MODEL-THEORETIC PROPERTIES OF FREE, PROJECTIVE AND FLAT S-ACTS

VICTORIA GOULD, ALEXANDER MIKHALEV, EVGENY PALYUTIN, AND ALENA STEPANOVA

Abstract. This is the second in a series of articles surveying the body of work on the model theory of S-acts over a monoid S. The first concentrated on the theory of *regular* S-acts. Here we review the material on model-theoretic properties of *free, projective* and *(strongly, weakly) flat* S-acts. We consider questions of axiomatisability, completeness, model completeness and stability for these classes. Most but not all of the results have already appeared; we remark that the description of those monoids S such that the class of free left S-acts is axiomatisable, is new.

1. Introduction

The interplay between model theory and other branches of mathematics is a fruitful and fascinating area. The model theory of modules over a ring R has long been a respectable branch of both model theory and ring theory: an excellent introduction to the subject may be found in the book of Prest [19]. The model theory of acts over monoids is rather less developed but again exhibits a nice interplay between algebra and model theory, with its own distinct flavour. In an attempt to make the existing results available to wider audiences, a group of authors is engaged in writing a series of survey articles, of which this is the second.

A left S–act over monoid S is a set A upon which S acts unitarily on the left. Thus, left S–acts can be considered as a natural generalisation of left modules over rings. Certainly many questions that can be asked and answered in the model theory of modules can be asked for S-acts, but often have rather different answers. For example, there is a finite ring R such that the class of free left R-modules is not axiomatisable, whereas if S is a finite monoid then the class of free left S-acts is *always* axiomatisable. At a basic level, the major difference is that there is no underlying group structure to an S-act, so that congruences cannot in general be determined by special subsets.

A class of L-structures $\mathcal{C}$ for a first order language L is *axiomatisable* or *elementary* if there is a set of sentences Π in L such that an L-structure $\mathbf{A}$ lies in $\mathcal{C}$ if and only if every sentence in Π is true in $\mathbf{A}$. Eklof and Sabbah [ES] characterise those rings R such that the class of all flat (projective, free) left R-modules is axiomatisable (in the natural first order language associated with R-modules). What is at scrutiny here is the power of a first order language to characterise categorical notions. Naturally enough the conditions that arise are finitary conditions on the ring R. In the theory of S–acts there are three contenders to the notion of flatness, called here weakly flat, flat

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and strongly flat, for which corresponding analogues for modules over a ring coincide. We devote Sections 5 to 9 to characterising those monoids S such that the classes of weakly flat, flat, strongly flat, projective and free S-acts are axiomatisable. The material for Sections 5 to 8 is taken from the papers [10] and [2] of the first author and Bulman-Fleming, and the paper [21] of the fourth author. Section 9 contains a new result of the first author characterising those monoids such that the class of free left S-acts is axiomatisable, and some specialisations taken from [21].

A theory T in L is *complete* if for each sentence φ of L we have either $\varphi \in T$ or $\neg\varphi \in T$. This is equivalent to saying that for any models $\mathbf{A}$ and $\mathbf{B}$ of T, that is, L-structures $\mathbf{A}$ and $\mathbf{B}$ in which all sentences of T are true, we have that $\mathbf{A}$ and $\mathbf{B}$ are 'the same' in some sense; precisely, they are *elementarily equivalent*. Related notions are those of *model completeness* and *categoricity*. We can define associated notions of completeness, model completeness and categoricity for classes of L-structures. Section 11, the results of which are taken from [21], considers the question of when the classes of strongly flat, projective and free left S-acts are complete, model complete or categorical, given that they are axiomatisable.

Sections 13 and 14 investigate the crucial properties of *stability* (surveyed in Section 12) for our classes of S-acts. We remarked above that although a parallel can be drawn between some problems in the model theory of modules and that for S–acts, the theories soon diverge. This is immediately apparent when stability comes into question. Each complete theory of an R-module over a ring R is stable, whereas there are S-acts with unstable theories [17, 9, 22]. We consider questions of stability for classes of strongly flat, projective and free left S-acts. Let us say that a class $\mathcal{C}$ of structures is stable (superstable, ω–stable) if for any $\mathbf{A} \in \mathcal{C}$ the set of sentences true in $\mathbf{A}$ (which is a complete theory) is stable (superstable, ω–stable). It is proved that if the class of strongly flat left S-acts is axiomatisable and S satisfies an additional finitary condition known as Condition (A), then the class is superstable. Moreover, if the class of projective (free) left S-acts is axiomatisable, then it is superstable. Finally we consider the question of ω-stability for an axiomatisable class of strongly flat left S-acts, given that S is countable and satisfies Condition (A). Consequently, if the class of projective (free) left S–acts is axiomatisable for a countable monoid S, then it is ω–stable.

We have endeavoured to make this paper as self-contained as possible, giving all necessary definitions and results, with proofs or careful references to background results where they are not immediately available. The article should be accessible to readers with only a rudimentary knowledge of semigroup theory, acts over monoids and model theory - little more than the notions of ideal, subsemigroup, S-act, first-order language and interpretation. We devote Section 2 to an introduction to the necessary general background for monoids and acts and Section 3 to a specific discussion of how the classes of flat, projective and free S-acts arise, their properties etc. Section 4 gives further details concerning axiomatisable classes, including a discussion of ultraproducts of specific kinds. Section 10 gives the necessary background for complete, model complete and categorical theories, and as mentioned above, Section 12 contains a discussion of the notion of stability. For a more comprehensive treatment we recommend the reader to [11] (for semigroup theory), [13] (for the theory of S-acts) and [4] (for model theory).

The finitary conditions that arise from considerations of axiomatisability etc., and the monoids that satisfy them, are of interest in themselves. They are currently the subject of the PhD of L. Shaheen, a student of the first author. The only serious omission of material in this article is that we have preferred not to recall the examples presented in the papers from which much of this article is constituted, and refer the reader to the references for further information.

2. Monoids and acts

Throughout this paper S will denote a monoid with identity 1 and set of idempotents E. Maps will be written on the *left* of their arguments. We make frequent use of the five equivalences on S known as *Green's relations* $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$. For the convenience we recall here that the relation $\mathcal{R}$ is defined on S by the rule that for any $a, b \in S$,

$$a \,\mathcal{R}\, b \text{ if and only if } aS = bS.$$

Clearly, $a \,\mathcal{R}\, b$ if and only if a and b are mutual left divisors, and $\mathcal{R}$ is a left compatible equivalence relation. The relation $\mathcal{L}$ is defined dually. The meet $\mathcal{H}$ of $\mathcal{R}$ and $\mathcal{L}$ (in the lattice of equivalences on S) is given by $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. Immediately from Proposition 2.1.3 of [11], the join $\mathcal{D}$ of $\mathcal{R}$ and $\mathcal{L}$ is given by

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

The fifth relation, $\mathcal{J}$, is defined by the rule that for any $a, b \in S$, $a \,\mathcal{J}\, b$ if and only if $SaS = SbS$.

The following result is due to Green.

Theorem 2.1. *Theorem 2.2.5,* [11] *If H is an $\mathcal{H}$-class of S, then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S.*

It is standard convention to write K_a for the $\mathcal{K}$-class of an element $a \in S$, where $\mathcal{K}$ is one of Green's relations. Theorem 2.1 gives immediately that H_e is a subgroup, for any $e \in E$. The *group of units* of S is H_1; we say that S is *local* if $S \setminus H_1$ is an ideal. The following lemma is straightforward.

Lemma 2.2. *The monoid S is local if and only if*

$$R_1 = L_1 = H_1.$$

Clearly $\mathcal{D} \subseteq \mathcal{J}$; it is not true for a general monoid that $\mathcal{D} = \mathcal{J}$. However, the equality holds for finite monoids and, more generally, for epigroups. We say a monoid S is an *epigroup* (or *group bound*) if every element of S has a positive power that lies in a subgroup. The following result is well known but we include for completeness its proof.

Proposition 2.3. *Let S be a group bound monoid. Then S is local and $\mathcal{D} = \mathcal{J}$.*

Proof. Let $a, b \in S$ with $a \,\mathcal{J}\, b$. Then there exist $x, y, u, v \in S$ with $a = xby, b = uav$. We have that

$$a = xby = x(uav)y = (xu)a(vy) = (xu)^n a(vy)^n$$

for any $n \in \omega$. By hypothesis we may pick $n \geq 1$ with $(xu)^n$ lying in a subgroup. By Theorem 2.1 we have that $(xu)^n \,\mathcal{L}\, (xu)^{2n}$, whence as $S(xu)^{2n} \subseteq S(xu)^{n+1} \subseteq S(xu)^n$ we must have that $(xu)^{n+1} \,\mathcal{L}\, (xu)^n$. Since $\mathcal{L}$ is a left congruence we have that

$$a = (xu)^n a(vy)^n \,\mathcal{L}\, (xu)^{n+1} a(vy)^n = xua.$$

By the same argument as above, $a \,\mathcal{L}\, ua$; dually, $a \,\mathcal{R}\, av$. Hence

$$a \,\mathcal{L}\, ua \,\mathcal{R}\, uav = b$$

so that $a\,\mathcal{D}\,b$ and $\mathcal{D}=\mathcal{J}$ as required.

A similar argument yields that S has the 'rectangular' property, that is, if $a\,\mathcal{D}\,b\,\mathcal{D}\,ab$, then $a\,\mathcal{R}\,ab\,\mathcal{L}\,b$.

If $1\,\mathcal{D}\,a$, then as certainly $1\,a=a$, we have that $1\,\mathcal{R}\,a$; dually, $1\,\mathcal{L}\,a$. Hence $1\,\mathcal{H}\,a$ and

$$J_1=D_1=L_1=R_1=H_1.$$

□

A monoid S is *regular* if for any $a\in S$ there exists an $x\in S$ with $a=axa$. Notice that in this case, $ax,xa\in E$ and

$$ax\,\mathcal{R}\,a\,\mathcal{L}\,xa.$$

A regular monoid S is *inverse* if, in addition, $ef=fe$ for all $e,f\in E$. In an inverse monoid the idempotents form a commutative subsemigroup which we refer to as the *semilattice of idempotents* of S.

A monoid S is *right collapsible* if for any $s,t\in S$ there exists $u\in S$ such that $su=tu$.

Lemma 2.4. *Let S be a right collapsible monoid and let $a_1,\dots,a_n\in S$. Then there exists $r\in S$ with*

$$a_1r=a_2r=\ldots=a_nr.$$

Proof. Certainly the result is true for $n=1$ or $n=2$. If $a_1r=a_2r=\ldots=a_ir$ for some i with $2\leq i<n$, then pick $s\in S$ with $a_irs=a_{i+1}rs$ and note that $a_1rs=\ldots=a_irs=a_{i+1}rs$. The result follows by finite induction.

□

A submonoid T of S is *right unitary* if for any $t\in T,s\in S$, if $st\in T$, then $s\in T$. We say that a monoid has the *condition of finite right solutions*, abbreviated by *(CFRS)*, if

$$\forall s\in S\,\exists n_s\in\mathbb{N}\,\forall t\in S|\{x\in S|\,sx=t\}|\leq n_s.$$

The *starred* analogues $\mathcal{L}^*$ and $\mathcal{R}^*$ of $\mathcal{L}$ and $\mathcal{R}$ also figure significantly in this work. We recall that elements a,b of S are $\mathcal{R}^*$-related if for all $x,y\in S$,

$$xa=ya \text{ if and only if } xb=yb.$$

Clearly, $\mathcal{R}^*$ is a left congruence on S containing $\mathcal{R}$; it is not hard to see that $\mathcal{R}=\mathcal{R}^*$ if S is regular. The relation $\mathcal{L}^*$ is defined dually.

This article is concerned with model theoretic aspects of the representation theory of monoids via morphisms to monoids of self-mappings of sets. We recall that a set A is a *left S-act* if there is a map $S\times A\to A,(s,a)\mapsto sa$, such that for all $a\in A$ and $s,t\in S$,

$$1a=a \text{ and } s(ta)=(st)a.$$

Terminology for S-acts has not been consistent in the literature: they are known variously as *S-sets, S-systems, S-operands* and *S-polygons*. The definitive reference [13] uses the term S-act, as do we here.

To say that a set A is a left S-act is equivalent to there being a morphism from S to the full transformation monoid $\mathcal{T}_A$ on A. For the record, we denote the identity map on a set X by I_X, so that such a morphism must take $1\in S$ to $I_A\in\mathcal{T}_A$. An *S-morphism* from a left S-act A to a left S-act B is a map $\theta:A\to B$ such that $\theta(sa)=s\theta(a)$, for all $a\in A,s\in S$. Clearly, the class of left S-acts together with S-morphisms forms a category, **S-Act**. It is clear that in **S-Act** the coproduct of S-acts $A_i, i \in I$ is simply disjoint union, denoted $\coprod_{i\in I} A_i$. Right S-acts, and the category **Act-S**, are defined dually.

An S-*subact* of a left S-act A is a subset B of A closed under the action of S. Clearly S may be regarded as a left S-act, and any left ideal becomes an S-subact. A left S-act A is *finitely generated* if there exists $a_1, \dots, a_n \in A$ such that $A = \bigcup_{i=1}^{i=n} Sa_i$ and *cyclic* if $A = Sa$ for some $a \in A$. If $x \in X$, where X is a set disjoint from S, then the set of formal expressions $Sx = \{sx \mid s \in S\}$ becomes a cyclic left S-act in an obvious way. Notice that $Sx \cong S$. The proof of the following lemma is immediate.

Lemma 2.5. *Let A, B be a left S-acts, let $a \in A$ and $b \in B$. Then there is an S-isomorphism $\theta : Sa \to Sb$ such that $\theta(a) = b$ if and only if for all $x, y \in S$,*

$$xa = ya \text{ if and only if } xb = yb.$$

Lemma 2.5 gives in particular that if $e \in E$, then $Sa \cong Se$ under an isomorphism θ such that $\theta(a) = e$, if and only if $ea = a$ and for all $x, y \in S$, $xa = ya$ implies that $xe = ye$; such an a is said to be *right e-cancellable.*

From [8], or Lemma 2.5 above, we deduce:

Lemma 2.6. [8] *Let $a, b \in S$. The left ideals Sa and Sb are isomorphic as left S-acts, under an isomorphism θ such that $\theta(a) = b$, if and only if $a\,\mathcal{R}^*\,b$.*

A monoid S is called *right cancellative* if every element from S is right 1-cancellable. The notions of a left e-cancellable element and a left cancellative monoid are defined dually.

A *congruence* on a left S-act A is an equivalence relation ρ on A such that $(a, a') \in \rho$ implies $(sa, sa') \in \rho$ for $a, a' \in A$, $s \in S$. To prevent ambiguity, a congruence on S *regarded as a left S-act* will be referred to as a *left congruence*. If ρ is a congruence on A we shall also write $a\,\rho\,a'$ for $(a, a') \in \rho$ and a/ρ for the ρ-class of $a \in A$. If $X \subseteq A \times A$ then by $\rho(X)$ we denote the smallest congruence on A containing X.

Proposition 2.7. [13] *Let A be an S–act, $X \subseteq A \times A$ and $\rho = \rho(X)$. Then for any $a, b \in A$, one has $a\,\rho\,b$ if and only if either $a = b$ or there exist $p_1, \dots, p_n, q_1, \dots, q_n \in A, s_1, \dots, s_n \in S$ such that $(p_i, q_i) \in X$ or $(q_i, p_i) \in X$ for any i, $1 \le i \le n$, and*

$$a = s_1p_1,\ s_1q_1 = s_2p_2,\ \dots\ , s_nq_n = b.$$

Elements x, y of a left S–act A are *connected* (denoted by $x \sim y$) if there exist $n \in \omega$, $a_0, \dots, a_n \in A$, $s_1, \dots, s_n \in S$ such that $x = a_0$, $y = a_n$, and $a_i = s_ia_{i-1}$ or $a_{i-1} = s_ia_i$ for any i, $1 \le i \le n$. An S-subact B of a left S–act A is a *connected* if we have $x \sim y$ for any $x, y \in B$. It is easy to check that $\sim$ is a congruence relation on a left S–act A, with classes that are S-subacts, and maximal connected components of A. Thus, A is a coproduct of connected S-subacts.

3. Free, Projective and Flat Acts

For the convenience of the reader we discuss in this section the classes of free, projectives and flat left S-acts; as explained below, there are several candidates for the notion of a flatness. What we give is a skeleton survey; further details may be found in [13].

We remind the reader that a left S-act F is *free* (on a subset X, in **S-Act**) if and only if for any left S-act A and map $\theta : X \to A$, there is a unique S-morphism $\overline{\theta} : F \to A$ such that $\overline{\theta}\iota = \theta$, where $\iota : X \to F$ is the inclusion mapping.

Theorem 3.1. [13] *A left S-act F is free on X if and only if $F \cong \coprod_{x\in X} Sx$.*

Notice from the above that a free left S-act is isomorphic to a coproduct of copies of the left S-act S. From the remark following Lemma 2.5 we have our next corollary.

Corollary 3.2. *A cyclic left S-act A is free if and only if $A = Sa$ for some right 1-cancellable $a \in A$.*

A left S-act P is *projective* if given any diagram of left S-acts and S-morphisms

$$\begin{tikzcd} & P \arrow[d, "\theta"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

where $\phi : M \to N$ is onto, there exists an S-morphism $\psi : P \to M$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "\psi"'] \arrow[d, "\theta"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

is commutative.

It is clear that a free left S-act F is projective; this is also an immediate consequence of Theorem 3.1 and Theorem 3.3 below.

Theorem 3.3. [15, 5] *A left S-act P is projective if and only if $P \cong \coprod_{i\in I} Se_i$, where $e_i \in E$ for all $i \in I \neq \emptyset$.*

Corollary 3.4. *A cyclic S-set A is projective if and only if $A = Sa$ for some right e-cancellable $a \in S$, for some $e \in E$.*

To define classes of flat left S-acts we need the notion of *tensor product* of S-acts. If A is a *right* S-act and B a left S-act then the tensor product of A and B, written $A \otimes B$, is the set $A \times B$ factored by the equivalence generated by $\{((as, b), (a, sb)) : a \in A, b \in B, s \in S\}$. For $a \in A$ and $b \in B$ we write $a \otimes b$ for the equivalence class of (a, b).

For a left S-act B, the map $- \otimes B$ is a functor from the category **Act–S** to the category **Set** of sets. It is from this functor that the various notion of *flatness* are derived.

A left S-act B is *weakly flat* if the functor $-\otimes B$ preserves embeddings of right ideals of S into S, *flat* if it preserves arbitrary embeddings of right S-acts, and *strongly flat* if it preserves pullbacks (equivalently, equalisers and pullbacks [1]). We give the reader a warning that terminology has changed over the years; in particular, left S-acts B such that $-\otimes B$ preserves equalisers and pullbacks are called weakly flat in [7] and [20], flat in [10], and pullback flat in [13].

Stenström was instrumental in forwarding the theory of flat acts, by producing interpolation conditions, later labelled (P) and (E), that are together equivalent to strong flatness.

Theorem 3.5. [20] *A left S-act B is strongly flat if and only if B satisfies conditions (P) and (E):*

(P): if $x, y \in B$ and $s, t \in S$ with $sx = ty$, then there is an element $z \in B$ and elements $s', t' \in S$ such that $x = s'z, y = t'z$ and $ss' = tt'$;

(E): if $x \in B$ and $s, t \in S$ with $sx = tx$, then there is an element $z \in B$ and $s' \in S$ with $x = s'z$ and $ss' = ts'$.

The classes of free, projective, strongly flat, flat and weakly flat left S-acts will be denoted by $\mathcal{F}r, \mathcal{P}, \mathcal{SF}, \mathcal{F}$ and $\mathcal{WF}$ respectively. We remark here that 'elementary' descriptions (that is, not involving arrows) of $\mathcal{F}$ and $\mathcal{WF}$, along the lines of Theorems 3.1, 3.3 and 3.5 for $\mathcal{F}r, \mathcal{P}$ and $\mathcal{SF}$, are not available. From those results it is immediate that projective left S-acts are strongly flat. Clearly flat left S-acts are weakly flat and since embeddings are equalisers in **Act–S**, strongly flat left S-acts are flat. Thus

$$\mathcal{F}r \subseteq \mathcal{P} \subseteq \mathcal{SF} \subseteq \mathcal{F} \subseteq \mathcal{WF}.$$

A congruence θ on a left S-act A is called *strongly flat* if $A/\theta \in \mathcal{SF}$. For the purposes of later sections we state a straightforward consequence of Theorem 3.5.

Corollary 3.6. [3] *A congruence θ on the left S-act S is strongly flat if and only if for any $u, v \in S$, $u \,\theta\, v$ if and only if there exists $s \in S$ such that $s \,\theta\, 1$ and $us = vs$.*

4. Axiomatisability

Any class of algebras $\mathcal{A}$ has an associated first order language L. One can then ask whether a property P, defined for members of $\mathcal{A}$, is expressible in the language L. In other words, is there a set of sentences Π in L such that a member A of $\mathcal{A}$ has property P if and only if all the sentences of Π are true in A, that is, if and only if A is a model of Π, which we denote by $A \vDash \Pi$. If Π exists we say that $\mathcal{B}$ is *axiomatisable*, where $\mathcal{B}$ is the subclass of $\mathcal{A}$ whose members have property P. Questions of axiomatisability are the first step in investigating the *model theory* of a class of algebras.

We concentrate in this article on aspects of the first-order theory of left S-acts, the next five sections considering questions of axiomatisability of the classes of free, projective and (strongly, weakly) flat acts. Our language is the first order language with equality L_S which has no constant or relation symbols and which has a unary function symbol λ_s for each $s \in S$: we write sx for $\lambda_s(x)$. The class of left S-acts is axiomatised by the set of sentences Π where

$$\Pi = \{(\forall x)(1x = x)\} \cup \{\mu_{s,t} : s, t \in S\}$$

where $\mu_{s,t}$ is the sentence

$$(\forall x)((st)x = s(tx)).$$

Certain classes of left S-acts are axiomatisable for *any* monoid S. For example the torsion free left S-acts are axiomatised by $\Pi \cup \Sigma_{\mathcal{TF}r}$ where

$$\Sigma_{\mathcal{TF}r} = \{(\forall x)(\forall y)(sx = sy \to x = y) : s \in T\}$$

where T is the set of left cancellable elements of S. Indeed in the context of S-acts the sentences of Π are understood, and we say more succinctly that $\Sigma_{\mathcal{TF}r}$ axiomatises $\mathcal{TF}r$. Other natural classes of left S-acts are axiomatisable for some monoids and not for others.

One of our main tools throughout will be that of an ultraproduct, which we now briefly recall.

For any set A we denote the set of all subsets of A by $P(A)$. A family $C \subseteq P(A)$ is *centred* if for any $X_1, \dots, X_n \in C$ the intersection $X_1 \cap \dots \cap X_n$ is not empty. We remark that if C is centred, then $\emptyset \notin C$. A non-empty family $F \subseteq P(A)$ is called a *filter* on A if the follow conditions are true:

(a) $\emptyset \notin F$;

(b) if $U, V \in F$ then $U \cap V \in F$;

(c) if $U \in F$ and $U \subseteq X \subseteq A$, then $X \in F$.

A filter F on a set A is said to be *uniform* if $|X| = |A|$ for any $X \in F$ and an *ultrafilter* if $X \in F$ or $A \setminus X \in F$ for any $X \subseteq A$. The following facts concerning these concepts follow easily from the definitions.

(1) A filter F on a set A is a centred set and $A \in F$.

(2) If F is an ultrafilter on A and $U_0 \cup \dots \cup U_n \in F$ then $U_i \in F$ for some $i \leqslant n$.

(3) If A is an infinite set then the set

$$\Phi_A = \{X \mid X \subseteq A,\ |A \setminus X| < |A|\}$$

is a filter (it is called the Fréchet filter).

(4) An ultrafilter F on an infinite set A is uniform if and only if F contains the filter Φ_A.

We now argue, that (with the use of Zorn's Lemma), every centred family can be extended to an ultrafilter. Hence, there are ultrafilters on every non-empty set.

Proposition 4.1. *Let A be a non-empty set. Then:*

(1) a centred family $C \subseteq P(A)$ extends to a maximal centred family D;

(2) a centred family is an ultrafilter if and only if it is a maximal centred family;

(3) any centred family $C \subseteq P(A)$ is contained in some ultrafilter F on A;

(4) a filter is an ultrafilter if and only if it is a maximal filter;

(5) if A is infinite then there is a uniform ultrafilter on A.

Proof. (1) Let S be the set of all centered families of subsets of A containing the family C. It is clear that the union of an ascending chain of centered families is a centered family. Thus the poset $\langle S, \subseteq\rangle$ satisfies the maximal condition required to invoke Zorn's Lemma. We deduce that there exists a centered family D which is maximal in $\langle S, \subseteq\rangle$.

(2) Let C be a maximal centred family. We have observed that $\emptyset \notin C$, and clearly C is closed under intersection. If $X \in C$ and $X \subseteq Y \subseteq A$, then clearly $C \cup \{Y\}$ is centred. By maximality of C we deduce that $Y \in C$ and so C is a filter. If $X \notin C$, then $C \cup \{X\}$ is not centred, so that $X \cap Y = \emptyset$ for some $Y \in C$. Hence $Y \subseteq A \setminus X$ so that $A \setminus X \in C$ since C is a filter. We deduce that C is an ultrafilter.

Conversely, any ultrafilter F is centred, and if $F \subset G$ for a centred family G, then taking $X \in G \setminus F$, we have that $X, A \setminus X \in G$, contradicting the fact that G is centred. Hence F is maximal among centred families.

(3) This follows immediately from (1) and (2).

(4) Clearly, an ultrafilter is a maximal filter. On the other hand, if F is a maximal filter, then by (1) F can be extended to a maximal centred family G. But G is an ultrafilter by (2) so that $F = G$ by the maximality of F.

(5) This follows from (3) and remark (4) above. $\square$

We now give the construction of an ultraproduct of left S-acts; we could, of course, define ultraproducts for any class of interpretations or structures of a given first order language L, that is, any class of L-*structures*, but we prefer to be specific and leave the reader to extrapolate. A point of notation: for any arbitrary language L we make a distinction between an L-structure $\mathbf{A}$ and the underlying set A of $\mathbf{A}$, whereas for S-acts we do not.

If $B = \prod_{i\in I} B_i$ is a product of left S-acts B_i, $i \in I$, and Φ is an ultrafilter on I, then we define a relation $\equiv_\Phi$ on B by the rule that

$$(a_i) \equiv_\Phi (b_i) \text{ if and only if } \{i \in I : a_i = b_i\} \in \Phi.$$

It is a fact that $\equiv_\Phi$ is an equivalence and moreover an S-act congruence, so that putting

$$U = (\prod_{i\in I} B_i)/\Phi = (\prod_{i\in I} B_i)/\equiv_\Phi$$

and denoting the $\equiv_\Phi$-class of $f \in \prod\limits_{i\in I} B_i$ by f/Φ, U is an S-act under the operation

$$s\,(a_i)/\Phi = (sa_i)/\Phi,$$

which we call the *ultraproduct* of left S-acts B_i, $i \in I$, under the ultrafilter Φ.

Ultraproducts are of central importance to us, due to the celebrated theorem of Łos.

Theorem 4.2. [4] *Let L be a first order language and $\mathcal{A}$ a class of L-structures. If $\mathcal{A}$ is axiomatisable, then $\mathcal{A}$ is closed under the formation of ultraproducts.*

Let $\varkappa$ be an infinite cardinal; thus $\varkappa$ is a limit ordinal and we may regard $\varkappa$ as the union of all smaller ordinals. A filter F is called *$\varkappa$-regular* if there exists a family $R = \{S_\alpha \mid \alpha \in \varkappa\}$ of distinct elements of F such that any intersection of any infinite subset of the family R is empty.

Proposition 4.3. *For any infinite cardinal $\varkappa$ there exists an $\varkappa$-regular ultrafilter on set I, where $|I| = \varkappa$.*

Proof. According to Proposition 4.1 it is enough to construct a set I, $|I| = \varkappa$, and a centred family $F = \{S_\alpha \mid \alpha \in \varkappa\}$ of distinct subsets of I such that the intersection of any infinite subset of F is empty.

Consider the set $I = \{v \mid v \subseteq \varkappa, v \text{ is finite}\}$. Clearly the cardinality of I is equal to $\varkappa$. Let $S_\alpha = \{v \mid v \in I, \alpha \in v\}$ and $R = \{S_\alpha \mid \alpha \in \varkappa\}$. If $S_{\alpha_1}, \ldots, S_{\alpha_n} \in R$ then

$$\{\alpha_1, \ldots, \alpha_n\} \in S_{\alpha_1} \cap \ldots \cap S_{\alpha_n}.$$

Therefore the family R is centred and the intersection of any infinite collection of its elements is empty. $\square$

We require one further technical result concerning ultraproducts that will be needed for later sections.

Theorem 4.4. *Let F be an $\varkappa$-regular filter on a set I, $|I| = \varkappa$, and let A_i ($i \in I$) be S-acts of infinite cardinality λ. Then the cardinality of the filtered product $(\prod_{i\in I} A_i)/F$ is equal to $\lambda^\varkappa$.*

Proof. Let us denote the set of all finite sequences of elements of the cardinal λ by $\lambda^{<\omega}$. It is clear that the cardinality of the set $\lambda^{<\omega}$ coincides with λ. Since the cardinality of the filtered products does not depend on the fact that the A_i's are S-acts, but merely on their cardinality, we can suppose that $A_i = \lambda^{<\omega}$ for all $i \in I$. Since $|(\lambda^{<\omega})^I| = \lambda^\varkappa$ then $|(\lambda^{<\omega})^I/F| \leqslant \lambda^\varkappa$. Thus it is enough to construct an embedding ϕ of the set $\lambda^\varkappa = \{f \mid f : \varkappa \to \lambda\}$ in the set $(\lambda^{<\omega})^I/F$.

Let the family $R = \{S_\alpha \mid \alpha \in \varkappa\}$ of elements from the filter F satisfy the required condition from the definition of $\varkappa$-regular filter, i.e. any intersection of an infinite subset of the family R is empty; clearly, for any $i \in I$, we must have that $\{\alpha : i \in S_\alpha\}$ is finite. We may add the set $I \setminus \bigcup R$ to the set S_0, so we can consider that $I = \bigcup R$. Let $f : \varkappa \to \lambda$ be an arbitrary map. Define the map $f^* : I \to \lambda^{<\omega}$ in the following way: $f^*(i) = \langle f(\alpha_1), \ldots, f(\alpha_n)\rangle$, where $i \in I$, $\{\alpha_1, \ldots, \alpha_n\} = \{\alpha \mid i \in S_\alpha\}$ and $\alpha_i < \alpha_{i+1}$ $(1 \leq i \leq n-1)$. We consider $\phi(f) = f^*/F$. Let $f_1, f_2 \in \lambda^\varkappa$ with $f_1 \neq f_2$, so that $f_1(\alpha) \neq f_2(\alpha)$ for some $\alpha \in \varkappa$. Then $f_1^*(i) \neq f_2^*(i)$ for any $i \in S_\alpha$; since $S_\alpha \in F$ we have that $f_1^*/F \neq f_2^*/F$ and so ϕ is one-one as required.

□

5. Axiomatisability of $\mathcal{WF}$

We begin our considerations of axiomatisability with the class $\mathcal{WF}$; the results of this section are all taken from [2].

At this point it is useful to give further details on tensor products.

Lemma 5.1. [13] *Let A be a right S-act and B a left S-act. Then for $a, a' \in A$ and $b, b' \in B$, $a \otimes b = a' \otimes b'$ if and only if there exist $s_1, t_1, s_2, t_2, \ldots, s_m, t_m \in S$, $a_2, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$ such that*

$$\begin{array}{rclrcl} & & & b & = & s_1 b_1 \\ a s_1 & = & a_2 t_1 & t_1 b_1 & = & s_2 b_2 \\ a_2 s_2 & = & a_3 t_2 & t_2 b_2 & = & s_3 b_3 \\ & \vdots & & & \vdots & \\ a_m s_m & = & a' t_m & t_m b_m & = & b'. \end{array}$$

The sequence presented in Lemma 5.1 will be called a *tossing* (or scheme) $\mathcal{T}$ of length m over A and B connecting (a, b) to (a', b'). The *skeleton* of $\mathcal{T}$, $\mathcal{S} = \mathcal{S}(\mathcal{T})$, is the sequence

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m) \in S^{2m}.$$

The set of all skeletons is denoted by $\mathbb{S}$. By considering trivial acts it is easy to see that $\mathbb{S}$ consists of all sequences of elements of S of even length.

Let $a, a' \in S$ and let $\mathcal{S} = (s_1, t_1, ..., s_m, t_m)$ be a skeleton of length m. We say that the triple $(a, \mathcal{S}, a')$ is *realised* if there are elements $a_2, a_3, \ldots, a_m \in S$ such that

$$\begin{array}{rcl} a s_1 & = & a_2 t_1 \\ a_2 s_2 & = & a_3 t_2 \\ & \vdots & \\ a_m s_m & = & a' t_m. \end{array}$$

We let $\mathbb{T}$ denote the set of realised triples.

A realised triple $(a, \mathcal{S}, a')$ is *witnessed by* $b, b_1, \ldots, b_m, b'$ where $b, b_1, \ldots,$ b_m, b' are elements of a left S-act B if

$$\begin{array}{rcl} b & = & s_1 b_1 \\ t_1 b_1 & = & s_2 b_2 \\ & \vdots & \\ t_m b_m & = & b'. \end{array}$$

We know that if $a, a' \in A$ and $b, b' \in B$, where A is a right S-act and B a left S-act, then $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exists a tossing $\mathcal{T}$ from (a, b) to (a', b') over A and B, with skeleton $\mathcal{S}$, say. If the equality $a \otimes b = a' \otimes b'$ holds also in $(aS \cup a'S) \otimes B$, and is determined by some tossing $\mathcal{T}'$ from (a, b) to (a', b') over $aS \cup a'S$

and B with skeleton $\mathcal{S}' = \mathcal{S}(\mathcal{T}')$ then we say that $\mathcal{T}'$ is a *replacement tossing* for $\mathcal{T}$, $\mathcal{S}'$ is a *replacement skeleton for* $\mathcal{S}$ and (in case $A = S$) triples $(a, \mathcal{S}', a')$ will be called *replacement triples* for $(a, \mathcal{S}, a')$.

Note that, for any left S-act B, right S-act A and $(a, b), (a', b') \in A \times B$, if $\mathcal{S} = (s_1, t_1, ..., s_m, t_m)$ is the skeleton of a tossing from (a, b) to (a', b') over A and B, then $\gamma_{\mathcal{S}}(b, b')$ holds in B, where $\gamma_{\mathcal{S}}$ is the formula

$$\gamma_{\mathcal{S}}(y, y') \leftrightharpoons (\exists y_1)(\exists y_2) \cdots (\exists y_m)(y = s_1 y_1 \wedge t_1 y_1 = s_2 y_2 \wedge \cdots \wedge t_m y_m = y').$$

For any $\mathcal{S} \in \mathbb{S}$ we define $\psi_{\mathcal{S}}$ to be the sentence

$$\psi_{\mathcal{S}} \leftrightharpoons (\forall y)(\forall y') \neg \gamma_{\mathcal{S}}(y, y').$$

Theorem 5.2. [2] *The following conditions are equivalent for a monoid S:*

(1) the class $\mathcal{WF}$ is axiomatisable;

(2) the class $\mathcal{WF}$ is closed under ultraproducts;

(3) for every skeleton $\mathcal{S}$ over S and $a, a' \in S$ there exist finitely many skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\alpha(a,\mathcal{S},a')}$ over S, such that for any weakly flat left S-act B, if $(a, b), (a', b') \in S \times B$ are connected by a tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1 \ldots, \alpha(a, \mathcal{S}, a')\}$.

Proof. That (1) implies (2) follows from Theorem 4.2 (Łos's theorem).

Suppose now that (2) holds but that (3) is false. Let J denote the set of all finite subsets of $\mathbb{S}$ and suppose that, for some skeleton $\mathcal{S}_0 = (s_1, t_1, \ldots, s_m, t_m) \in \mathbb{S}$ and $a, a' \in S$, for every $f \in J$, there is a weakly flat left S-act B_f and $b_f, b'_f \in B_f$ with the pairs $(a, b_f), (a', b'_f) \in S \times B_f$ connected by a tossing $\mathcal{T}_f$ with skeleton $\mathcal{S}_0$, but such that no tossing over $aS \cup a'S$ and B_f connecting (a, b_f) and (a', b'_f) has a skeleton belonging to the set f.

For each $\mathcal{S} \in \mathbb{S}$ let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$. Each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty (because $\mathbb{S}$ is infinite), so there exists an ultrafilter Φ over J such that each $J_{\mathcal{S}}$ ($\mathcal{S} \in \mathbb{S}$) belongs to Φ. Notice that $a \otimes \underline{b} = a' \otimes \underline{b'}$ in $S \otimes B$ where $B = \prod_{f \in J} B_f$, $\underline{b}(f) = b_f$ and $\underline{b'}(f) = b'_f$, and that this equality is determined by a tossing over S and B having skeleton $\mathcal{S}_0$. It follows that the equality $a \otimes (\underline{b}/\Phi) = a' \otimes (\underline{b'}/\Phi)$ holds also in $S \otimes U$, where $U = (\prod_{f \in J} B_f)/\Phi$, and is also determined by a tossing over S and U with skeleton $\mathcal{S}_0$.

By our assumption, U is weakly flat, so that $(a, \underline{b}/\Phi)$ and $(a', \underline{b'}/\Phi)$ are connected via a replacement tossing $\mathcal{T}'$ over $aS \cup a'S$ and U, say

$$\begin{array}{rclrcl} & & & \underline{b}/\Phi & = & u_1 \underline{d_1}/\Phi \\ a u_1 & = & c_2 v_1 & v_1 \underline{d_1}/\Phi & = & u_2 \underline{d_2}/\Phi \\ c_2 u_2 & = & c_3 v_2 & v_2 \underline{d_2}/\Phi & = & u_3 \underline{d_3}/\Phi \\ \vdots & & & & \vdots & \\ c_n u_n & = & a' v_n & v_n \underline{d_n}/\Phi & = & \underline{b'}/\Phi, \end{array}$$

where $\underline{d_i}(f) = d_{i,f}$ for any $f \in J$ and $i \in \{1, \ldots, n\}$. We put $\mathcal{S}' = \mathcal{S}(\mathcal{T}')$.

As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$\begin{array}{rclrcl} & & & b_f & = & u_1 d_{1,f} \\ au_1 & = & c_2 v_1 & v_1 d_{1,f} & = & u_2 d_{2,f} \\ c_2 u_2 & = & c_3 v_2 & v_2 d_{2,f} & = & u_3 d_{3,f} \\ & \vdots & & & \vdots & \\ c_n u_n & = & a' v_n & v_n d_{n,f} & = & b'_f \end{array}$$

whenever $f \in D$. Now, suppose $f \in D \cap J_{\mathcal{S}'}$. Then, from the tossing just considered, we see that $\mathcal{S}'$ is a replacement skeleton for skeleton $\mathcal{S}_0$, connecting (a, b_f) and (a', b'_f) over $aS \cup a'S$ and B_f. But $\mathcal{S}' \in f$, contradicting the choice of (a, b_f) and (a', b'_f). Thus (3) holds.

Finally, suppose that (3) holds. We introduce a sentence corresponding to each element of $\mathbb{T}$ in such a way that the resulting set of sentences axiomatises the class $\mathcal{WF}$.

We let $\mathbb{T}_1$ be the set of realised triples that are not witnessed in any weakly flat left S-act B, and put $\mathbb{T}_2 = \mathbb{T} \setminus \mathbb{T}_1$. For $T = (a, \mathcal{S}, a') \in \mathbb{T}_1$ we let ψ_T be the sentence $\psi_{\mathcal{S}}$ defined before the statement of this theorem. If $T = (a, \mathcal{S}, a') \in \mathbb{T}_2$, then $\mathcal{S}$ is the skeleton of some scheme joining (a, b) to (a', b') over S and some weakly flat left S-act B. By our assumption (3), there is a finite list of replacement skeletons $\mathcal{S}_1, ..., \mathcal{S}_{\alpha(T)}$. Then, for each $k \in \{1, ..., \alpha(T)\}$, if $\mathcal{S}_k = (u_1, v_1, ..., u_h, v_h)$, there exist a weakly flat left S-act C_k elements $c, c', c_1, ..., c_h \in C_k$, and elements $q_2, ..., q_h \in aS \cup a'S$ such that

$$\text{(i)} \qquad \begin{array}{rclrcl} & & & c & = & u_1 c_1 \\ au_1 & = & q_2 v_1 & v_1 c_1 & = & u_2 c_2 \\ q_2 u_2 & = & q_3 v_2 & v_2 c_2 & = & u_3 c_3 \\ & \vdots & & & \vdots & \\ q_h u_h & = & a' v_h & v_h c_h & = & c'. \end{array}$$

For each k, we fix such a list $q_2, ..., q_h$ of elements, for future reference, and let φ_T be the sentence

$$\varphi_T \leftrightharpoons (\forall y)(\forall y')(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \cdots \vee \gamma_{\mathcal{S}_{\alpha(T)}}(y, y')).$$

Let

$$\Sigma_{\mathcal{WF}} = \{\psi_T : T \in \mathbb{T}_1\} \cup \{\varphi_T : T \in \mathbb{T}_2\}.$$

We claim that $\Sigma_{\mathcal{WF}}$ axiomatises $\mathcal{WF}$.

Suppose first that D is any weakly flat left S-act. Let $T \in \mathbb{T}_1$. Then $T = (a, \mathcal{S}, a')$ is a realised triple. Since T is not witnessed in *any* weakly flat left S-act, T is certainly not witnessed in D so that $D \models \psi_T$.

On the other hand, for $T = (a, \mathcal{S}, a') \in \mathbb{T}_2$, where $\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$, if $d, d' \in D$ are such that $\gamma_{\mathcal{S}}(d, d')$ is true, then there are elements $d_1, \ldots d_m \in D$ such that

$$\begin{array}{rcl} d & = & s_1 d_1 \\ t_1 d_1 & = & s_2 d_2 \\ & \vdots & \\ t_m d_m & = & d,' \end{array}$$

which together with the fact that T is a realised triple, gives that (a, d) is connected to (a', d') over S and D via a tossing with skeleton $\mathcal{S}$. Because D is weakly flat, (a, d) and (a', d') are connected over $aS \cup a'S$ and D, and by assumption (3), we can take the tossing to have skeleton one of $\mathcal{S}_1, \ldots, \mathcal{S}_{\alpha(T)}$, say $\mathcal{S}_k$. Thus $D \models \gamma_{\mathcal{S}_k}(d, d')$ and it follows that $D \models \varphi_T$. Hence D is a model of $\Sigma_{\mathcal{WF}}$.

Conversely, we show that every model of $\Sigma_{\mathcal{WF}}$ is weakly flat. Let $C \models \Sigma_{\mathcal{WF}}$ and suppose that $a, a' \in S, c, c' \in C$ and we have a tossing

$$\begin{array}{rclrcl} & & & c & = & s_1c_1 \\ as_1 & = & a_2t_1 & t_1c_1 & = & s_2c_2 \\ a_2s_2 & = & a_3t_2 & t_2c_2 & = & s_3c_3 \\ & \vdots & & & \vdots & \\ a_ms_m & = & a't_m & t_mc_m & = & c' \end{array}$$

with skeleton $\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$ over S and C. Then the triple $T = (a, \mathcal{S}, a')$ is realised, so that $T \in \mathbb{T}$. Since $\gamma_{\mathcal{S}}(c, c')$ holds, C cannot be a model of ψ_T. Since $C \models \Sigma_{\mathcal{WF}}$ it follows that $T \in \mathbb{T}_2$. But then φ_T holds in C so that for some $k \in \{1 \ldots, \alpha(T)\}$ we have that $\gamma_{\mathcal{S}_k}(c, c')$ is true. Together with the equalities on the left hand side of (i) we have a tossing over $aS \cup a'S$ and C connecting (a, c) to (a', c'). Thus C is weakly flat. $\square$

6. Axiomatisability of $\mathcal{F}$

We now turn our attention to the class $\mathcal{F}$ of flat left S-acts. The results of this section are again taken from [2]. First we consider the *finitely presented flatness lemma* of [2], which is crucial to our arguments.

Let $\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m) \in \mathbb{S}$ be a skeleton. We let F^{m+1} be the free right S-act

$$xS \amalg x_2S \amalg ... \amalg x_mS \amalg x'S$$

and let $\rho_{\mathcal{S}}$ be the congruence on F^{m+1} generated by

$$\{(xs_1, x_2t_1), (x_2s_2, x_3t_2), ..., (x_{m-1}s_{m-1}, x_mt_{m-1}), (x_ms_m, x't_m)\}.$$

We denote the $\rho_{\mathcal{S}}$-class of $w \in F^{m+1}$ by $[w]$. If B is a left S-act and $b, b_1, \ldots, b_m, b' \in B$ are such that

$$b = s_1b_1, t_1b_1 = s_2b_2, \ldots, t_mb_m = b'$$

then the tossing

$$\begin{array}{rclrcl} & & & b & = & s_1b_1 \\ [x]\, s_1 & = & [x_2]\, t_1 & t_1b_1 & = & s_2b_2 \\ [x_2]\, s_2 & = & [x_3]\, t_2 & t_2b_2 & = & s_3b_3 \\ & \vdots & & & \vdots & \\ [x_{m-1}]\, s_{m-1} & = & [x_m]\, t_{m-1} & t_{m-1}b_{m-1} & = & s_mb_m \\ [x_m]\, s_m & = & [x']\, t_m & t_mb_m & = & b'. \end{array}$$

over $F^{m+1}/\rho_{\mathcal{S}}$ and B is called a *standard tossing* with skeleton $\mathcal{S}$ connecting $([x], b)$ to $([x'], b')$.

We refer the reader to [2] for the proof of the following lemma.

Lemma 6.1. [2] *The following are equivalent for a left S-act B:*

(1) B is flat;

(2) $-\otimes B$ preserves all embeddings of A in C, where A is a finitely generated subact of a finitely presented right S-act C;

(3) $-\otimes B$ preserves the embedding of $[x]\, S \cup [x']\, S$ into $F^{m+1}/\rho_{\mathcal{S}}$, for all skeletons $\mathcal{S}$;

(4) if $([x], b)$ *and* $([x'], b')$ *are connected by a standard tossing over* $F^{m+1}/\rho_{\mathcal{S}}$ *and* B *with skeleton* $\mathcal{S}$*, then they are connected by a tossing over* $[x]\, S \cup [x']\, S$ *and* B*.*

The construction of $F^{m+1}/\rho_{\mathcal{S}}$ enables us to observe that for any left S-act B and any $b, b' \in B$, a skeleton $\mathcal{S} = (s_1, t_1, ..., s_m, t_m)$ is the skeleton of a tossing from (a, b) to (a', b') over A and B for some A and some $a, a' \in A$ if and only if $\gamma_{\mathcal{S}}(b, b')$ holds in B, where $\gamma_{\mathcal{S}}$ is the sentence defined before Theorem 5.2. We also note that if $(a, b), (a', b')$ are connected via a tossing with skeleton $\mathcal{S}$, then $([x], b), ([x'], b) \in F^{m+1}/\rho_{\mathcal{S}}$ are connected via the standard tossing with skeleton $\mathcal{S}$.

Theorem 6.2. [2] *The following conditions are equivalent for a monoid* S*:*

(1) the class $\mathcal{F}$ *is axiomatisable;*

(2) the class $\mathcal{F}$ *is closed under formation of ultraproducts;*

(3) for every skeleton $\mathcal{S}$ *over* S *there exist finitely many replacement skeletons* $\mathcal{S}_1, ..., \mathcal{S}_{\alpha(\mathcal{S})}$ *over* S *such that, for any right* S*-act* A *and any flat act left* S*-act* B*, if* $(a, b), (a', b') \in A \times B$ *are connected by a tossing* $\mathcal{T}$ *over* A *and* B *with* $\mathcal{S}(\mathcal{T}) = \mathcal{S}$*, then* (a, b) *and* (a', b') *are connected by a tossing* $\mathcal{T}'$ *over* $aS \cup a'S$ *and* B *such that* $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$*, for some* $k \in \{1, ..., \alpha(\mathcal{S})\}$*.*

Proof. The implication (1) implies (2) is clear from Theorem 4.2.

The proof of (2) implies (3) follows the pattern set by that of Theorem 5.2, in particular, J, the sets $J_{\mathcal{S}}$ for $\mathcal{S} \in \mathbb{S}$ and the ultrafilter Φ are defined as in that theorem.

Suppose that $\mathcal{F}$ is closed under formation of ultraproducts, but that assertion (3) is false. Let J denote the family of all finite subsets of $\mathbb{S}$. Suppose $\mathcal{S}_0 = (s_1, t_1, ..., s_m, t_m) \in \mathbb{S}$ is such that, for every $f \in J$, there exist a right S-act A_f, a flat left S-act B_f, and pairs (a_f, b_f), $(a'_f, b'_f) \in A_f \times B_f$ such that (a_f, b_f) and (a'_f, b'_f) are connected over A_f and B_f by a tossing $\mathcal{T}_f$ with skeleton $\mathcal{S}_0$, but no replacement tossing over $a_f S \cup a'_f S$ and B_f connecting (a_f, b_f) and (a'_f, b'_f) has a skeleton belonging to the set f. Note that $\underline{a} \otimes \underline{b} = \underline{a}' \otimes \underline{b}'$ in $A \otimes B$, where for each $f \in J$, $\underline{a}(f) = a_f, \underline{b}(f) = b_f$, $A = \prod_{f \in J} A_f$ and $B = \prod_{f \in J} B_f$, and that this equality is determined by a tossing over A and B (the "product" of the $\mathcal{T}_f$'s) having skeleton $\mathcal{S}_0$. If follows that the equality $\underline{a} \otimes (\underline{b}/\Phi) = \underline{a}' \otimes (\underline{b}'/\Phi)$ holds also in $A \otimes U$, where $U = (\prod_{f \in J} B_f)/\Phi$, and is determined by a tossing over A and U with skeleton $\mathcal{S}_0$. Because U is flat, $\mathcal{S}_0$ has a replacement skeleton $\mathcal{S}(\mathcal{T}') = \mathcal{S}' = (u_1, v_1, ..., u_n, v_n)$, where $\mathcal{T}'$ is a tossing

$$\begin{array}{rclrcl} & & & \underline{b}/\Phi & = & u_1 \underline{d_1}/\Phi \\ \underline{a} u_1 & = & \underline{c_2} v_1 & v_1 \underline{d_1}/\Phi & = & u_2 \underline{d_2}/\Phi \\ \underline{c_2} u_2 & = & \underline{c_3} v_2 & v_2 \underline{d_2}/\Phi & = & u_3 \underline{d_3}/\Phi \\ & \vdots & & & \vdots & \\ \underline{c_n} u_n & = & \underline{a}' v_n & v_n \underline{d_n}/\Phi & = & \underline{b}'/\Phi \end{array}$$

where for $2 \leq j \leq n$ and $f \in J$ we have $\underline{c_j}(f) = c_{j,f} \in a_f S \cup a'_f S$ and for $1 \leq j \leq n$ and $f \in J$ we have $\underline{d_j}(f) = d_{j,f} \in B_f$. As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$\begin{array}{rclrcl} & & & b_f & = & u_1 d_{1,f} \\ a_f u_1 & = & c_{2,f} v_1 & v_1 d_{1,f} & = & u_2 d_{2,f} \\ c_{2,f} u_2 & = & c_{3,f} v_2 & v_2 d_{2,f} & = & u_3 d_{3,f} \\ & \vdots & & & \vdots & \\ c_{n,f} u_n & = & a'_f v_n & v_n d_{n,f} & = & b'_f \end{array}$$

whenever $f \in D$. Now, suppose $f \in D \cap J_{\mathcal{S}'}$. Then, from the tossing just considered, we see that $\mathcal{S}'$ is a replacement skeleton for skeleton $\mathcal{S}_0$, the latter being the skeleton of tossing $\mathcal{T}_f$ connecting the pairs $(a_{f,}b_f)$ and (a'_f, b'_f) over A_f and B_f. But because $\mathcal{S}'$ belongs to f, this is impossible. This completes the proof that (2) implies (3).

Finally, we show that (3) implies (1). Suppose every skeleton requires only finitely many replacement skeletons, as made precise in the statement of (3) above. We aim to use this condition to construct a set of axioms for $\mathcal{F}$.

Let $\mathbb{S}_1$ denote the set of all elements of $\mathbb{S}$ that are *not* the skeleton of any tossing connecting two elements of $A \times B$, where A ranges over all right S-acts and B over all flat left S-acts, and let $\mathbb{S}_2 = \mathbb{S} \backslash \mathbb{S}_1$.

For $\mathcal{S} \in \mathbb{S}_2$, the comments preceding the theorem yield that $\mathcal{S}$ is the skeleton of a standard tossing joining $([x], b)$ to $([x'], b')$ over $F^{m+1}/\rho_{\mathcal{S}}$ and B where B is flat, $b, b' \in B$, and F^{m+1} and $\rho_{\mathcal{S}}$ are as defined in Lemma 6.1.

Let $\mathcal{S}_1, ..., \mathcal{S}_{\alpha(\mathcal{S})}$ be a set of replacement skeletons for $\mathcal{S}$ as provided by assertion (3) and without loss of generality suppose that replacements for standard tossings may be chosen to have skeletons from $\{\mathcal{S}_1, \ldots, \mathcal{S}_{\alpha'(\mathcal{S})}\}$, where $\alpha'(\mathcal{S}) \leq \alpha(\mathcal{S})$. Hence for each $k \in \{1, ..., \alpha'(\mathcal{S})\}$ if $\mathcal{S}_k = (u_1, v_1, ..., u_h, v_h)$, there exist a flat left S-act C_k, elements $c, c', c_1, ..., c_h \in C_k$, and elements $p_2, ..., p_h \in [x] S \cup [x'] S$ such that

$$\text{(ii)} \qquad \begin{array}{rclrcl} & & & c & = & u_1 c_1 \\ [x] u_1 & = & p_2 v_1 & v_1 c_1 & = & u_2 c_2 \\ p_2 u_2 & = & p_3 v_2 & v_2 c_2 & = & u_3 c_3 \\ & \vdots & & & \vdots & \\ p_h u_h & = & [x'] v_h & v_h c_h & = & c'. \end{array}$$

For each k, we fix such a list $p_2, ..., p_h$ of elements, for future reference, and define $\varphi_{\mathcal{S}}$ to be the sentence

$$\varphi_{\mathcal{S}} \leftrightharpoons (\forall y)(\forall y')(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \cdots \vee \gamma_{\mathcal{S}_{\alpha'(\mathcal{S})}}(y, y')).$$

Let

$$\Sigma_{\mathcal{F}} = \{\psi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_1\} \cup \{\varphi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_2\}.$$

We claim that $\Sigma_{\mathcal{F}}$ axiomatises $\mathcal{F}$.

Suppose first that D is any flat left S-act.

For $\mathcal{S} \in \mathbb{S}_1$, if D did not satisfy $\psi_{\mathcal{S}}$, then we would have $\gamma_{\mathcal{S}}(d, d')$ for some $d, d' \in D$, and so, by the comments preceding the statement of this theorem, $\mathcal{S}$ is the skeleton of some tossing joining (a, d) to (a', d') over some right S-act A and flat left S-act D, contrary to the fact that $\mathcal{S} \in \mathbb{S}_1$. Therefore, $D \vDash \psi_{\mathcal{S}}$.

Now take any $\mathcal{S} \in \mathbb{S}_2$, and suppose $d, d' \in D$ are such that D satisfies $\gamma_{\mathcal{S}}(d, d')$. Then, as noted earlier, $([x], d)$ and $([x'], d')$ are joined over $F^{m+1}/\rho_{\mathcal{S}}$ and D by a standard tossing with skeleton $\mathcal{S}$, and therefore, by assumption, by a tossing over $[x] S \cup [x'] S$ and D with skeleton $\mathcal{S}_k$ for some $k \in \{1, ..., \alpha'(\mathcal{S})\}$. It is now clear that $\gamma_{\mathcal{S}_k}(d, d')$ holds in D, as required. We have now shown that $D \vDash \Sigma_{\mathcal{F}}$.

Finally, we show that a left S-act C that satisfies $\Sigma_{\mathcal{F}}$ must be flat. We need only show that condition (4) of Lemma 6.1 holds for C. Let $\mathcal{S} \in \mathbb{S}$ and suppose we have a standard tossing

$$\text{(iii)} \qquad \begin{array}{rclrcl} & & & c & = & s_1c_1 \\ [x]\,s_1 & = & [x_2]\,t_1 & t_1c_1 & = & s_2c_2 \\ & \vdots & & & \vdots & \\ [x_m]\,s_m & = & [x']\,t_m & t_mc_m & = & c' \end{array}$$

over $F^{m+1}/\rho_{\mathcal{S}}$ and C. If $\mathcal{S}$ belonged to $\mathbb{S}_1$, then C would satisfy the sentence $(\forall y)(\forall y')\neg\gamma_{\mathcal{S}}(y,y')$, and so $\neg\gamma_{\mathcal{S}}(c,c')$ would hold, contrary to the sequence of equalities in the right-hand column of (ii). Therefore, $\mathcal{S}$ belongs to $\mathbb{S}_2$. Because C satisfies $\varphi_{\mathcal{S}}$ and because $\gamma_{\mathcal{S}}(c,c')$ holds, it follows that $\gamma_{\mathcal{S}_k}(c,c')$ holds for some $k\in\{1,...,\alpha'(\mathcal{S})\}$. If $\mathcal{S}_k=(u_1,v_1,...,u_h,v_h)$, then

$$\text{(iv)} \qquad \begin{array}{rcl} c & = & u_1e_1 \\ v_1e_1 & = & u_2e_2 \\ & \vdots & \\ v_he_h & = & c' \end{array}$$

for certain $e_1,...,e_h\in C$. Equalities (iv) and the left hand side of (ii) together constitute a tossing over $[x]\,S\cup[x']\,S$ and C connecting $([x]\,,c)$ and $([x']\,,c')$, showing that C is indeed flat. The proof is now complete. □

7. Axiomatisability of $\mathcal{SF}$

The earliest axiomatisability result, and certainly the most straightforward, in the sequence of those described in this paper, is the characterisation of those monoids S such that $\mathcal{SF}$ is an axiomatisable class. The results described in this section appear (in amalgamated form) in [10]. The reader should note that in [10], strongly flat acts are referred to as flat acts.

For any elements s,t of a monoid S, we define right annihilators $R(s,t)$ and $r(s,t)$ as follows:

$$R(s,t)=\{u,v)\in S\times S: su=tv\},$$

and

$$r(s,t)=\{u\in S: su=tu\}.$$

Where non-empty, it is clear that $R(s,t)$ and $r(s,t)$ are, respectively, an S-subact of the right S-act $S\times S$ and a right ideal of S.

Proposition 7.1. *The following conditions are equivalent for a monoid S:*
(1) the class of left S-acts satisfying condition (E) is axiomatisable;
(2) the class of left S-acts satisfying condition (E) is closed under ultraproducts;
(3) every ultrapower of S as a left S-act satisfies condition (E);
(4) for any $s,t\in S$, $r(s,t)=\emptyset$ or is finitely generated as a right ideal of S.

Proof. That (1) implies (2) is immediate from Theorem 4.2; clearly (3) follows from (2) since S is easily seen to satisfy (E).

Suppose now that every ultrapower of S satisfies condition (E). Let $s,t\in S$ and suppose that $r(s,t)\neq\emptyset$ and is *not* finitely generated as a right ideal.

Let $\{u_\beta:\beta<\gamma\}$ be a generating set of $r(s,t)$ of minimum cardinality γ; we identify the cardinal γ with its initial ordinal; since γ is infinite it must therefore be a limit ordinal. Let $\underline{u}\in\prod_{\beta<\gamma}S_\beta$, where each S_β is a copy of S, be such that $\underline{u}(\beta)=u_\beta$. We may suppose that for any $\beta<\gamma$, $u_\beta\notin\bigcup_{\alpha<\beta}u_\alpha S$.

From Proposition 4.1 we can choose a uniform ultrafilter Φ on γ. Put $U = (\prod_{\beta<\gamma} S_\beta)/\Phi$ so that by our assumption (3), U satisfies condition (E).

Since $su_\beta = tu_\beta$ for all $\beta < \gamma$, clearly $s\underline{u} = t\underline{u}$ and so $s\,\underline{u}\,/\Phi = t\,\underline{u}\,/\Phi$. Now U has (E), so that there exist $s' \in S$ and $\underline{v}\,/\Phi \in U$ such that $ss' = ts'$ and $\underline{u}\,/\Phi = s'\,\underline{v}\,/\Phi$.

From $ss' = ts'$ we have that $s' \in r(s,t)$, so that $s' = u_\beta w$ for some $\beta < \gamma$ and $w \in S$. Let $T = \{\alpha < \gamma : u_\alpha = s'v_\alpha\}$; from the uniformity of Φ, we can pick $\sigma \in T$ with $\sigma > \beta$. Then

$$u_\sigma = s'v_\sigma = u_\beta w v_\sigma \in u_\beta S,$$

a contradiction. We deduce that $r(s,t)$ is finitely generated.

Finally we assume that (3) holds and find a set of axioms for the class of left S-acts satisfying (E).

For any element ρ of $S \times S$ with $r(\rho) \neq \emptyset$ we choose and fix a set of generators

$$w_{\rho 1}, \ldots, w_{\rho\, m(\rho)}$$

of $r(\rho)$. For $\rho = (s,t)$ we define a sentence ξ_ρ of L_S as follows: if $r(\rho) = \emptyset$ then

$$\xi_\rho \leftrightharpoons (\forall x)(sx \neq tx)$$

and on the other hand, if $r(\rho) \neq \emptyset$ we put

$$\xi_\rho \leftrightharpoons (\forall x)\Bigg(sx = tx \rightarrow (\exists z)\Bigg(\bigvee_{i=1}^{m(\rho)} x = w_{\rho i} z\Bigg)\Bigg).$$

We claim that

$$\Sigma_E = \{\xi_\rho : \rho \in S \times S\}$$

axiomatises the class of left S-acts satisfying condition (E).

Suppose first that the left S-act A satisfies (E), and let $\rho = (s,t) \in S \times S$. If $r(\rho) = \emptyset$ and $sa = ta$ for some $a \in S$, then since A satisfies (E) we have an element $s' \in S$ such that $ss' = ts'$, a contradiction. Thus $A \models \xi_\rho$. On the other hand, if $r(\rho) \neq \emptyset$ and $sa = ta$ for some $a \in S$, then again we have that $ss' = ts'$ for some $s' \in S$, and $a = s'b$ for some $b \in A$. Now $s' \in r(\rho)$ so that $s' = w_{\rho i} v$ for some $i \in \{1, \ldots, m(\rho)\}$ and $v \in S$. Consequently, $a = w_{\rho i} c$ for $c = vb \in A$. Thus $A \models \xi_\rho$ in this case also. Thus A is a model of $\Sigma_{\mathcal{E}}$.

Finally, suppose that $A \models \Sigma_{\mathcal{E}}$ and $sa = ta$ for some $s,t \in S$ and $a \in A$. Put $\rho = (s,t)$; since $A \models \xi_\rho$ we are forced to have $r(\rho) \neq \emptyset$ and $a = w_{\rho i} b$ for some $i \in \{1, \ldots, m(\rho)\}$. By very choice of $w_{\rho i}$ we have that $sw_{\rho i} = tw_{\rho i}$. Hence A satisfies condition (E) as required. □

Similarly, and argued in full in [10], we have the corresponding result for condition (P).

Proposition 7.2. *The following conditions are equivalent for a monoid S:*

(1) the class of left S-acts satisfying condition (P) is axiomatisable;

(2) the class of left S-acts satisfying condition (P) is closed under ultraproducts;

(3) every ultrapower of S as a left S-act satisfies condition (P);

(4) for any $s,t \in S$, $R(s,t) = \emptyset$ or is finitely generated as an S-subact of the right S-act $S \times S$.

We may put together Propositions 7.1 and 7.2 to obtain the following result for $\mathcal{SF}$, taken from [10].

Theorem 7.3. *The following conditions are equivalent for a monoid S:*

(1) $\mathcal{SF}$ is axiomatisable;

(2) $\mathcal{SF}$ is closed under ultraproducts;

(3) every ultrapower of S as a left S-act is strongly flat;

(4) for any $s, t \in S$, $r(s,t) = \emptyset$ or is a finitely generated right ideal of S, and $R(s,t) = \emptyset$ or is finitely generated as an S-subact of the right S-act $S \times S$.

8. Axiomatisability of $\mathcal{P}$

Those monoids for which $\mathcal{P}$ is axiomatisable were determined by the fourth author in [21], using preliminary results of the first author from [10]. In fact, $\mathcal{P}$ is axiomatisable if and only if $\mathcal{SF}$ is axiomatisable and $\mathcal{P} = \mathcal{SF}$. Monoids for which $\mathcal{P} = \mathcal{SF}$ are called left perfect. Since left perfect monoids figure largely in this and subsequent sections, we devote some time to them here, developing on the way some new properties of such monoids.

A left S-act B is called a *cover* of a left S-act A if there exists an S-epimorphism $\theta : B \to A$ such that the restriction of θ to any proper S-subact of B is not an epimorphism to A. If B is in addition projective, then B is a *projective cover* for A. A monoid S is *left perfect* if every left S-act has a projective cover.

We now give a number of finitary conditions used in determining left perfect monoids, and in subsequent arguments.

(A) Every left S-act satisfies the ascending chain condition for cyclic S-subacts.

(D) Every right unitary submonoid of S has a minimal left ideal generated by an idempotent.

$(M_R)/(M_L)$ The monoid S satisfies the descending chain condition for principal right/left ideals.

$(M^R)/(M^L)$ The monoid S satisfies the ascending chain condition for principal right/left ideals.

Theorem 8.1. [7, 12, 13] *The following conditions are equivalent for a monoid S:*

(1) S is left perfect;

(2) S satisfies Conditions (A) and (D);

(3) S satisfies Conditions (A) and (M_R);

(4) $\mathcal{SF} = \mathcal{P}$.

Proposition 8.2. *Let S be a left perfect monoid. Then*

(1) S is group bound;

(2) if Sb (bS) is a minimal left (right) ideal of S, then bS (Sb) is a minimal right (left) ideal of S;

(3) if $Sb_1 \subseteq Sb_0$ and $Sb_1 \cong Sb_0$, then $Sb_0 = Sb_1$;

(4) any minimal left (right) ideal of S is generated by an idempotent.

Proof. (1) Let S be a left perfect monoid. From Theorem 8.1 S satisfies (M_R), so that for any $a \in S$, $a^m S = a^{m+1} S$ for some $m \in \mathbb{N}$. On the other hand, consider the descending chain

$$Sa \supseteq Sa^2 \supseteq Sa^3 \supseteq \dots$$

of principal left ideals. Let Φ be a uniform ultrafilter on $\mathbb{N}$ and consider the ultrapower $U = S^{\mathbb{N}}/\Phi$. For each $n \in \mathbb{N}$ let

$$\underline{u_n} = (1, 1, \dots, a, a^2, a^3, \dots)/\Phi$$

where the first a occurs in the n'th place. Clearly $\underline{u_n} = a\underline{u_{n+1}}$ for any $n \in \mathbb{N}$, so that

$$S\underline{u_1} \subseteq S\underline{u_2} \subseteq \dots$$

By Theorem 8.1, U has the ascending chain condition on cyclic S-subacts, so that $S\underline{u_h} = S\underline{u_{h+1}}$ for some h. Consequently, $s\underline{u_h} = \underline{u_{h+1}}$ for some $s \in S$; since Φ is uniform, we deduce that for some $i \geq h+1$, $sa^{i-h+1} = a^{i-h}$. Putting $k = i-h$ we deduce that $a^k \mathcal{L}\, a^{k+1}$. Now take n to be the bigger of m and k; clearly $a^n\, \mathcal{H}\, a^{n+1}\, \mathcal{H} a^{2n}$, whence by Theorem 2.1, a^n lies in a subgroup of S.

(2) Suppose now that Sb is a minimal left ideal of S; since S has (M_R) we can choose $c \in S$ with $cS \subseteq bS$ and cS minimal. Notice that $Scb = Sb$ and so $cb\, \mathcal{L}\, b$; consequently $cb\, \mathcal{L}^*\, b$. If $d \in bS$, then as $c^2S = cS = cdS$ we have $cd = ccd'$ for some $d' \in S$. Now, $d = bx, cd' = by$ for some $x, y \in S$, and so $cbx = cby$, giving that

$$d = bx = by = cd',$$

that is, $d \in cS$. Hence $bS = cS$ is minimal.

To prove (3), let us assume that $Sb_1 \subseteq Sb_0$ and $Sb_1 \cong Sb_0$. Let $\phi : Sb_0 \to Sb_1$ be an S-isomorphism. Then $\phi(b_0) = sb_1 = b_2$ for some $s \in S$. Then $Sb_2 \subseteq Sb_1 \subseteq Sb_0$ and $b_2 = tb_0$ for some $t \in S$. Since ϕ is an isomorphism we have that $tb_0\, \mathcal{R}^*\, b_0$ and as $\mathcal{R}^*$ is a left congruence and S is group bound, $b_0\, \mathcal{R}^*\, t^n b_0$ for some $n \in \mathbb{N}$ such that t^n lies in a subgroup. Let s be the inverse of t in this subgroup. Then $st^nt^n = t^n$, so that

$$b_0 = st^nb_0 = st^{n-1}tb_0 = st^{n-1}b_2$$

whence $Sb_2 = Sb_1 = Sb_0$ as required.

We now prove the second part of (2). Suppose $b \in S$ and bS is a minimal right ideal. Let $Sc \subseteq Sb$. In view of the minimality of bS we have that $bc\, \mathcal{R}\, b$ and so $bc\, \mathcal{R}^*\, b$. By Lemma 2.6, $Sbc \cong Sb$. But $Sbc \subseteq Sc \subseteq Sb$. Now (3) gives that $Sbc = Sc = Sb$ as required.

To see that (4) holds, note that if Sb is a minimal left ideal, then $Sb = Sb^n$ for all $n \in \mathbb{N}$; since S is group bound, $b^n\, \mathcal{H}\, e$ for some $n \in \mathbb{N}$ and some $e \in E$. Hence $Sb^n = Se$; dually for principal right ideals. ☐

From Propositions 2.3 and 8.2 the following is immediate.

Corollary 8.3. *Let S be a left perfect monoid. Then S is local and $\mathcal{D} = \mathcal{J}$.*

Before stating the main result of this section, we require a preliminary lemma, due to the fourth author, that has significant consequences. We recall from Section 2 say that a monoid satisfies (CFRS) if

$$\forall s \in S\, \exists n_s \in \mathbb{N}\, \forall t \in S |\{x \in S|\, sx = t\}| \leq n_s.$$

Lemma 8.4. [21] *Let S be a monoid such that every ultrapower of S as a left S-act is projective. Then S satisfies (CFRS).*

Proof. Suppose to the contrary that $t \in S$ exists for which the condition is *not* true. That is, for each $n \in \mathbb{N}$, there exists an element $a_n \in S$ such that $|\{x \in S : tx = a_n\}| > n$. Let Φ be a uniform ultrafilter on $\mathbb{N}$ and for $n \in \mathbb{N}$ choose $b_{n,1}, \dots, b_{n,n} \in S$ such that $tb_{n,i} = a_n$, $1 \leq i \leq n$. Put

$$\underline{c_n} = (1, \dots, 1, b_{n,n}, b_{n+1,n}, b_{n+2,n}, \dots)$$

with $b_{n,n}$ occuring in the n'th place. Now, $\underline{c_i}\,/\Phi \neq \underline{c_j}\,/\Phi$ for any $i \neq j$, but $t\underline{c_i}\,/\Phi = \underline{a}\,/\Phi$ where $\underline{a} = (a_1, a_2, \dots)$. Since $U = \prod_{n \in \mathbb{N}} S/\Phi$ is projective and $\underline{a}\,/\Phi \in S\underline{c}\,/\Phi \cong Se$ for some $e \in E$, we deduce that there exists $d \in S$ such that $A = |\{x \in S : tx = d\}|$ is infinite.

Now choose a cardinal $\alpha > |S|$. Combining Propositions 4.3 and 4.4, we can choose an ultrafilter Θ over α such that $|A^\alpha/\Theta| = |A|^\alpha > |S|$ (any set may be regarded as an S-act over a trivial monoid). Put $V = S^\alpha/\Theta$ and let $\underline{d} \in S^\alpha$ be such that $\underline{d}(i) = d$ for all $i \in \alpha$. If $\underline{x} \in A^\alpha$ then clearly $t\,\underline{x}\,/\Theta = \underline{d}\,/\Theta$. But V is projective by assumption, so that $\underline{d}\,/\Theta \in S\underline{g}\,/\Theta \cong Sf$ for some $f \in E$. Consequently, there is an element $u \in S$ such that the equation $tx = u$ has more than $|S|$ solutions in S, which is clearly nonsense. Hence S has (CFRS). □

Corollary 8.5. *Let S be such that any ultrapower of S as a left S-set is projective. Then for all $s \in S$ there exists $n_s \in \mathbb{N}$ such that for any $P \in \mathcal{P}$ and $t \in P$,*

$$|\{x \in S|\, sx = t\}| \leq n_s.$$

We now set out to prove the main result of this section, due to the first and fourth authors.

Theorem 8.6. [10, 21, 2] *The following conditions are equivalent for a monoid S:*
(1) every ultrapower of the left S-set S is projective;
(2) $\mathcal{SF}$ is axiomatisable and S is left perfect.
(3) $\mathcal{P}$ is axiomatisable.

Proof. If $\mathcal{SF}$ is axiomatisable and S is left perfect, then by Theorem 8.1, $\mathcal{P} = \mathcal{SF}$ is an axiomatisable class. Clearly if $\mathcal{P}$ is axiomatisable, then Theorem 4.2 gives that every ultrapower of S is projective.

Suppose that every ultrapower of S as a left S-set is projective; by Theorem 7.3, certainly $\mathcal{SF}$ is an axiomatisable class. We proceed via a series of subsidiary lemmas.

Lemma 8.7. [10] *Let S be such that every ultrapower of the left S-act S is projective. Then S has (M_R).*

Proof. Let $a_1S \supseteq b_2S \supseteq b_3S \supseteq \ldots$ be a decreasing sequence of principal right ideals of S, so that for $i \geq 2$ we have $b_i = b_{i-1}a_i$ for some $a_i \in S$, putting $b_1 = a_1$. Thus $b_2 = a_1a_2, b_3 = b_2a_3 = a_1a_2a_3, \ldots$.

Let Φ be a uniform ultrafilter over $\mathbb{N}$ and put $U = S^{\mathbb{N}}/\Phi$; by assumption, U is projective.

Define elements $\underline{u_i} \in S^{\mathbb{N}}$, $i \in \mathbb{N}$, by

$$\underline{u_i} = (1, 1, \ldots, 1, a_i, a_ia_{i+1}, a_ia_{i+1}a_{i+2}, \ldots),$$

where the entry a_i is the i'th coordinate. Then, for any $i, j \in \mathbb{N}$ with $i < j$ we have

$$\underline{u_i}\,/\Phi = a_ia_{i+1}\ldots a_{j-1}\underline{u_j}\,/\Phi.$$

Since U is projective, Proposition 3.3 and Corollary 3.4 give that

$$S\underline{u_1}\,/\Phi \subseteq S\underline{u_2}\,/\Phi \subseteq \ldots S\underline{c}\,/\Phi$$

where $\underline{c}\,/\Phi$ is left e-cancellable for some $e \in E$. Put $\underline{c} = (c_1, c_2, \ldots)$, and let $d_i \in S$ be such that $\underline{u_i}\,/\Phi = d_i\underline{c}\,/\Phi$ for each $i \in \mathbb{N}$. For any $i < j$ we have that

$$d_i\underline{c}\,/\Phi = a_i \ldots a_{j-1}d_j\underline{c}\,/\Phi$$

whence from the left e-cancellability of $\underline{c}\,/\Phi$,

$$d_ie = a_i \ldots a_{j-1}d_je.$$

Choose $i \in \mathbb{N}$ such that $a_1 \dots a_i = d_1 c_i$, and $ec_i = c_i$. Then for any $j > i$,

$$a_1 \dots a_i S = d_1 e c_i S = a_1 \dots a_j d_{j+1} e c_i S \subseteq a_1 \dots a_j S \subseteq a_1 \dots a_i S,$$

whence

$$b_i S = b_{i+1} S = \dots$$

as required. □

Lemma 8.8. [21] *Let S be such that every ultrapower of the left S-act S is projective. Then S has (M^L).*

Proof. Consider an ascending chain

$$Sa_1 \subseteq Sa_2 \subseteq \cdots$$

of principal left ideals of S. Let $u_2, u_3, \dots \in S$ be such that $a_i = u_{i+1} a_{i+1}$, from which it follows that for any i, j with $i < j$,

$$a_i = u_{i+1} u_{i+2} \cdots u_j a_j.$$

Consider an ultrapower $U = S^{\mathbb{N}}/\Phi$ where Φ is a uniform ultrafilter on $\mathbb{N}$. By assumption, U is projective as a left S-act. For each $i \geq 2$ define

$$\underline{v_i}\,/\Phi = (1, 1, \dots, 1, u_i, u_i u_{i+1}, u_i u_{i+1} u_{i+2}, \dots)/\Phi,$$

the entry u_i occurring in the ith position. Observe that

$$\underline{v_i}\,/\Phi = u_i \underline{v_{i+1}}\,/\Phi,$$

for each i, and as in Lemma 8.7 the projectivity of U ensures that there exist $\underline{f}\,/\Phi = (f_1, f_2, ...)/\Phi$ and $s_1, s_2, \dots \in S$ such that

$$\underline{v_k}\,/\Phi = s_k \underline{f}\,/\Phi$$

for each $k \in \mathbb{N}$. For each natural number k it follows that the set

$$E_k = \{j \in \mathbb{N} \mid j > k,\, u_k u_{k+1} \cdots u_j = s_k f_j\}$$

belongs to Φ. For each i we put

$$\underline{g_i}\,/\Phi = f_i \underline{v_{i+1}}\,/\Phi = (f_i, ..., f_i, f_i u_{i+1}, f_i u_{i+1} u_{i+2}, ...)\,/\Phi.$$

Suppose that for each $i \in \mathbb{N}$, $\left\{ j \in \mathbb{N} \mid \underline{g_i}\,/\Phi = \underline{g_j}\,/\Phi \right\} \notin \Phi$. In this case, for each $i \in \mathbb{N}$ put $T_i = \left\{ j \in \mathbb{N} \mid \underline{g_i}\,/\Phi \neq \underline{g_j}\,/\Phi \right\}$, which by assumption belongs to Φ. Now choose $j_1, j_2, ... \in \mathbb{N}$ as follows:

$$\begin{array}{lcl} j_1 & \in & E_2 \\ j_2 & \in & E_2 \cap T_{j_1},\, j_2 > j_1 \\ j_3 & \in & E_2 \cap T_{j_1} \cap T_{j_2},\, j_3 > j_2 > j_1 \\ & \vdots & \end{array}$$

For any k,

$$\begin{array}{rcl} \underline{v_2}\,/\Phi & = & u_2 u_3 \cdots u_{j_k} \underline{v_{j_k+1}}\,/\Phi \\ & = & s_2 f_{j_k} \underline{v_{j_k+1}}\,/\Phi \\ & = & s_2 \underline{g_{j_k}}\,/\Phi. \end{array}$$

By definition of the sets T_i, all of the elements $\underline{g_{j_k}}/\Phi$ are distinct. This contradicts Lemma 8.5.

In view of the above, there exist $i_0 \in \mathbb{N}$ such that $D := \left\{j \in \mathbb{N} \mid \underline{g_j}/\Phi = \underline{g_{i_0}}/\Phi\right\}$ belongs to Φ. For any $k \in D$ with $k > i_0$, pick $j \in D \cap E_k$. Then we have

$$S_j := \{m \in \mathbb{N} \mid m > j,\ f_j u_{j+1} \cdots u_m = f_{i_0} u_{i_0+1} \cdots u_m\} \in \Phi.$$

Now take any $m \in S_j$ and calculate

$$\begin{aligned} a_{k-1} &= u_k \cdots u_j u_{j+1} \cdots u_m a_m \\ &= s_k f_j u_{j+1} \cdots u_m a_m \\ &= s_k f_{i_0} u_{i_0+1} \cdots u_m a_m \\ &= s_k f_{i_0} a_{i_0}. \end{aligned}$$

In summary we have shown in this case that, for any $k \in D$ with $k > i_0$, the left ideals Sa_{k-1} and Sa_{i_0} are equal. If now $l > i_0$ is arbitrary, take any $k \in D$ with $k > l$ and note that $Sa_{i_0} \subseteq Sa_l \subseteq Sa_{k-1} = Sa_{i_0}$, and so the chain terminates, as required.

□

In order to complete the proof of the implication (1) ⇒ (2) in Theorem 8.6 we need one further lemma. To this end, the following alternative characterisation of Condition (A) will be useful.

Lemma 8.9. [12] *A monoid S satisfies (A) if and only if for any elements $a_1, a_2, \ldots$ of S, there exists $n \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, $i \geq n$, there exists $j_i \in \mathbb{N}, j_i \geq i+1$, with*

$$Sa_i a_{i+1} \ldots a_{j_i} = Sa_{i+1} \ldots a_{j_i}.$$

Lemma 8.10. [10] *Let S be such that every ultrapower of the left S-act S is projective. Then S satisfies Condition (A).*

Proof. Let $a_i \in S, i \in \mathbb{N}$ and define Φ, $U, \underline{u_i}/\Phi, d_i (i \in \mathbb{N})$ and $\underline{c}/\Phi$ as in the proof of Lemma 8.7. For any i we have that $d_i e = a_i d_{i+1} e$ so that $Sd_1 e \subseteq Sd_2 e \subseteq \ldots$. By Lemma 8.8 we know that $Sd_n e = Sd_{n+1} e = \ldots$ for some $n \in \mathbb{N}$ and since $e\underline{c}/\Phi = \underline{c}/\Phi$ it follows that $S\underline{u_n}/\Phi = S\underline{u_{n+1}}/\Phi = \ldots$. Now let $i \geq n$, so that $\underline{u_{i+1}}/\Phi = s\underline{u_i}/\Phi$ for some $s \in S$. Since Φ is uniform there exists $j_i \geq i+1$ such that $a_{i+1} \ldots a_{j_i} = sa_i a_{i+1} \ldots a_{j_i}$ and so

$$Sa_{i+1} \ldots a_{j_i} = Sa_i a_{i+1} \ldots a_{j_i},$$

so that by Lemma 8.9, S satisfies Condition (A).

□

We now proceed with the proof of Theorem 8.6. If every ultrapower of S is projective, then from Lemmas 8.7 and 8.10 we know that S satisfies (M_R) and Condition (A). By Theorem 8.1, S is left perfect. □

9. Axiomatisability of $\mathcal{F}r$

The question of axiomatisability of $\mathcal{F}r$ was solved in some special cases by the fourth author in [21], and most recently by the first author as below.

For convenience, we introduce some new terminology. Let $e \in E$ and $a \in S$. We say that $a = xy$ is an *e-good factorisation of a through x* if $y \neq wz$ for any w, z with $e = xw$ and $w \,\mathcal{L}\, e$.

Theorem 9.1. *The following conditions are equivalent for a monoid S:*

(1) every ultrapower of the left S-act S is free;

(2) $\mathcal{P}$ is axiomatisable and S satisfies (*): for all $e \in E \setminus \{1\}$, there exists a finite set $f \in S$ such that any $a \in S$ has an e-good factorisation through w, for some $w \in f$.

(3) $\mathcal{F}r$ is axiomatisable.

Proof. If $\mathcal{F}r$ is axiomatisable, then certainly (1) holds. On the other hand, if (1) holds, then by Theorem 8.6, $\mathcal{P}$ is axiomatisable and S is left perfect. Note that by Corollary 8.3, S is local. We show that (*) holds.

Let $e \in E$ with $e \neq 1$. For any $a \in S$, $a = a \cdot 1$; if $e = av$ with $v \,\mathcal{L}\, e$, then as S is local, $1 \neq vc$ for any c.

We proceed by contradiction. Let J denote the set of finite subsets of S. Suppose that for any $f \in J$ there exists an element $w_f \in S$ such that w_f does not have an e-good factorisation through w, for any $w \in f$. Clearly S and J must be infinite.

For each $w \in S$ let $J_w = \{f \in J : w \in f\}$; since $\{w_1, \ldots, w_n\} \in J_{w_1} \cap \ldots \cap J_{w_n}$, there exists an ultrafilter Φ over J such that $J_w \in \Phi$ for all $w \in S$.

Consider $U = S^J/\Phi$; by assumption (1), U is free. Let $\underline{x} \in S^J$ be such that $\underline{x}(f) = w_f$. Since U is free, Theorem 3.1 and Corollary 3.2 give that $\underline{x}/\Phi = w\underline{d}/\Phi$ for some w, where $\underline{d}/\Phi$ is right 1-cancellable. Suppose that $\underline{d}(f) = d_f$, for any $f \in J$.

We claim that

$$D = \{f \in J \mid wd_f \text{ is an } e\text{-good factorisation through } w\} \in \Phi.$$

Suppose to the contrary. Then

$$D' = \{f \in J \mid d_f = vz \text{ for some } v, z \text{ with } e = wv \text{ and } v \,\mathcal{L}\, e\} \in \Phi.$$

By Lemma 8.4, there are only finitely many $v_1, \ldots, v_n$ such that $e = wv_i$ and $v_i \,\mathcal{L}\, e$. For $1 \leq i \leq n$ let

$$D_i = \{f \in J : d_f = v_i z \text{ for some } z\},$$

so that $D' = D_1 \cup \ldots \cup D_n$. Consequently, $D_i \in \Phi$ for some $i \in \{1, \ldots, n\}$. We know that $v_i \,\mathcal{L}\, e$, so that v_i is regular and $v_i \,\mathcal{R}\, g$ for some $g \in S$; as S is local, $g \neq 1$. But $gv_i = v_i$ so that $gd_f = d_f$ for all $f \in D_i \in \Phi$. Hence $g\underline{d}/\Phi = \underline{d}/\Phi$, so that as $\underline{d}/\Phi$ is right 1-cancellable, $g = 1$, a contradiction. We conclude that $D \in \Phi$.

Let $T = \{f \in J : w_f = wd_f\}$, so that $T \in \Phi$; now pick $f \in D \cap T \cap J_w$. We have that $w \in f$, and as $f \in T$, $w_f = wd_f$; moreover, as $f \in D$, this is an e-good factorisation of w_f through w. This contradicts the choice of w_f. We deduce that (*) holds.

Finally, we suppose that $\mathcal{P}$ is axiomatisable, and S satisfies (*). Let $\Sigma_{\mathcal{P}} = \Sigma_{\mathcal{SF}}$ be a set of sentences axiomatising $\mathcal{P}$. Let $e \in E, e \neq 1$. Choose a finite set $f = \{u_1, \ldots, u_n\}$ guaranteed by (*), such that every $a \in S$ has an e-good factorisation through u_i, for some $i \in \{1, \ldots, n\}$. Since $\mathcal{P}$ is axiomatisable, Lemma 8.4 tells us that for each $i \in \{1, \ldots, n\}$ there exist finitely many $v_{i1}, \ldots, v_{im_i} \in L_e$, $m_i \geq 0$, such that $e = u_i v_{ij}, 1 \leq j \leq m_i$. Let

$$\varphi_{e,i} \leftrightharpoons (\exists b)(a = u_i b \wedge (\bigwedge_{1 \leq j \leq m_i} b \neq v_{ij} a)).$$

We now define ϕ_e as

$$(\forall a) \bigvee_{1 \leq i \leq n} \varphi_{e,i}.$$

Put

$$\Sigma_{\mathcal{F}r} = \Sigma_{\mathcal{P}} \cup \{\varphi_e \mid e \in E \setminus \{1\}\}.$$

We claim that $\Sigma_{\mathcal{F}r}$ axiomatises $\mathcal{F}r$.

Let F be a free S-set; certainly $F \vDash \Sigma_{\mathcal{P}}$. Say that F is free on X, let $e \in E, e \neq 1$ and let $a \in F$. Then $a = sx$ for some $x \in X$. By choice of $u_1, \ldots, u_n$, we can write $s = u_i t$ for some $t \in S$ with $t \neq vw$ for any $w \in S$ and $v \in L_e$ such that $e = u_i v$. Put $b = tx$; clearly then $F \vDash \varphi_e$.

Conversely, let A be an S-set and suppose that $A \models \Sigma_{\mathcal{F}r}$. Since A is therefore projective, we know that A is a coproduct of maximal indecomposable S-subsets of the form Sa, where there exists an $e \in E$ such that a is right e-cancellable; notice that $ea = a$. Suppose that $e \neq 1$. Since $A \vDash \varphi_e$ we have that $a = u_i b$ for some b such that $b \neq va$ for any $v \in L_e$ with $e = u_i v$. But $b = wa$ say, giving that $a = u_i wa$ and so $e = u_i we$. Clearly $e \,\mathcal{L}\, we$, and $b = wa = wea$, a contradiction. Thus $e = 1$ and we deduce that A is free. Consequently, $\Sigma_{\mathcal{F}r}$ axiomatises $\mathcal{F}r$ as required.

□

For some restricted classes of monoids, we can simplify the condition given in Theorem 9.1. We say that the group of units H_1 of a monoid S has *finite right index* if there exists $u_1, \ldots, u_n \in S$ such that $S = u_1 H_1 \cup \ldots u_n H_1$. Note that if in addition S is local, then for any $e \in E, e \neq 1$, any $a \in S$ has an e-good factorisation through u_i, for some $i \in \{1, \ldots, n\}$.

Proposition 9.2. *Let S be a monoid such that*

$$S \setminus R_1 = s_1 S \cup \ldots \cup s_m S$$

for some $s_1, \ldots, s_m \in S$. Then $\mathcal{F}r$ is an axiomatisable class if and only if $\mathcal{P}$ is axiomatisable and H_1 has finite right index in S.

Proof. Suppose that $\mathcal{P}$ is axiomatisable and H_1 has finite right index in S. By Theorem 8.6 Corollary 8.3, S is local, so that by the comments above, condition (*) of Theorem 9.1 holds, and so $\mathcal{F}r$ is an axiomatisable class.

Conversely, if $\mathcal{F}r$ is axiomatisable, it remains only to show that H_1 has finite right index. Suppose for contradiction that there exists $a_1, a_2, \ldots$ in S with $a_i U \cap a_j U = \emptyset$ for all $i \neq j$. Let Φ be a uniform ultrafilter on $\mathbb{N}$, let $U = S^{\mathbb{N}}/\Phi$ and let $\underline{a} \in S^{\mathbb{N}}$ be given by $\underline{a}(i) = a_i$. Since U is free, $\underline{a}/\Phi = w\underline{d}/\Phi$ for some right 1-cancellable $\underline{d}/\Phi$ generating the connected component in which $\underline{a}/\Phi$ lies. Say $\underline{d}(i) = d_i$.

By Theorem 9.1, and Corollary 8.3, we know that S is local so that $R_1 = H_1$. Hence $\mathbb{N} = T_1 \cup \ldots T_m \cup T$ where $T_i = \{i \in \mathbb{N} : d_i \in s_i S\}$ and $T = \{i \in \mathbb{N} : d_i \in H_1\}$. If $T_i \in \Phi$, then $\underline{d}/\Phi = s_i \underline{f}/\Phi$ for some $\underline{f}/\Phi$; but $\underline{f}/\Phi = v\underline{d}/\Phi$ so that as $\underline{d}/\Phi$ is 1-cancellable, we obtain $1 = s_i v$, a contradiction. Hence $T \in \Phi$. Let $D = \{i \in \mathbb{N} : a_i = wd_i\}$, and pick distinct $i, j \in D \cap T$. Then $a_i = wd_i, a_j = wd_j$ and so

$$a_i H_1 = wd_i H_1 = wH_1 = wd_j H_1 = a_j H_1,$$

a contradiction. We deduce that H_1 has finite right index in S.

□

Our final corollary is now straightforward.

Corollary 9.3. [21] *Let S be an inverse monoid. Then $\mathcal{F}r$ is an axiomatisable class if and only if $\mathcal{P}$ is axiomatisable and H_1 has finite right index in S.*

Proof. The converse holds as in Proposition 9.2.

Suppose now that $\mathcal{F}r$ is axiomatisable. Since S has M_R we can pick a minimal principal right ideal; as S is regular, this is generated by $e \in E$. For any $f \in E$ we have that $eS = efS$, so that $e\mathcal{R}ef$. But S is inverse, so that E is a semilattice, and every $\mathcal{R}$-class contains a unique idempotent. Hence $e = ef$ for all $f \in E$; by Lemma 8.4 we deduce that E is finite. Since every principal right ideal is idempotent generated, S has only finitely many principal right ideals. The result follows by Proposition 9.2. □

10. Completeness, model completeness and categoricity

At this point we need to present a little more model theory as motivation for the remaining sections. We remind the reader that throughout, L denotes a first order language.

An *elementary theory* or simply a *theory* of a first order language L is a set of sentences T of L, which is closed under deduction. We recall from Section 4 that an L-structure $\mathbf{A}$ in which all sentences of the theory T are true, is called a *model* of the theory T and we write $\mathbf{A} \models T$. A theory T is *consistent* if for any sentence $\varphi \in L$ we do not have both φ and $\neg\varphi \in T$. A consistent theory T is *complete* if $\varphi \in T$ or $\neg\varphi \in T$ for any sentence φ of the language L. By the *Extended Completeness Theorem* (c.f. Theorem 1.3.21 of [4]), a theory T is consistent if and only if it has a model. Clearly, for any L-structure $\mathbf{A}$, the *theory* $\mathrm{Th}(\mathbf{A})$ *of* $\mathbf{A}$, defined by

$$\mathrm{Th}(\mathbf{A}) = \{\varphi : \varphi \text{ is a sentence, } \mathbf{A} \models \varphi\}$$

is a consistent complete theory and for a class of L-structures $\mathcal{K}$,

$$\mathrm{Th}(\mathcal{K}) = \bigcap_{\mathbf{A}\in\mathcal{K}} \mathrm{Th}(\mathbf{A})$$

is consistent.

Structures $\mathbf{A}, \mathbf{B}$ for L are *elementarily equivalent*, denoted $\mathbf{A} \equiv \mathbf{B}$, if $\mathrm{Th}(\mathbf{A}) = \mathrm{Th}(\mathbf{B})$. One of the basic tenets of model theory tells us that if T is a theory and φ is a sentence such that $\varphi \notin T$, then $T \cup \{\neg\varphi\}$ is consistent, and hence has a model. Thus we deduce

Lemma 10.1. *A consistent theory T is complete if and only if $\mathbf{A} \equiv \mathbf{B}$ for any models $\mathbf{A}, \mathbf{B}$ of T.*

The previous few chapters have concentrated on axiomatisable classes of S-acts. We remark here that if $\mathcal{K}$ is a class axiomatised by T, then $\mathrm{Th}(\mathcal{K}) = T$.

For an L-structure $\mathbf{A}$ with universe A and subset B of A we often consider the augmented or *enriched* language L_B, which is obtained from L by adding a set of constants $\{b' : b \in B\}$, where $b'_1 \neq b'_2$ for distinct b_1 and b_2 from B. We will write $\mathbf{A}_B$ for the corresponding enriched L_B-structure. So, $\mathbf{A}_B$ is obtained from $\mathbf{A}$ by interpreting the constant b', where $b \in B$, by b. We denote by $\mathrm{Th}(\mathbf{A}, b)_{b\in B}$ the set of sentences of L_B true in $\mathbf{A}_B$. Another useful tool is that of the *diagram* of an L-structure $\mathbf{A}$, denoted by Diag $\mathbf{A}$, which is the set of atomic and negated atomic formulae of L_A that are true in $\mathbf{A}_A$. From Proposition 2.1.8 of [4], an L-structure $\mathbf{A}$ embeds into an L-structure $\mathbf{B}$ if and only if $\mathbf{B}$ has an enriching to the language L_A, such that $\mathbf{B}_A$ is a model of Diag $\mathbf{A}$.

We adopt the standard convention of writing $\bar{x} \in X$ to indicate that $\bar{x} = (x_1, \dots, x_n)$ for some finite set $\{x_1, \dots, x_n\} \subseteq X$. A substructure $\mathbf{A}$ of an L-structure $\mathbf{B}$ is said to be *elementary* (denoted $\mathbf{A} \preccurlyeq \mathbf{B}$), if for any formula $\varphi(\bar{x})$ of the language L and any $\bar{a} \in A$

$$\mathbf{A} \models \varphi(\bar{a}) \Leftrightarrow \mathbf{B} \models \varphi(\bar{a}).$$

Note that in this definition the condition"$\Leftrightarrow$" can be exchanged to "$\Leftarrow$" or "$\Rightarrow$" (consider the negation of the formula and bear in mind that for any L-structure $\mathbf{C}$, $\mathrm{Th}(\mathbf{C}, c)_{c \in C}$ is complete). It is easy to see that if $\mathbf{A}$ is a substructure of $\mathbf{B}$, then

$$\mathbf{A} \preccurlyeq \mathbf{B} \text{ if and only if } \mathbf{B}_A \models \mathrm{Th}(A, a)_{a \in A}.$$

We can now state a crucial result, known as the upward and downward Löwenheim-Skolem-Tarski theorem.

Theorem 10.2. *Corollary 2.1.6 and Theorem 3.1.6,* [4] *Let T be a theory in L with an infinite model. Then for any cardinal $\varkappa \geq |L|$, T has a model $\mathbf{A}$ with $|A| = \varkappa$.*

If $\mathbf{B}$ is a model of T with $|B| = \varkappa \geq \alpha \geq |L|$ and $X \subseteq B$ with $|X| \leq \alpha$, then there is an elementary substructure $\mathbf{C}$ of $\mathbf{B}$ (so that certainly $\mathbf{C} \models T$) such that $X \subseteq C$ and $|C| = \alpha$.

A consistent theory T of the language L is called *model complete* if

$$\mathbf{A} \subseteq \mathbf{B} \Rightarrow \mathbf{A} \preccurlyeq \mathbf{B}$$

for any models $\mathbf{A}, \mathbf{B}$ of T.

Lemma 10.3. *Proposition 3.1.9,* [4] *Let T be model complete and such that any two models of T are isomorphically embedded in a third. Then T is complete.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be models of T; by hypothesis there exists a model $\mathbf{C}$ of T such that $\mathbf{A}$ and $\mathbf{B}$ embed into $\mathbf{C}$. Since T is model complete,

$$\mathbf{A} \preccurlyeq \mathbf{C} \text{ and } \mathbf{B} \preccurlyeq \mathbf{C}$$

so that certainly $\mathbf{A} \equiv \mathbf{B}$ and from Lemma 10.1, T is complete. □

By writing a formula φ of L as $\varphi(\bar{x})$, we indicate that the free variables of φ lie amongst those of $\bar{x}$. We can also write $\varphi(\bar{x}; \bar{y})$ to indicate that the free variables lie amongst those of the distinct tuples $\bar{x}$ and $\bar{y}$. A formula of the form $(\exists \bar{x})\psi(\bar{x}; \bar{y})$ for a quantifier–free formula $\psi(\bar{x}; \bar{y})$ is called *existential.* A structure $\mathbf{A}$ in a class $\mathcal{K}$ of L-structures is called *existentially closed in* $\mathcal{K}$ if for every extension $\mathbf{B} \in \mathcal{K}$ of $\mathbf{A}$ and every existential formula $(\exists \bar{x})\varphi(\bar{x}; \bar{a})$ with $\bar{a} \in A$, if $\mathbf{B}_A \models (\exists \bar{x})\varphi(\bar{x}; \bar{a})$, then $\mathbf{A}_A \models (\exists \bar{x})\varphi(\bar{x}; \bar{a})$.

Theorem 10.4. *Proposition 3.1.7,* [4] *A theory T is model complete if and only if whenever $\mathbf{A}, \mathbf{B} \models T$ and $\mathbf{A} \subseteq \mathbf{B}$, then for any existential formula $(\exists \bar{y})\psi(\bar{y})$ of L_A,*

$$\mathbf{B}_A \models (\exists \bar{y})\psi(\bar{y}) \Rightarrow \mathbf{A}_A \models (\exists \bar{y})\psi(\bar{y}).$$

Let $\varkappa$ be a cardinal. We recall that a theory T in L is *categorical in* $\varkappa$, or *$\varkappa$-categorical*, if T has a model of cardinality $\varkappa$, and any two models of cardinality $\varkappa$ are isomorphic. The next result is known as the *Los-Vaught test*; its proof is straightforward, relying upon the Löwenheim-Skolem-Tarski theorems.

Proposition 10.5. *Proposition 3.1.10,* [4] *Suppose that T is a consistent theory with only infinite models, and that T is $\varkappa$-categorical for some $\varkappa \geq |L|$. Then T is complete.*

When applying the notions of categoricity, completeness and model completeness to a class of L-structures, we have to be a little careful. A class $\mathcal{K}$ of L-structures is called *categorical in cardinality* $\varkappa$ or $\varkappa$*-categorical* if all structures from $\mathcal{K}$ of cardinal $\varkappa$ are isomorphic. The class $\mathcal{K}$ is called *categorical* if $\mathcal{K}$ is categorical in some cardinal $\varkappa \geq |L|$.

Let $\mathcal{K}$ be class of the L-structures. We denote class of the infinite structures of $\mathcal{K}$ by $\mathcal{K}_\infty$. The class $\mathcal{K}$ is called *complete (model complete)*, if the theory $\mathrm{Th}(\mathcal{K}_\infty)$ of the infinite structures of this class is complete (model complete).

Lemma 10.6. *Let $\mathcal{K}$ be a class of L-structures axiomatised by a theory T. For $n \in \mathbb{N}$ we let φ_n be the sentence*

$$\varphi_n \leftrightharpoons (\exists x_1 \dots x_n) \bigwedge_{1 \leq i \neq j \leq n} (x_i \neq x_j)$$

and let T_∞ be the deductive closure of

$$T \cup \{\varphi_n : n \in \mathbb{N}\}.$$

Then T_∞ axiomatises $\mathcal{K}_\infty$, so that $Th(\mathcal{K}_\infty) = T_\infty$.

Lemma 10.7. *Let $\mathcal{K}$ be an axiomatisable class of L-structures and $\varkappa$ an infinite cardinal, $\varkappa \geqslant |L|$.*

(1) If $\mathcal{K}$ is closed under the union of increasing chains then there is an existentially closed structure $\mathbf{A} \in \mathcal{K}$, $|\mathbf{A}| = \varkappa$.

(2) If there is an infinite structure in $\mathcal{K}$ which is not existentially closed, then there is a structure $\mathbf{A} \in \mathcal{K}$, $|\mathbf{A}| = \varkappa$ and such that $\mathbf{A}$ is not existentially closed.

Proof. (1) Note that for any L-structure $\mathbf{A}$ and existential formula with parameters from $\mathbf{A}$ which is true in $\mathbf{A}$, this formula is true in any L-structure $\mathbf{B}$, $\mathbf{A} \subseteq \mathbf{B}$. By Theorem 10.2, there is an L-structure $\mathbf{A}_0 \in \mathcal{K}$ with $|\mathbf{A}_0| = \varkappa$.

Enumerate the existential formulae of L_{A_0} by $\{(\exists \bar{x})\varphi_i(\bar{x}; \bar{a}) : i < \varkappa\}$ and define L-structures $\mathbf{B}_i \in \mathcal{K}$, $0 \leq i < \varkappa$ inductively as follows. We put $\mathbf{B}_0 = \mathbf{A}_0$. If $\mathbf{B}_i \models (\exists \bar{x})\varphi_i(\bar{x}; \bar{a})$, then put $\mathbf{B}_{i+1} = \mathbf{B}_i$. If $\mathbf{B}_i \models \neg(\exists \bar{x})\varphi_i(\bar{x}; \bar{a})$ and there is no $\mathbf{B} \in \mathcal{K}$ with $\mathbf{B}_i \subset \mathbf{B}$ and $\mathbf{B} \models (\exists \bar{x})\varphi_i(\bar{x}; \bar{a})$, then again we put $\mathbf{B}_{i+1} = \mathbf{B}_i$. On the other hand, if we can find a $\mathbf{B} \supset \mathbf{B}_i$ with $\mathbf{B} \models (\exists \bar{x})\varphi_i(\bar{x}; \bar{a})$, put $\mathbf{B}'_{i+1} = \mathbf{B}$. By Theorem 10.2, there is an elementary substructure $\mathbf{B}_{i+1}$ of $\mathbf{B}'_{i+1}$ such that $B_i \subseteq B_{i+1}$ and $|B_{i+1}| = \varkappa$. Certainly $\mathbf{B}_{i+1} \models (\exists \bar{x})\varphi_i(\bar{x}; \bar{a})$. For a limit ordinal α we let $\mathbf{B}_\alpha = \bigcup_{i<\alpha} \mathbf{B}_i$. Since $\mathcal{K}$ is closed under unions of increasing chains, it is clear that $\mathbf{B}_\varkappa$ is in $\mathcal{K}$, $|B_\varkappa| = \varkappa$ and $\mathbf{B}_\varkappa$ satisfies the condition:

(*) for any existential formula φ with parameters from structure $\mathbf{A}_0$ if φ is true in some structure $\mathbf{B} \in \mathcal{K}$, $\mathbf{B} \supseteq \mathbf{B}_\varkappa$, then φ is true in $\mathbf{B}_\varkappa$. Put $\mathbf{A_1} = \mathbf{B}_\varkappa$. Continuing this procedure we receive an increasing chain of L-structures from $\mathcal{K}$ with cardinality $\varkappa$:

$$\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \dots \subseteq \mathbf{A}_n \subseteq \dots$$

where each pair of structures $\mathbf{A}_n$, $\mathbf{A}_{n+1}$, $n \in \omega$, satisfies the condition (*) with $\mathbf{A}_0$ and $\mathbf{A}_1$ replaced by $\mathbf{A}_n$ and $\mathbf{A}_{n+1}$ accordingly. It is clear that the structure $\mathbf{A} = \bigcup\{\mathbf{A}_n \mid n \in \omega\} \in \mathcal{K}$ is existentially closed and $|\mathbf{A}| = \varkappa$.

(2) Let $\mathbf{A}_0$ and $\mathbf{B}_0$ be infinite structures from $\mathcal{K}$ with $\mathbf{A}_0 \subseteq \mathbf{B}_0$, for which there exists an existential formula $(\exists \bar{x})\varphi(\bar{x}; \bar{a})$, where $\bar{a} \subseteq A$, such that $\mathbf{A}_0 \models \neg(\exists \bar{x})\varphi(\bar{x}; \bar{a})$ and $\mathbf{B}_0 \models (\exists \bar{x})\varphi(\bar{x}; \bar{a})$. We consider $(\exists \bar{x})\varphi(\bar{x}; \bar{a})$ to be a sentence in $L_{\bar{a}}$, and note that $\varkappa \geq |L_{\bar{a}}|$. From Proposition 4.3 and Theorem 4.4 there is an ultrafilter D over $\varkappa$ such

that $|\mathbf{A}| \geq \varkappa$, where $\mathbf{A} = \mathbf{A}_0^{\varkappa}/D$. Then $\mathbf{A} \models \neg(\exists\bar{x})\varphi(\bar{x};\bar{a})$ and $\mathbf{B} \models (\exists\bar{x})\varphi(\bar{x};\bar{a})$, where $\mathbf{B} = \mathbf{B}_0^{\varkappa}/D$, moreover $\mathbf{A} \subseteq \mathbf{B}$.

By Theorem 3.1.6 of [4] there exists an elementary substructure $\mathbf{E} \preccurlyeq \mathbf{A}$ of cardinality $\varkappa$. Clearly $\mathbf{E} \models \neg(\exists\bar{x})\varphi(\bar{x};\bar{a})$, $\mathbf{E} \in \mathcal{K}$, and $\mathbf{E}$ is not existentially closed. □

The next result is known as Lindström's Theorem.

Theorem 10.8. *Theorem 3.1.12,* [4]. *Let a class $\mathcal{K}$ of infinite L-structures be axiomatisable, categorical in some infinite cardinality $\varkappa \geqslant |L|$ and closed under unions of increasing chains. Then $\mathcal{K}$ is model complete.*

Proof. It is clear that the property of being existentially closed is preserved by isomorphism. Since all structures from $\mathcal{K}$ of the cardinality $\varkappa$ are isomorphic then from Lemma 10.7, all infinite structures from $\mathcal{K}$ are existentially closed. Theorem 10.4 now gives that the class K is model complete. □

11. Completeness of $\mathcal{SF}$, $\mathcal{P}$ and $\mathcal{F}r$

We investigate here the monoids with axiomatisable classes of strongly flat, projective and free S–acts, asking for conditions under which these classes are complete and model complete. The results of this section are all taken from [21].

Theorem 11.1. *Let S be a commutative monoid. Suppose that the class $\mathcal{SF}$ is axiomatisable. Then the following conditions are equivalent:*
(1) the class $\mathcal{SF}$ is complete;
(2) the class $\mathcal{SF}$ is model complete;
(3) the class $\mathcal{SF}$ is categorical;
(4) $\mathcal{SF} = \mathcal{F}r$;
(5) S is an Abelian group.

Proof. The implication (2)⇒(1) follows from Lemma 10.3 and the closure of the class $\mathcal{SF}$ under the coproducts. The statement (4)⇒(2) follows from Theorem 10.8, (3)⇒(1) is an immediate consequence of Proposition 10.5. It is clear that any two free S-acts of cardinality $\alpha > |L|$ are isomorphic so that (4)⇒(3) follows.

(4)⇔ (5) This is immediate from Theorem 2.6 of [14].

(1)⇒(5) Fix an arbitrary $a \in A$; it is enough to prove that $aS = S$. Let Φ be a uniform ultrafilter on ω and for $k \in \mathbb{Z}$ define $f_k \in S^\omega$ by

$$f_k(i) = \begin{cases} a^{k+i} & k+i > 0; \\ 1 & k+i \leq 0, \end{cases}$$

We will show that the left S–act $U = \bigcup_{k\in Z} Sf_k/\Phi$, which is a subact of a left S–act S^ω/Φ, is strongly flat. Since $f_k/\Phi = af_{k-1}/\Phi$ then $Sf_k/\Phi \subseteq Sf_{k-1}/\Phi$. Assume $r\,g_1/\Phi = s\,g_2/\Phi$, where $r, s \in S$, $g_1/\Phi, g_2/\Phi \in U$. There exists $k \in \mathbb{Z}$ such that $g_1/\Phi, g_2/\Phi \in S\,f_k/\Phi$, i.e. $g_1/\Phi = t_1\,f_k/\Phi$, $g_2/\Phi = t_2\,f_k/\Phi$ for some $t_1, t_2 \in S$. Hence there exists $i \in \omega$ such that $i + k > 0$, $r\,g_1(i) = s\,g_2(i)$, $g_1(i) = t_1 f_k(i) = t_1 a^{k+i}$, $g_2(i) = t_2 f_k(i) = t_2 a^{k+i}$. Thus $rt_1 a^{k+i} = st_2 a^{k+i}$, $g_1/\Phi = t_1\,a^{k+i} f_{-i}/\Phi$, $g_2/\Phi = t_2 a^{k+i}\,f_{-i}/\Phi$. Thus U satisfies condition (P); a minor adjustment yields condition (E) also. Theorem 3.5 now implies that U is strongly flat.

As S is a commutative monoid then

$$\coprod_{i\in\omega} U_i \models (\forall x)(\exists y)(x = ay),$$

where U_i are the copies of the left S–act U, $i \in \omega$. Since the class $\mathcal{SF}$ is complete and $S \in \mathcal{SF}$ then

$$\coprod_{i\in\omega} S_i \models (\forall x)(\exists y)(x = ay),$$

where S_i are the copies of the left S–act S, $i \in \omega$. Therefore $aS = S$. □

Theorem 11.2. *Let S be a monoid such that $\mathcal{P}$ is an axiomatisable class. Then the following conditions are equivalent:*

(1) the class $\mathcal{P}$ is complete;
(2) the class $\mathcal{P}$ is model complete;
(3) the class $\mathcal{P}$ is categorical;
(4) $\mathcal{P} = \mathcal{F}r$;
(5) S is a group.

Proof. We remark that if S is a group, then again from [14] we have that $\mathcal{SF} = \mathcal{P} = \mathcal{F}r$. As in the proof of Theorem 11.1 it is then enough to prove the implication (1)⇒(5). The axiomatizability of the class $\mathcal{P}$ and Theorem 8.1 imply that S is a left perfect monoid and S therefore satisfies Condition (M_R). Consequently, S has a minimal right ideal which from Proposition 8.2 (iv) is of the form eS for some $e \in E$. Proposition 8.2 (ii) gives that left ideal Se is also minimal. Clearly, $Se \in \mathcal{P}$.

Since S is local, 1 is the *only* idempotent in the $\mathcal{R}$-class and in the $\mathcal{L}$-class of 1. Suppose $e \neq 1$ so that $Se \subset S$ and $eS \subset S$.

For any $a \in S$ we put

$$X_a = \{x \in S \mid ex = a\}$$

so that by Lemma 8.4 each set X_a is a finite subset of S. Let $X_e \cap Se = \{a_1, \dots, a_n\}$, $a_i \neq a_j$ $(i \neq j)$, and choose any $t \in Se$. Notice that $X_e \cap Se \subseteq L_e$.

We will show that $|X_{et} \cap Se| = n$. Clearly $a_i t \in X_{et} \cap Se$, $1 \leq i \leq n$. Since $Se = Sa_i$ is a minimal left ideal Proposition 8.2 gives that $a_i S$ is minimal right ideal for $i \in \{1, \dots, n\}$.

Suppose $a_i t = a_j t$ where $1 \leq i, j \leq n$. Since the ideals $a_i S$ and $a_j S$ are minimal right we have that $a_i S = a_j S$ and so $a_j = a_i k$ for some $k \in S$. Since $ea_i = ea_j = e$, we deduce that $ek = ea_i k = ea_j = e$, that is, $ek = e$. Since $Se = Sa_i$ then $a_i = a_i e$ and so

$$a_j = a_i k = a_i ek = a_i e = a_i.$$

Hence $|\{a_1 t, \dots, a_n t\}| = n$.

Assume there exists $c \in X_{et} \cap Se$ such that $c \neq a_i t$ for any i, $1 \leq i \leq n$. Since $c = ce$ and the left ideal Se is minimal, we have that $Se = Sce = Sc$, that is, Sc is a minimal left ideal. Consequently $Sc = Sec$ and $c = lec$ for some $l \in S$. The minimality of the ideal eS implies the equality $eS = ecS$. Hence, there is $d \in cS$ such that $ed = e$, that is, $d \in X_e$, and $d = cr$ for some $r \in S$. Suppose $d \in Se$. Then $d = a_i$ for some i, $1 \leq i \leq n$. The equalities $ecrt = edt = ea_i t = et = ec$ imply $ecrt = ec$. Let us multiply this equation by l from the left. Then $crt = c$, i.e. $c = a_i t$. This contradicts the choice of c. Thus, $d \in (X_e \cap cS) \setminus Se$. Since $d = cr$ we have $ecr = ed = e = ecre$. Let us multiply this equation by l from the left. Then $cr = cre$, that is $d \in Se$, a contradiction.

Thus, $X_{et} \cap Se = \{a_1 t, \cdots, a_n t\}$ and $a_i t \neq a_j t$ $(i \neq j)$ for any $t \in Se$. So $Se \models \psi$ where

$$\psi : (\forall x)(ex = x \to (\exists (y_1, \dots, y_n)\Big(\bigwedge_{1\leq i\leq n} (ey_i = x) \wedge (ey = x \to \bigvee_{1\leq i\leq n} y = y_i)\Big))).$$

Since the class $\mathcal{P}$ is complete then $A = \coprod_{i\in\omega} Se_i \equiv B = \coprod_{i\in\omega} S_i$, where S_i, Se_i, $i \in \omega$, are copies of the left S–acts S and Se respectively, $i \in \omega$. As $A \models \psi$ we must have that $B \models \psi$. In particular, there are exactly n solutions to $ey = e$; but $|X_e \cap Se| = n$ and $1 \in X_e \setminus Se$, a contradiction. So $e = 1$ and (as every principal right ideal contains a minimal one), $aS = S$ for all $a \in S$. Since S is a minimal right ideal, Proposition 8.2 gives that it is a minimal left ideal ideal and so $Sa = S$ for all $a \in S$. Consequently, S is a group. □

The final result of this section follows directly from the structure of free left S–acts.

Proposition 11.3. *Let S be a monoid such that $\mathcal{F}r$ is an axiomatisable class. Then $\mathcal{F}r$ is categorical, complete and model complete.*

12. Stability

Let T be a consistent theory in the language L, let $\{x_i \mid 1 \leq i \leq n\}$ be a fixed set of variables and let $L_n = L_{\{x_1,\dots,x_n\}}$. A consistent set of sentences p of L_n is called an *n–type* of the language L. If $p \cup T$ is consistent, that is, it has a model, then p is called an *n–type over T*. If p is complete, it is a *complete n-type* and if in addition $T \subseteq p$ we say that p is a *complete n–type over T*. The set of all complete n–types over T is denoted by $S_n(T)$.

Let $\mathbf{A}$ be an L-structure, let $X \subseteq A$ and let $a \in A$. The set

$$\mathrm{tp}(a, X) = \{\varphi(x) \in L_X \mid \mathbf{A} \models \varphi(a)\}$$

is called the *type of a over X*. Clearly $\mathrm{tp}(a, X)$ is a complete 1–type over $\mathrm{Th}(\mathbf{A}, x)_{x\in X}$; we say that it is *realised* by a. By $S_n(X)$ we denote $S_n(\mathrm{Th}(\mathbf{A}, x)_{x\in X})$. Often we will write $S(X)$ instead $S_1(X)$.

A complete theory T with no finite models is called *stable in a cardinal $\varkappa$* or *$\varkappa$–stable* if $|S(X)| \leq \varkappa$ for any model $\mathbf{A}$ of the theory T and any $X \subseteq A$ of cardinal $\varkappa$. If T is $\varkappa$–stable for some infinite $\varkappa$, then T is called *stable*. If T is $\varkappa$–stable for all $\varkappa \geq 2^{|T|}$, then T is called *superstable*. An *unstable* theory is one which is not stable!

Morley proved [16] that, if a countable theory T is ω-stable, then it stable in every infinite cardinality. If an arbitrary theory T in a language L is ω-stable, then $|S_n(T)| \leqslant \omega$ for all $n \in \omega$. It follows that T is essentially countable, in the following sense. There must be a sublanguage $L' \subseteq L$, such that $|L'| = \omega$ and for each formula φ of L there is a formula φ' of L' such that for any L-structure $\mathbf{A}$ of L with $\mathbf{A} \models T$, we have that $\mathbf{A} \models \varphi$ if and only if $\mathbf{A} \models \varphi'$. Consequent upon the result of Morley, if T is ω-stable, then it is stable in every infinite cardinality.

The notion of the monster model of a complete theory is a useful tool in our investigations. In order to define such a model, we need the notions of saturation and homogeneity. An L-structure $\mathbf{A}$ is *$\varkappa$-homogeneous* for a cardinal $\varkappa$ if for any $X \subseteq A$ with $|X| < \varkappa$, any map $f : X \to A$ with

$$\mathrm{tp}(\mathbf{A}, x)_{x\in X} = \mathrm{tp}(\mathbf{A}, f(x))_{x\in X}$$

can be extended to an automorphism of $\mathbf{A}$. An L-structure $\mathbf{A}$ is *$\varkappa$-saturated* for a cardinal $\varkappa$ if for any $X \subseteq A$ with $|X| < \varkappa$, every $p \in S(\mathrm{Th}(\mathbf{A}, x)_{x\in X})$ is realised in $\mathbf{A}$.

Suppose now that T is a complete theory in L. We may find a cardinal $\overline{\varkappa}$ greater than all others under consideration and a $\overline{\varkappa}$-saturated and $\overline{\varkappa}$-homogeneous model $\mathbf{M}$ of T. The convention is that all models of T will be elementary substructures of $\mathbf{M}$ of cardinality strictly less than $\overline{\varkappa}$, and all sets of parameters will be subsets of M, with again, cardinality strictly less that $\overline{\varkappa}$. With this convention, if $\mathbf{A}$ is a model of T, $X \subseteq A$ and $a \in A$, then

$$\mathrm{tp}(a, X) = \{\varphi(x) \in L_X \mid \mathbf{M} \models \varphi(a)\}.$$

The model $\mathbf{M}$ is called the *monster model* of T. Justification of the use of the monster model, and the following result, may be found in standard stability theory texts, such as [18].

Lemma 12.1. *Let T be a complete theory in L with monster model $\mathbf{M}$ and let X be a subset of M. Suppose also that $\bar{a}, \bar{a}' \subseteq M$. Then*

$$tp(\bar{a}, X) = tp(\bar{a}', X)$$

if and only if there exists an automorphism of $\mathbf{M}$, which acts identically on X and maps $\bar{a}$ into $\bar{a}'$.

Let S be a monoid and let $\mathcal{K}$ be a class of left S–acts. Then S is called a $\mathcal{K}$–*stabiliser* ($\mathcal{K}$–*superstabiliser*, $\mathcal{K}$–ω–*stabiliser*) if $\mathrm{Th}(A)$ is stable (superstable, ω–stable) for any infinite left S–act $A \in \mathcal{K}$. If $\mathcal{K}$ is the class of all left S-acts, then a $\mathcal{K}$–stabiliser ($\mathcal{K}$–superstabiliser, $\mathcal{K}$–ω–stabiliser) is referred to more simply as a stabiliser (superstabiliser, ω–stabiliser).

13. Superstability of $\mathcal{SF}$, $\mathcal{P}$ and $\mathcal{F}r$

Now we begin to consider the stability questions for S–acts. The results of this section are all taken from [22].

Lemma 13.1. *Let S be a monoid satisfying (CFRS). Then for any $A \in \mathcal{SF}$, $a \in A$, $s \in S$*

$$|\{x \in A \mid sx = a\}| \leq n_s.$$

Proof. Assume there exists $A \in \mathcal{SF}$, $s \in S$ and $a_0, \ldots, a_{n_s} \in A$ such that $sa_i = sa_j$, $a_i \neq a_j$ $(i \neq j)$. By induction on $n \leq n_s$ we will show that there exist $b \in A$, $r_0, \ldots, r_n \in S$ such that $a_i = r_i b$, $sr_i = sr_j$ for any $i, j \in \{0, \ldots, n\}$. Let $n = 1$. Since $A \in \mathcal{SF}$ there are $r_0', r_1' \in S$ and $b' \in A$ such that $sr_0' = sr_1'$, $a_0 = r_0'b'$ and $a_1 = r_1'b'$. Suppose there exist $r_0'', \ldots, r_{n-1}'' \in S$ and $b'' \in A$ such that $sr_i'' = sr_j''$ and $a_i = r_i''b''$ for any $i, j \in \{0, \ldots, n-1\}$. As $A \in \mathcal{SF}$ and $sr_0''b'' = sa_n$, there exist $r, r_n \in S$ and $b \in A$ such that $sr_0''r = sr_n$, $b'' = rb$ and $a_n = r_n b$. Let $r_i = r_i''r$ $(0 \leq i \leq n-1)$. Then $a_i = r_i''b'' = r_i''rb = r_i b$, $sr_i = sr_i''r = sr_j''r = sr_j$ for any $i, j \in \{0, \ldots, n-1\}$. Thus there exist $r_0, \ldots, r_{n_s} \in S$ such that $r_i \neq r_j$ $(i \neq j)$ and $sr_i = sr_j$ for any $i, j \in \{0, \ldots, n_s\}$, contradicting the fact that S satisfies (CFRS). $\square$

Lemma 13.2. *Let S be a monoid satisfying (CFRS), $B \in \mathcal{SF}$, $B \preccurlyeq C$. Then $\bigcup_{c \in C \setminus B} Sc \cap B = \emptyset$.*

Proof. Let the given conditions hold and suppose that $b \in \bigcup_{c \in C \setminus B} Sc \cap B$. Then there exists $c_1 \in C \setminus B$ and $s \in S$ such that $b = sc_1$. The formula

$$(\exists x_1 \ldots x_k)(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \bigwedge_{1 \leq i \leq k} y = sx_i)$$

will be denoted by $\varphi_k(y)$.

Clearly $C \models \varphi_1(b)$. We show by induction on k that $C \models \varphi_k(b)$ for all $k \geq 1$. Suppose that $k \geq 1$ and $C \models \varphi_k(b)$, that is $b = sc_i$, $1 \leq i \leq k$, where $c_1, \ldots, c_k$ are distinct elements of C. Since B is an elementary substructure of C then $B \models \varphi_k(b)$, that is $b = sb_i$, $1 \leq i \leq k$, where $b_1, \ldots, b_k$ are distinct elements of B. Since also $sc_1 = b$ and $c_1 \notin B$ we have that $C \models \varphi_{k+1}(b)$. Therefore $C \models \varphi_k(b)$ for any $k \geq 1$. Since B is an elementary substructure of C then $B \models \varphi_k(b)$ for any $k \geq 1$, contradicting Lemma 13.1. □

Lemma 13.3. *Let S be a monoid S satisfying (CFRS), $B \in \mathcal{SF}$, $B \preccurlyeq M$ where M is the monster model of $Th(B)$, and let $c_1, c_2 \in C \backslash B$. Then*

$$tp(c_1, B) = \ tp(c_2, B) \Leftrightarrow tp(c_1, \emptyset) = \ tp(c_2, \emptyset).$$

Consequently, for any subset $B' \subseteq B$, we have

$$tp(c_1, B') = \ tp(c_2, B') \Leftrightarrow tp(c_1, \emptyset) = \ tp(c_2, \emptyset).$$

Proof. Let the conditions of the Lemma hold. Necessity is obvious. To prove sufficiency, suppose that $\mathrm{tp}(c_1, \emptyset) = \mathrm{tp}(c_2, \emptyset)$. From Lemma 12.1 there exists an automorphism $\phi : M \to M$ such that $\phi(c_1) = c_2$. According to Lemma 13.2, M is the disjoint union of B, C_1 and C_2 and D, where for $i = 1, 2$, C_i is the connected component containing c_i, and $D = M \setminus (B \cup C_1 \cup C_2)$.

Since S-morphisms preserve the relation $\sim$ we must have that $\phi : C_1 \to C_2$ is an S-isomorphism. If $C_1 = C_2$ then we define $\psi : C \to C$ by

$$\psi|_{C_1} = \phi|_{C_1} \text{ and } \psi|_{B \cup D} = I_{B \cup D}$$

and if $C_1 \cap C_2 = \emptyset$ we define ψ by

$$\psi|_{C_1} = \phi|_{C_1}, \psi|_{C_2} = \phi^{-1}|_{C_2} \text{ and } \psi|_{B \cup D} = I_{B \cup D}.$$

Clearly in either case ψ is an S-automorphism of the left S–act M. Since $\psi(c_1) = c_2$ and $\psi|_B = I_B$ we have that $\mathrm{tp}(c_1, B) = \mathrm{tp}(c_2, B)$. □

Theorem 13.4. *Let S be a monoid such that $\mathcal{SF}$ is axiomatisable and S satisfies (CFRS). Then S is $\mathcal{SF}$-superstabiliser.*

Proof. Suppose the monoid S satisfies (CFRS), $T = \mathrm{Th}(A)$ is the complete theory of some infinite strongly flat left S–act A, M is the monster model of T and $B \subseteq M$ with $|B| = \varkappa \geq 2^{|T|}$. By Theorem 10.2, there is an S-act $B' \preccurlyeq M$ with $B \subseteq B'$ and $|B'| = \varkappa$. Since $\mathcal{SF}$ is axiomatisable, M and B' are strongly flat. According to Lemma 13.3, $|\{\mathrm{tp}(x, B) \mid x \in M \setminus B'\}| = |\{\mathrm{tp}(x, \emptyset) \mid x \in M \setminus B'\}| \leq 2^{|T|} \leq \varkappa$. Furthermore, $|\{\mathrm{tp}(x, B) \mid x \in B'\}| = \varkappa$. Thus, $|S(B)| \leq \varkappa$ and the theory T is superstable. □

Corollary 13.5. *If S is a left cancellative monoid such that $\mathcal{SF}$ is axiomatisable, then S is an $\mathcal{SF}$–superstabiliser.*

Proof. Let S be a left cancellative monoid, $s \in S$. Since s is a left 1-cancellable element then $|\{x \in S \mid sx = t\}| = 1$ for any $t \in S$. So S satisfies (CFRS) and by Theorem 13.4, S is an $\mathcal{SF}$-superstabiliser. □

For an example of a monoid S satisfying the conditions of Corollary 13.5 we can take the free monoid X^* on a set X, which is certainly left cancellative. Since X^* is also right cancellative, $r(s, t) = \emptyset$ if $s \neq t$, and if $s = t$ then $r(s, t) = X^*$, which is principal. If neither s nor t is a prefix of the other, then $R(s, t) = \emptyset$; on the other hand if $s = tw$ then $R(s, t) = (1, w)X^*$ and dually if s is a prefix of t.

Lemma 13.6. *Let S be a monoid such that $\mathcal{SF}$ is axiomatisable and Condition (A) holds. Then S satisfies (CFRS).*

Proof. Suppose to the contrary that there is $s_1 \in S$ such that for any $i \in \omega$ there exists $b_i \in S$ such that:

$$T_i = |\{x \in S \mid s_1 x = b_i\}| \geq i.$$

We let $x_{i,1}, \ldots, x_{i,i}$ be distinct elements of T_i. Let D be a uniform ultrafilter over ω, let $S_0 = S^\omega/D$, $\bar{a} \in S_0$, $\bar{a} = a/D$, $a(i) = b_i$ for any $i \in \omega$. For $i \in \omega$ we define $x_i \in S^\omega$ by

$$x_i(j) = \begin{cases} 1 & j < i \\ x_{j,i} & j \geq i \end{cases}$$

and put $\overline{x_i} = x_i/D$. It is clear that the $\overline{x_i}$'s are distinct and so $|\{x \in S_0 \mid s_1 x = \bar{a}\}| \geq \omega$. In view of the axiomatizability of the class $\mathcal{SF}$ we have $S_0 \in \mathcal{SF}$. Now choose a cardinal $\alpha > |S|$. According to Proposition 4.3 and Theorem 4.4 we can choose an ultrafilter Φ over α such that $|\{x \in S_0 \mid s_1 x = \bar{a}\}^\alpha/\Phi| > |S|$. Denote S_0^α/Φ by S_1, so that as $\mathcal{SF}$ is axiomatisable, S_1 is strongly flat, put $a_0 = a'/\Phi$, where $a'(\beta) = \bar{a}$ for any $\beta < \alpha$, and let $A_1 = \{x \in S_1 \mid s_1 x = a_0\}$, so that $|A_1| > |S|$. As $|Sa_0| \leq |S|$ there exists $a_1 \in A_1$ such that $Sa_0 \subset Sa_1$.

Let $k \in \mathbb{N}$. Assume that for $0 < i < k$ the sets $A_i \subseteq S_1$, elements $a_i \in A_i$ and $s_i \in S$ are defined such that $Sa_{i-1} \subset Sa_i$, $A_i = \{x \in S_1 \mid s_i x = a_{i-1}\}$ and $|A_i| > |S|$. Let us define $A_k \subseteq S_1$, $a_k \in A_k$, $s_k \in S$ such that $A_k \subseteq \{x \in S_1 \mid s_k x = a_{k-1}\}$, $Sa_{k-1} \subset Sa_k$ and $|A_k| > |S|$. From Theorem 7.3, since it is certainly not empty, $R(s_{k-1}, s_{k-1}) = \{(x,y) \in S^2 \mid s_{k-1}x = s_{k-1}y\} = \bigcup\limits_{0 \leq i \leq m} (u_i, v_i)S$ for some $m \in \omega$, $u_i, v_i \in S$ $(0 \leq i \leq m)$. Clearly $s_{k-1}a_{k-1} = s_{k-1}b$ for all $b \in A_{k-1}$ so there exists i, $0 \leq i \leq m$, such that $|\{x \in S_1 \mid u_i x = a_{k-1},\ v_i x \in A_{k-1}\}| > |S|$. Let $s_k = u_i$, $A_k = \{x \in S_1 \mid s_k x = a_{k-1}\}$. As $|A_k| > |S|$ and $|Sa_{k-1}| \leq |S|$ then there exists $a_k \in A_k$ such that $Sa_{k-1} \subset Sa_k$. Thus, there is the ascending chain of cyclic S-subacts Sa_i $(i \in \omega)$ of the left S–act S_1, contradicting our hypothesis that Condition (A) holds. □

From Lemma 13.6 and Theorem 13.4 our next result immediately follows.

Corollary 13.7. *For a monoid S, if the class $\mathcal{SF}$ is axiomatisable and S satisfies Condition (A), then S is an $\mathcal{SF}$–superstabiliser.*

Corollary 13.8. *If the class $\mathcal{P}$ is axiomatisable for a monoid S, then S is a $\mathcal{P}$–superstabiliser.*

Proof. Let the class $\mathcal{P}$ be axiomatisable. From Theorem 8.6, $\mathcal{SF}$ is axiomatisable and S is a left perfect monoid so that, according to Theorem 8.1, S satisfies Condition (A) and $\mathcal{SF} = \mathcal{P}$. Now Corollary 13.7 yields that S is a $\mathcal{P}$–superstabiliser. □

Corollary 13.9. *If the class $\mathcal{F}r$ is axiomatisable for a monoid S, then S is an $\mathcal{F}r$–superstabiliser.*

Proof. Let $\mathcal{F}r$ be axiomatisable. From Theorem 9.1 the class $\mathcal{P}$ is axiomatisable. Now Corollary 13.8 says that S is $\mathcal{P}$–superstabiliser, so that certainly S is an $\mathcal{F}r$–superstabiliser. □

14. ω-stability of $\mathcal{SF}$, $\mathcal{P}$ and $\mathcal{F}r$

All results from this section are again taken from [22].

Lemma 14.1. *If θ is a left congruence of a monoid S then $1/\theta$ is a submonoid of S.*

Proof. If $u, v \in 1/\theta$ then $1\,\theta\, u$ so that as θ is left compatible,

$$vu\,\theta\, v\,1 = v\,\theta\,1.$$

□

Lemma 14.2. *Let θ_1, θ_2 be strongly flat left congruences of a monoid S. Then*

$$\theta_1 = \theta_2 \text{ if and only if } 1/\theta_1 = 1/\theta_2.$$

Proof. The necessity is obvious. Suppose now that $1/\theta_1 = 1/\theta_2$. Let $u, v \in S$. If $u\,\theta_1\, v$ then from Theorem 3.4 there exists $s \in S$ such that $s\,\theta_1\,1$ and $us = vs$. Hence $s\,\theta_2\,1$ and again from Theorem 3.4, $u\,\theta_2\, v$. □

Lemma 14.3. *Let S be a monoid such that the class of left S-acts satisfying Condition (E) is axiomatisable. Let M be a left S-act satisfying Condition (E) and let Sa be a connected component of M. Then the relation $\theta_a = \{(s,t) \in S^2 \mid sa = ta\}$ is a strongly flat left congruence of the monoid S and the mapping $\phi : Sa \to S/\theta_a$ given by $\phi(sa) = s/\theta_a$ $(s \in S)$ is an isomorphism of left S–acts.*

Proof. It is obvious that the relation θ_a is a left congruence of S: we claim that θ_a is flat. Suppose that $s\theta_a t$, so that $sa = ta$. From Proposition 7.1 we have that $r(s,t) \neq \emptyset$ and the following sentence is true in M

$$(\forall x)(sx = tx \to (\exists y) \bigvee_{0 \le i \le k} x = u_i y)$$

where $\{u_0, \ldots, u_k\}$ is a set of generators of the right ideal $r(s,t)$. Then $a = u_i b$ for some i, $0 \le i \le k$, and $b \in M$. Since Sa is a connected component of a left S-act M, then $Sa = Sb$, i.e. $b = ka$ for some $k \in S$. Consequently $a = u_i ka$, that is, $1\theta_a u_i k$. Furthermore the equation $su_i = tu_i$ implies the equation $s(u_i k) = t(u_i k)$. According to Corollary 3.6, θ_a is a flat left congruence of the monoid S. The mapping $\phi : Sa \to S/\theta_a$, where $\phi(sa) = s/\theta_a$, is obviously an S-isomorphism. □

Lemma 14.4. *If for a monoid S the class $\mathcal{SF}$ is axiomatisable and S satisfies Condition (A), then any $M \in \mathcal{SF}$ is a coproduct cyclic left S–acts.*

Proof. Suppose that $\mathcal{SF}$ is axiomatisable and S satisfies Condition (A). Let $M \in \mathcal{SF}$ and write $M = \coprod_{i \in I} M_i$, where M_i is a connected component of M $(i \in I)$. It is clear that $M_i \in \mathcal{SF}$ $(i \in I)$. Assume that M_i is not a cyclic left S–act for some $i \in I$. Then for any $a \in M_i$ there is $b \in M_i$ such that $Sa \subset Sa \cup Sb$. It is easy to see that for elements u, v of a connected component of a strongly flat S-act there is an element w such that $Su \cup Sv \subseteq Sw$. Since $M_i \in \mathcal{SF}$ then there exists $c \in M_i$ such that $Sa \cup Sb \subseteq Sc$, so that $Sa \subset Sc$. Thus there exists $a_j \in M_i$ $(j \in \omega)$ such that $Sa_j \subset Sa_{j+1}$, contradicting the fact that S satisfies Condition (A). □

For a subset X of a monoid S we put $\rho_X = \rho(X \times X)$, that is, ρ_X is the left congruence on S generated by $X \times X$.

Lemma 14.5. *Let T be a submonoid of a monoid S. Then T is a class of ρ_T if and only if T is a right unitary submonoid.*

Proof. Suppose first that T is a class of ρ_T and $st \in T$, where $s \in S$, $t \in T$. Since $t\,\rho_T\,1$, then $st\,\rho_T s$ and $s \in T$. So T is a right unitary submonoid.

Conversely, suppose that T is a right unitary submonoid of S and $x\,\rho_T\,y$ where $x \in T$. We claim that $y \in T$. By Proposition 2.7, $x = y$ (so that certainly $y \in T$), or there exist $n \in \omega$, $t_0, \ldots, t_{2n+1} \in T$, $s_0, \ldots, s_n \in S$ such that

$$\text{(v)} \qquad x = s_0t_0,\ s_0t_1 = s_1t_2, \ldots, s_it_{2i+1} = s_{i+1}t_{2i+2}, \ldots, s_nt_{2n+1} = y$$

for any i, $0 \le i \le n-1$. By the induction on i we prove that $s_i \in T$. Note that $s_0t_0, t_0 \in T$ give that $s_0 \in T$ since T is right unitary. For $i > 0$, if $s_{i-1} \in T$, then the equality $s_{i-1}t_{2i-1} = s_it_{2i}$ implies $s_it_{2i} \in T$. Since T is a right unitary submonoid of S then $s_i \in T$. So $s_n \in T$ and thus $y = s_nt_{2n+1} \in T$. □

Lemma 14.6. *Let T be a right unitary submonoid of S. Then T is right collapsible if and only if the left congruence ρ_T is strongly flat.*

Proof. Let T be a right unitary submonoid of S, and suppose first that T is right collapsible and $x\,\rho_T\,y$ where $x, y \in S$. If $x = y$ then $x\,1 = y\,1$. Otherwise there exist $n \in \omega$, $t_0, \ldots, t_{2n+1} \in T$, $s_0, \ldots, s_n \in S$ such that (v) holds. From Lemma 2.4 there exists $r \in T$ such that $t_ir = t_jr$, $0 \le i, j \le 2n+1$. Hence

$$xr = s_0t_0r = s_0t_1r = s_1t_2r = \ldots = s_nt_{2n+1}r = yr$$

and certainly $r\,\rho_T\,1$. From Corollary 3.6, ρ_T is a strongly flat left congruence.

Conversely, assume that ρ_T is a strongly flat left congruence and $x, y \in T$. Then $x\,\rho_T\,y$ and from Corollary 3.6, and Lemma 14.5 there exists $z \in T$ such that $xz = yz$. Hence T is right collapsible. □

We will write CU^S for the set of all right collapsible and right unitary submonoids of a monoid S.

Theorem 14.7. *Let S be a monoid with $|S| \le \omega$. Suppose that the class $\mathcal{SF}$ is axiomatisable and S satisfies Condition (A). Then S is an $\mathcal{SF}$–ω–stabiliser if and only if $|CU^S| \le \omega$.*

Proof. Suppose the given hypotheses hold. Assume first that S is an $\mathcal{SF}$–ω–stabiliser. Let C denote the set of strongly flat left congruences on S, so that $C \neq \emptyset$ as the equality relation ι lies in C. Put

$$A = \coprod\{S/\rho \mid \rho \in C\}$$

and let

$$B = \coprod_{i \in \omega} A_i,$$

where the A_i's are disjoint copies of the left S–act A and $1_i/\rho \in A_i$ is a copy of $1/\rho \in A$. It is clear that A and B lie in $\mathcal{SF}$. Put $T = \mathrm{Th}(B)$.

By assumption, the theory T is ω–stable and $|S(\emptyset)| \le \omega$. Let $U \in CU^S$ so that from Lemma 14.6, ρ_U is a strongly flat left congruence of S, that is, $S/\rho \in \mathcal{SF}$, and from Lemma 14.5, $U = 1/\rho_U$. Suppose $\alpha : CU^S \to S(\emptyset)$ is the mapping such that $\alpha(U) = \mathrm{tp}(1_0/\rho_U, \emptyset)$. We claim that α is an injection. Let $U, V \in CU^S$ with $U \neq V$ and without loss of generality choose $t \in U \setminus V$. Then $t \cdot 1_0/\rho_U = 1_0/\rho_U$ and $t \cdot 1_0/\rho_V \neq 1_0/\rho_V$. Hence $tx = x$ lies in $\mathrm{tp}(1_0/\rho_U, \emptyset)$ but not in $\mathrm{tp}(1_0/\rho_V, \emptyset)$. Hence α is an injective mapping and consequently $|CU^S| \le \omega$.

Conversely, assume that $|CU^S| \leq \omega$. Let $s, t \in S$. For $d \in D \in \mathcal{SF}$, where Sd is connected, we denote the relation $\{(s,t) \in S^2 \mid sd = td\}$ by θ_d; clearly, θ_d is a left congruence on S. In view of the axiomatisability of $\mathcal{SF}$, Theorem 7.3 gives that the set $r(s,t)$ is empty or is finitely generated as a right ideal of S. Lemma 14.3 now gives that the relation θ_d is a strongly flat left congruence of S.

Let $F \in \mathcal{SF}$ with $|F| \geq \omega$ and let $T = \mathrm{Th}(F)$; we must show that T is ω-stable. To this end, let M be the monster model of T, so that as $\mathcal{SF}$ is axiomatisable, certainly M is a strongly flat left S-act. Let $A \subseteq M$ with $|A| \leq \omega$; by Theorem 10.2, there is a model B of T with $A \subseteq B$ and $|B| = \omega$. By construction of M, $B \preccurlyeq M$.

Let $c_1, c_2 \in M$. According to Lemma 14.4, M is a coproduct of cyclic left S-acts. Hence there exists $d_1, d' \in M$ such that $c_1 \in Sd_1$, $c_2 \in Sd'$ with Sd_1, Sd' connected components of M. It is clear that either $Sd_1 = Sd'$, or $Sd_1 \cap Sd' = \emptyset$. We claim that for $c_1, c_2 \in C \setminus B$, $\mathrm{tp}(c_1, A) = \mathrm{tp}(c_2, A)$ if and only if

(vi) $\exists d_2 \in M, t \in S$ such that $Sd_2 = Sd', 1/\theta_{d_1} = 1/\theta_{d_2}, c_1 = td_1, c_2 = td_2$.

Let $\mathrm{tp}(c_1, A) = \mathrm{tp}(c_2, A)$. Then there exists an S-automorphism $\phi : C \to C$ such that $\phi(c_1) = c_2$, and $\phi|_A = I_A$. Putting $d_2 = \phi(d_1)$ we have that Sd_2 is a connected component containing c_2, so that $Sd_2 = Sd'$. Now $c_1 = td_1$ for some $t \in S$, so that $c_2 = td_2$. Furthermore

$$ud_1 = vd_1 \Leftrightarrow ud_2 = vd_2$$

for any $u, v \in S$, i.e. $\theta_{d_1} = \theta_{d_2}$, in particular $1/\theta_{d_1} = 1/\theta_{d_2}$. Hence (vi) holds.

Suppose conversely that (vi) holds. From Lemma 14.1, the sets $1/\theta_{d_1}, 1/\theta_{d_2}$ are submonoids of S. Since they are strongly flat left congruences and $1/\theta_{d_1} = 1/\theta_{d_2}$, Lemma 14.2 gives that $\theta_{d_1} = \theta_{d_2}$ and so $S/\theta_{d_1} = S/\theta_{d_2}$. From Lemma 14.3 there is an isomorphism $\phi : Sd_1 \to Sd_2$ of left S–acts such that $\phi(d_1) = d_2$. Since $c_1 = td_1$ and $c_2 = td_2$, then $\phi(c_1) = c_2$. From Lemma 13.6, S has Condition (CFRS) so that by Lemma 13.2, $Sd_i \cap B = \emptyset$ for $i \in \{1, 2\}$. Define an automorphism $\psi : M \to M$ as follows: $\psi|_{Sd_1} = \phi|_{Sd_1}$, $\psi|_{Sd_2} = \phi^{-1}|_{Sd_2}$ (unless $Sd_1 = Sd_2$) and $\psi|_{M\setminus(Sd_1 \cup Sd_2)} = I$. Since in either case $\psi(c_1) = c_2$, and $\psi|_A = I_A$, then $\mathrm{tp}(c_1, A) = \mathrm{tp}(c_2, A)$.

Thus the type of any element $m \in M \setminus B$ over A is determined by some element of S and by some flat congruence θ_d, where Sd is a connected component and $m \in Sd$.

If Sm is a connected component of M, then as noted above, θ_m is a strongly flat left congruence of S. Let $U = 1/\theta_m$. We know that U is a submonoid of S; if $t, st \in U$ then $tm = m = stm$ so that $sm = m$ and $m \in U$. Hence U is right unitary. On the other hand, if $p, q \in U$, then $pm = m = qm$. As M is strongly flat we have that $pr = qr$ for some $r \in S$ and with $m = rk$ for some $k \in Sm$. Since Sm is a connected component we deduce that $k = r'm$ for some $r' \in S$. Then $prr' = qrr'$ and $m = rr'm$, so that $rr' \in U$ and U is right collapsible, that is, $U \in CU^S$. From Lemma 14.2,

$$|\{\theta_c \mid Sc \text{ is a connected component of } M\}| \leq |CU^S| \leq \omega.$$

Since

$$|\{\mathrm{tp}(b, A) \mid b \in B\}| \leq |B| = \omega$$

we deduce that $|S(A)| \leq \omega$ as required. □

From Theorem 7.3 and Theorem 14.7 we have

Corollary 14.8. *If S is a finite monoid then S is an $\mathcal{SF}$–ω–stabiliser.*

Corollary 14.9. *If S is a countable group then S is an $\mathcal{SF}$–ω–stabiliser.*

Proof. Let S be a countable group. As remarked in [10], both $\mathcal{SF}$ and $\mathcal{P}$ are axiomatisable, so that S is left perfect and Condition (A) holds. Let U be a right collapsible submonoid of S; then for any $u, v \in U$ we have that $ur = vr$ for some $r \in S$. We deduce that $U = \{1\}$ and $|CU^S| = 1$. From Theorem 14.7, the monoid S is an $\mathcal{SF}$–ω–stabiliser. □

Corollary 14.10. *If $|S| \leq \omega$ and the class $\mathcal{P}$ is axiomatisable, then S is an $\mathcal{P}$–ω–stabiliser.*

Proof. Suppose that $|S| \leq \omega$ and $\mathcal{P}$ is axiomatisable. From Theorem 8.6, $\mathcal{SF}$ is axiomatisable and S is a left perfect monoid. Hence from Theorem 8.1, $\mathcal{SF} = \mathcal{P}$ and S satisfies Condition (A). Let us construct an embedding ϕ of the set CU^S into the set E of idempotents of S.

Let $U \in CU^S$. From Lemma 14.5, $U = 1/\rho_U$ and from Lemma 14.6, ρ_U is a strongly flat left congruence. In view of the equality $\mathcal{SF} = \mathcal{P}$ there exist an idempotent $e \in E$ and an S-isomorphism $\alpha : S/\rho_U \to Se$. Put $a = 1/\rho_U$, so that $S/\rho_U = Sa$ and $u\,\rho_U\,v$ if and only if $ua = va$. Now $\alpha(a) = se$ and $\alpha(ta) = e$ for some $s, t \in S$. It is easy to see that consequently, $e = tse$, $eta = ta$ and $seta = a = sta$. Hence $setset = seet = set$, i.e. $g = set$ is an idempotent of S and $ga = a$. Moreover, for any $u, v \in S$,

$$\begin{aligned} u\,\rho_U\,v &\Leftrightarrow ua = va \\ &\Rightarrow use = vse \\ &\Rightarrow ug = vg \\ &\Rightarrow useta = uga = vga = vseta \\ &\Rightarrow ua = va \end{aligned}.$$

Now define $\phi : CU^S \to E$ by $\phi(U) = g$. If $\phi(U) = \phi(V)$, then $\rho_U = \rho_V$, so that $U = 1/\rho_U = 1/\rho_V = V$. Thus ϕ is an injection and $|CU^S| \leq |E| \leq |S| \leq \omega$. □

Corollary 14.11. *If $|S| \leq \omega$ and the class $\mathcal{F}r$ is axiomatisable than S is an $\mathcal{F}r$–ω–stabiliser.*

Proof. If $\mathcal{F}r$ is axiomatisable then by Theorem 9.1, $\mathcal{P}$ is axiomatisable. From Corollary 14.10, S is a $\mathcal{P}$–ω–stabiliser, and hence in particular an $\mathcal{F}r$–ω–stabiliser. □

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Department of Mathematics, University of York, Heslington, York YO1 5DD, UK
E-mail address: `varg1@york.ac.uk`

Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow, Russia
E-mail address: `mikhalev@shade.msu.ru`

Department of Algebra, Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail address: `palyutin@math.nsc.ru`

Institute of Mathematics and Computer Science, Far East State University, Vladivostok, Russia
E-mail address: `stepltd@mail.primorye.ru`