = \emptyset$ or is finitely generated. Moreover, this is equivalent to the property that every ultrapower of copies of S is strongly flat as a left S–set. It is easy to see directly that such monoids must be weakly right coherent. On the other hand, this follows from Proposition 5.6 below.

In view of Łos's theorem the problem arises of describing those monoids S such that ultraproducts of certain left S–sets retain a given flatness property, and here again coherency plays a part. One can of course ask the corresponding question for products and it transpires that the answers are related . On the other hand we recall that in [4] a monoid S is *defined* to be (weakly) right coherent if every non–empty product of (weakly) flat left S–sets is (weakly) flat.

Proposition 5.2 *The following conditions are equivalent for a monoid S :*
(i)every product $S^I (I \neq \emptyset)$ is strongly flat as a left S–set;
(ii)every product $\prod\{A_i : i \in I\}(I \neq \emptyset)$ of strongly flat left S–sets $A_i, i \in I$ is strongly flat ;
(iii)for all $(a,b) \in S \times S, R(a,b) = \emptyset$ or is cyclic and $r(a,b) = \emptyset$ or is principal.

Proof (i)$\Rightarrow$ (iii)Suppose that $a,b \in S$ and $R(a,b) \neq \emptyset$. Let I index the set $R(a,b)$ and write $R(a,b) = \{(x_i,y_i) : i \in I\}$. Let $\underline{x}, \underline{y}$ be the elements of S^I whose i'th components are x_i, y_i respectively. Then $a\underline{x} = b\underline{y}$ in S^I and as S^I is strongly flat by assumption, we have that $au = bv, \underline{x} = u\underline{z}$ and $\underline{y} = v\underline{z}$ for some $u,v \in S$ and $\underline{z} \in S^I$. Thus $(u,v) \in R(a,b)$ so that $(u,v) = (x_j,y_j)$ for some $j \in I$. If $i \in I$ we have $(x_i,y_i) = (u,v)z_i = (x_j,y_j)z_i$ where z_i is the i'th component of $\underline{z}$. This gives that $R(a,b)$ is cyclic. A similar argument applies to $r(a,b)$.

(iii)⇒(ii) Let $P = \prod\{A_i : i \in I\}, I \neq \emptyset$, be a product of strongly flat left S–sets. Suppose that $a\underline{x} = b\underline{y}$ where $s, t \in S$ and $\underline{x} = (x_i), \underline{y} = (y_i) \in P$. For each $i \in I$, $ax_i = by_i$ and so as A_i is flat there are elements $h_i, k_i \in S$ and $z_i \in A_i$ with $ah_i = bk_i, x_i = h_i z_i, y_i = k_i z_i$. Certainly $R(a, b) \neq \emptyset$ and so by assumption is cyclic, say $R(a, b) = (p, q)S$. Thus for each $i \in I, (h_i, k_i) = (p, q)r_i$ for some $r_i \in S$. We now have $ap = bq$ and $x_i = pr_i z_i, y_i = qr_i z_i$ for each $i \in I$. If $\underline{w} = (r_i z_i) \in P$, then $\underline{x} = p\underline{w}$ and $\underline{y} = q\underline{w}$. Together with a similar argument for equalities of the form $a\underline{x} = b\underline{x}$, this gives that P is strongly flat.

We now consider a further notion of coherency, which appears naturally when considering weakly flat ultraproducts of left S–sets and is moreover related to the work in [4].

A monoid S is *bounded weakly right coherent* if given any S–homomorphism $\phi : \bigcup\{x_i S : 1 \leq i \leq n\} \to S$, then $ker\phi = \rho(H)$ for some finite set H such that the minimal lengths of the $\rho(H)$–sequences connecting pairs in $ker\phi$ is bounded.

Given ϕ as above and a generating set K of $ker\phi$, then if $(x_i s, x_j t) \in ker\phi$ and

$$x_i s = c_1 t_1, d_1 t_1 = c_2 t_2, ..., d_\ell t_\ell = x_j t$$

is $\rho(K)$–sequence, we say that

$$P = (c_1, d_1), ..., (c_\ell, d_\ell)$$

is a *path* connecting $x_i s$ and $x_j t$.

If S is such that given any S–homomorphism ϕ as above then $ker\phi =< K >$ for some K with the property that for any $i, j \in \{1, ..., n\}$ then at most one path is needed to connect all elements of $ker\phi$ of the form$(x_i s, x_j t)$, then S is *path weakly right coherent*. Clearly, if S is path weakly right coherent then S is bounded weakly right coherent.

The proof of the following lemma is similar to that of Proposition 2 of [4] and so is left to the reader.

Lemma 5.3 *A monoid S is path weakly right coherent if and only if for every finitely generated right ideal I of S and non–empty set Γ, the natural map $I \otimes S^\Gamma \to I^\Gamma$ is one–one.*

The next result is now immediate from the definitions in [4].

Proposition 5.4 *All left S–sets $S^\Gamma, \Gamma \neq \emptyset$, are weakly flat if and only if S is path weakly right coherent*

Thus if a monoid S is weakly right coherent in the sense of [4] then it is weakly right coherent in our sense; this also follows indirectly from Corollary 3.3 and Theorem 4 of [4].

Corollary 5.5 *Every left absolutely flat monoid is weakly right coherent.*

We remark that Corollary 5.5 extends the result of Section 3 which states that every inverse monoid is weakly right coherent, for it is shown in [1] that inverse monoids are absolutely flat. Proposition 5.6 below compares with Proposition 5.4 in a manner analogous to that in which Theorem 3.1 of [11] compares with Proposition 5.2.

We adopt the notation $\underline{\underline{x}}$ for the natural image in an ultrapower S^γ/E of an element $\underline{x} \in S^\gamma$.

Proposition 5.6 *Every ultrapower of S is weakly flat as a left S-set if and only if S is bounded weakly right coherent.*

Proof We suppose first that every ultrapower of S is weakly flat in S–$\underline{Ens}$. Let $\phi : \dot{\bigcup}\{x_iS : 1 \leq i \leq n\} \to S$ be an S–homomorphism and put $x_i\phi = c_i, 1 \leq i \leq n$ and $\rho = ker\phi$. Assume that ρ is not finitely generated.

Let $H = \{(x_{f(\beta)}u_\beta, x_{g(\beta)}v_\beta : \beta < \gamma\}$ be a generating set for ρ of minimum cardinality γ such that if $\rho_\alpha =< \{(x_{f(\beta)}u_\beta, x_{g(\beta)}v_\beta) : \beta < \alpha\} >$, then $(x_{f(\alpha)}u_\alpha, x_{g(\alpha)}v_\alpha) \notin \rho_\alpha$, for any $\alpha < \gamma$. Here f, g are functions from γ to $\{1, ..., n\}$. Let D be a uniform ultrafilter on γ, that is, D is an ultrafilter such that all sets in D have cardinality γ. Put $\mathcal{U} = S^\gamma/D, \underline{u} = (u_\beta)$ and $\underline{v} = (v_\beta)$, so that $\underline{\underline{u}}, \underline{\underline{v}} \in \mathcal{U}$.

For each $i, j \in \{1, ..., n\}$ put $T_{ij} = \{\beta : f(\beta) = i, g(\beta) = j\}$. Then $\gamma = \cup\{T_{ij} : 1 \leq i, j \leq n\}$ and it follows that for some pair $(i, j), T_{ij} \in D :$ we fix for the moment this pair. For $\beta \in T_{ij}, c_iu_\beta = c_jv_\beta$ so that $c_i\underline{\underline{u}} = c_j\underline{\underline{v}}$ and it follows that $c_i \otimes \underline{\underline{u}} = c_j \otimes \underline{\underline{v}}$ in $S \otimes \mathcal{U}$. Then by assumption there is a scheme

$$\begin{array}{rr} & \underline{\underline{u}} = s_1\underline{\underline{b}}_1 \\ c_is_1 = a_2t_1, & t_1\underline{\underline{b}}_1 = s_2\underline{\underline{b}}_2 \\ a_2s_2 = a_3t_2, & \\ \cdot & \cdot \\ \cdot & \cdot \\ a_ms_m = c_jt_m, & t_m\underline{\underline{b}}_m = \underline{\underline{v}} \end{array}$$

with $a_2, ..., a_m \in c_iS \cup c_jS$. It is easy to see that we may assume $a_2, ..., a_m \in \{c_i, c_j\}$. Pick a function $\{1, ..., m+1\} \to \{i, j\}$, written $k \to i_k$, such that $i_1 = i, i_{m+1} = j$ and for $k \in \{2, ..., m\}, a_k = c_{i_k}$. Then for $k \in \{1, ..., m\}, x_{i_k}s_k\rho x_{i_{k+1}}t_k$. As H generates ρ and γ is a limit ordinal, there is some $\alpha < \gamma$ with $x_{i_k}s_k\rho_\alpha x_{i_{k+1}}t_k$ for all $k \in \{1, ..., m\}$.

For $\ell \in \{1, ..., m\}$ we write $\underline{b_\ell} = (b_{\ell,\beta})$ and define $V_0 = \{\beta : u_\beta = s_1b_{1,\beta}\}, V_1 = \{\beta : t_1b_{1,\beta} = s_2b_{2,\beta}\}, ..., V_m = \{\beta : t_mb_{m,\beta} = v_\beta\}$. Thus $T = T_{ij} \cap V_0 \cap ... \cap V_m$ is in D and as D is uniform T contains an element θ with $\alpha < \theta$. We now have that

$$x_{f(\theta)}u_\theta = x_iu_\theta = x_is_1b_{1,\theta} = x_{i_1}s_1b_{1,\theta}\rho_\alpha x_{i_2}t_1b_{1,\theta} = x_{i_2}s_2b_{2,\theta}\rho_\alpha ... \rho_\alpha$$

$$x_{i_m}t_{m-1}b_{m-1,\theta} = x_{i_m}s_m b_{m,\theta}\rho_\alpha x_j t_m b_{m,\theta} = x_j v_\theta = x_{g(\theta)}v_\theta$$

so that $x_{f(\theta)}u_\theta\rho_\theta x_{g(\theta)}v_\theta$, a contradiction. Hence $ker\phi$ is finitely generated and S is weakly right coherent.

To show that S is bounded weakly right coherent, we consider ϕ as above and let K be a finite set of generators of $ker\phi$. We suppose by way of contradiction that for any $m \in N$ there is an element (y_m, z_m) of $ker\phi$ such that y_m cannot be connected to z_m by a $\rho(K)$–sequence of length equal to or less than m. Let $y_m = x_{f(m)}y'_m, z_m = x_{g(m)}z'_m$, where $f, g : N \to \{1, ..., n\}$ and $y'_m, z'_m \in S, m \in N$. Let D be a non–principal ultrafilter over N, $\mathcal{U} = S^N/D$ and put $\underline{y} = (y'_m), \underline{z} = (z'_m)$.

For each $i, j \in \{1, ..., n\}$ let $T_{ij} = \{m : f(m) = i, g(m) = j\}$ so that $N = \bigcup\{T_{ij} : 1 \leq i, j \leq n\}$ and we may choose and fix some $T_{ij} \in D$. If $m \in T_{ij}$ then $(y_m, z_m) = (x_i y'_m, x_j z'_m)$ and since $(y_m, z_m) \in ker\phi, c_i y'_m = c_j z'_m$, giving that $c_i\underline{\underline{y}} = c_j\underline{\underline{z}}$ and $c_i \otimes \underline{\underline{y}} = c_j \otimes \underline{\underline{z}}$ in $S \otimes \mathcal{U}$. Again we have a typical scheme

$$\begin{array}{rr} & \underline{\underline{y}} = s_1\underline{\underline{b}}_1 \\ c_i s_1 = a_2 t_1, & t_1\underline{\underline{b}}_1 = s_2\underline{\underline{b}}_2 \\ a_2 s_2 = a_3 t_2, & . \\ . & . \\ . & . \\ a_m s_m = c_j t_m, & t_m\underline{\underline{b}}_m = \underline{\underline{z}}. \end{array}$$

where we take $a_2, ..., a_m \in \{c_i, c_j\}$; we pick a function $\{1, ..., m+1\} \to \{i, j\}$, written $k \mapsto i_k$, as in the first part of the proof. Then for $k \in \{1, ..., m\}, x_{i_k}s_k\rho x_{i_{k+1}}t_k$ and amongst the $\rho(K)$–sequences joining $x_{i_k}s_k$ to $x_{i_{k+1}}t_k$ there must be one of shortest length p_k. Choose and fix such a sequence for each $k \in \{1, ..., m\}$.

For $\ell \in \{1, ..., m\}$ put $\underline{b}_\ell = (b_{\ell,r})$ and let $V_0 = \{r : y'_r = s_1 b_{1,r}\}, V_1 = \{r : t_1 b_{1,r} = s_2 b_{2,r}\}, ..., V_m = \{r : t_m b_{m,r} = z'_m\}$. Then $T = T_{ij} \cap V_0 \cap ... \cap V_m \in D$ and so there is an element q in T with $q > p_1 + ... + p_m$.

By assumption, $(y_q, z_q) = (x_i y'_q, x_j z'_q)$ is such that y_q cannot be related to z_q by a $\rho(K)$–sequence of length equal to or less than q. But $x_i y'_q = x_{i_1}s_1 b_{1,q}\rho x_{i_2}t_1 b_{1,q} = x_{i_2}s_2 b_{2,q}\rho ... \rho x_{i_{m+1}}t_m b_{m,q} = x_j z'_q$, which gives a $\rho(K)$–sequence of length $p_1 + ... + p_k$, a contradiction. It follows that S is bounded weakly right coherent.

Conversely, we assume that S is bounded weakly right coherent and consider an ultrapower $\mathcal{U} = S^\gamma/D$. In view of Exercises 4.1.6 and 4.1.21 of [6] we assume that γ is infinite and D is non–principal. We suppose that $a \otimes \underline{\underline{x}} = b \otimes \underline{\underline{y}}$ in $S \otimes \mathcal{U}$ so that in $\mathcal{U}$ we have $a\underline{\underline{x}} = b\underline{\underline{y}}$. Define $\phi : wS\dot{\cup}zS \to S$ by $w\phi = a$ and $z\phi = b$. Then there is a finite set $H \subseteq ker\phi$ such that $ker\phi = \rho(H)$ and the minimal lengths of $\rho(H)$–sequences connecting the co–ordinates of elements of $ker\phi$ is bounded. Writing $\underline{x} = (x_i)$ and $\underline{y} = (y_i)$ we let $T = \{i \in \gamma : ax_i = by_i\}$, so that $T \in D$. Let $P_1, ..., P_r$ label the paths of $\rho(H)$–sequences needed to connect elements of $ker\phi$. For each $i \in T, wx_i\rho(H)zy_i$ and so we choose and fix a path $P_{f(i)}$ such that wx_i is connected to zy_i via this path.

Define $T_k = \{i \in T : f(i) = k\}, k \in \{1, ..., r\}$, so that $T = T_1 \cup ... \cup T_r$ and for some $k \in \{1, ..., r\}, T_k \in D$. Let us fix this k. Then for any $j \in T_k$ there is a path

$$wx_j = c_1t_{1j}, d_1t_{1j} = c_2t_{2j}, ..., d_\ell t_{\ell j} = zy_j$$

where $(c_i, d_i) \in H \cup H^{-1}, 1 \leq i \leq \ell$. For $i \in \{1, ..., \ell\}$ let $\underline{t}_\ell \in S^\gamma$ have j'th co-ordinate $t_{\ell j}$ for $j \in T_k$ (and be arbitrary otherwise). Write (c_i, d_i) as $(x_{g(i)}u_i, x_{h(i)}v_i)$. This gives that

$$a \otimes \underline{\underline{x}} = a \otimes u_1\underline{\underline{t}}_1 = au_1 \otimes \underline{\underline{t}}_1 = ... = bv_\ell \otimes \underline{\underline{t}}_\ell = b \otimes v_\ell\underline{\underline{t}}_\ell = b \otimes \underline{\underline{y}}$$

and so $a \otimes \underline{\underline{x}} = b \otimes \underline{\underline{y}}$ in $aS \cup bS \otimes \mathcal{U}$. Thus $\mathcal{U}$ is weakly flat.

It is a consequence of the proof of Proposition 5.6 that if S is bounded weakly right coherent, then given an S–homomorphism $\phi : x_1S \cup ... \cup x_nS \to S$, it is true that for *any* finite set H of generators of $ker\phi$, the minimal lengths of the $\rho(H)$–sequences needed to connect pairs in $ker\phi$ is bounded.

Corollary 5.7 *If the weakly flat (flat,strongly flat) left S–sets are an axiomatisable class, then S is bounded weakly right coherent.*

Proof This follows from Theorem 5.1 and Proposition 5.6.

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