FINITE PROPER COVERS IN A CLASS OF FINITE SEMIGROUPS WITH COMMUTING IDEMPOTENTS

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Abstract. Weakly left ample semigroups are a class of semigroups that are $(2,1)$-subalgebras of semigroups of partial transformations, where the unary operation takes a transformation α to the identity map in the domain of α. It is known that there is a class of *proper* weakly left ample semigroups whose structure is determined by unipotent monoids acting on semilattices or categories. In this paper we show that for every finite weakly left ample semigroup S, there is a *finite proper* weakly left ample semigroup $\widehat{S}$ and an onto morphism from $\widehat{S}$ to S which separates idempotents. In fact, $\widehat{S}$ is actually a $(2,1)$-subalgebra of a symmetric inverse semigroup, that is, it is a left ample semigroup (formerly, left type A).

1. Introduction

The powerful technique of covers is now well established in semigroup theory. The idea is, given a semigroup S, to find a semigroup $\widehat{S}$ having a better understood structure than that of S, and an onto morphism ϕ of a specific kind from $\widehat{S}$ to S. With the right condition on ϕ, the behaviour of S is closely linked to that of $\widehat{S}$.

The celebrated papers [15] and [16] of McAlister are the flagship of this theory. In [15] McAlister showed that, given any inverse semigroup S, there is a proper inverse semigroup $\widehat{S}$ and an idempotent separating morphism from $\widehat{S}$ to S. A structure theorem for proper inverse semigroups in terms of groups acting on semilattices is given in [16].

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Many authors have extended and adapted the work of McAlister to wider classes of semigroups. See, for example [18],[19] and [9]. Powerful new techniques have been introduced along the way, most notably the construction of proper inverse semigroups using groups acting on categories [13]. One direction in which to generalise McAlister's work is to keep the condition that S be regular but weaken the condition that the idempotents of S commute, that is, they form a semilattice (see [18]). Another path, and the one we follow, is to drop the regularity of S but retain commutativity of the idempotents. Specifically, we consider a class of semigroups we call weakly left ample. Weakly left ample semigroups are $(2,1)$-subalgebras of semigroups of partial transformations $\mathcal{PT}_X$ of a set X. The binary operation is composition of functions and the unary operation $\alpha \mapsto \alpha^+$ is given by $\alpha^+ = I_{\text{dom } \alpha}$, the identity map in the domain of α ([12],[11],[10]). Of course, $\mathcal{PT}_X$ contains idempotents other than those of the form I_Y, $Y \subseteq X$. Weakly left ample semigroups are precisely the $(2,1)$-subalgebras of $\mathcal{PT}_X$ containing only idempotents of the form I_Y. On the other hand, all the idempotents of the symmetric inverse semigroup $\mathcal{I}_X$ on a set X are of the form I_Y. The $(2,1)$-subalgebras of $\mathcal{I}_X$ are called *left ample* (formerly, left type A) semigroups. Clearly, inverse semigroups are left ample and left ample semigroups are weakly left ample. The technical definitions of weakly left ample and left ample semigroups in terms of the equivalence relations $\widetilde{\mathcal{R}}$ and $\mathcal{R}^*$ and quasi-identities are reserved for the next section.

A (weakly) left ample semigroup S has a least congruence σ such that S/σ is a right cancellative monoid (respectively, a unipotent monoid). We say that S is *proper* if $\mathcal{R}^* \cap \sigma$ (respectively, $\widetilde{\mathcal{R}} \cap \sigma$) is trivial. The structure of proper (weakly) left ample semigroups in terms of right cancellative (respectively, unipotent) monoids acting on semilattices, or alternatively, on categories, is given in [7],[4]. Moreover every (weakly) left ample semigroup S has a proper cover $\widehat{S}$. To be precise, there is a proper (weakly) left ample semigroup $\widehat{S}$ and an onto morphism $\phi : \widehat{S} \to S$ which is idempotent separating [8],[4].

In this paper we concentrate on *finite* weakly left ample semigroups. All the techniques to date for constructing proper covers in this case produce *infinite* covers. McAlister [15] introduced an approach in the inverse case that generates finite proper covers of finite inverse semigroups, and Fountain [3] extended this to the left ample case. However, this construction fails in the weakly left ample case. The main aim of this paper is to show that every finite weakly left ample semigroup has a *finite* proper cover. On the way we discuss connections with E-unitary weakly left ample semigroups and give a number of examples.

The layout of the paper is as follows.

Section 2 contains details of the relations $\widetilde{\mathcal{R}}$ and $\mathcal{R}^*$ and the quasi-identities that determine (weakly) left ample semigroups. We define σ and the concept of proper for (weakly) left ample semigroups. A proper (weakly) left ample semigroup is E-unitary, that is, the idempotents form a unitary subsemigroup. However, unlike the corresponding situation for inverse semigroups, the converse is not true. In [3] Fountain gives an example of an *infinite* left ample semigroup that is E-unitary but not proper. Since this paper is concerned with the finite case, we provide a finite example. The significance of this example is that the standard way of constructing a cover via the natural morphism onto the greatest group morphic image, does not produce the finite proper cover for which we seek.

Any unipotent monoid that is not left cancellative is proper weakly left ample but not left ample. Such a monoid must of course be infinite. In Section 3 we show that if S is a finite proper weakly left ample semigroup, then S is left ample. We also give an example of a finite weakly left ample semigroup that is not left ample.

In Section 4 we consider the special case of finite weakly left ample semigroups that have a single generator (as a $(2,1)$-subalgebra of $\mathcal{PT}_X$). We show that all such semigroups are left ample and if such a semigroup is E-unitary, then it is proper.

The final section contains our main result. We show that, given any finite weakly left ample semigroup S we can find a finite proper left ample cover. We proceed by taking a set of generators of S as a $(2,1)$-subalgebra of $\mathcal{PT}_X$. If all the generators are one to one, then S is left ample and the existence of a finite left ample cover is known [3]. Otherwise, we find a finite $(2,1)$-subalgebra S' of $\mathcal{PT}_Y$ for some Y, having a set of generators which are 'more one to one' than those of S (a concept explained precisely in Section 5) and an onto morphism from S' to S that separates idempotents. We know that S' is weakly left ample and may obtain the required result by finite induction.

2. Preliminaries

There are two ways of approaching (weakly) left ample semigroups, both of which we describe in this section. We also give an example of a finite E-unitary left ample semigroup that is not proper. Some of the material in this section falls into the category of folklore. Details may be found on the homepage of the second author at *http://www-users.york.ac.uk/~varg1/* under '(Weakly) left E-ample semigroups'.

We begin with the relations $\widetilde{\mathcal{R}}$ and $\mathcal{R}^*$. The relation $\widetilde{\mathcal{R}}$ is defined on a semigroup S by the rule that for any $a, b \in S$, $a \,\widetilde{\mathcal{R}}\, b$ if and only if for all $e \in E(S)$,

$$ea = a \text{ if and only if } eb = b$$

where $E(S)$ denotes the set of idempotents of S. The relation $\mathcal{R}^*$ is defined by the rule that for any $a, b \in S$, $a \,\mathcal{R}^*\, b$ if and only if for all $x, y \in S^1$,

$$xa = ya \text{ if and only if } xb = yb.$$

The relation $\mathcal{R}^*$ has the alternative description that elements of S are $\mathcal{R}^*$-related if and only if they are $\mathcal{R}$-related in an oversemigroup of S [14].

For any semigroup S, we have

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$$

and if S is regular, these relations are equal. They are, in general, distinct. This may be easily seen by considering any commutative cancellative monoid that is not a group, and any unipotent monoid that is not right cancellative.

Clearly, $\widetilde{\mathcal{R}}$ and $\mathcal{R}^*$ are equivalence relations. A semigroup S is *left semiadequate* if the idempotents of S form a semilattice and every element of S is $\widetilde{\mathcal{R}}$-related to an idempotent. It is easy to see that in this case, there is a *unique* idempotent in each $\widetilde{\mathcal{R}}$-class and we denote by a^+ the idempotent in the $\widetilde{\mathcal{R}}$-class of a. Notice that for any $a, b \in S$, $a \,\widetilde{\mathcal{R}}\, b$ if and only if $a^+ = b^+$. We regard left semiadequate semigroups as algebras of type $(2, 1)$, where the binary operation is the semigroup operation and the unary operation is $a \mapsto a^+$. One consequence of this is than any morphism between such semigroups *must preserve* $+$.

A semigroup S is *left adequate* if the idempotents of S form a semilattice and every element of S is $\mathcal{R}^*$-related to an idempotent. Since $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$ it is clear that a left adequate semigroup is left semiadequate. Indeed the following is easy to check.

Lemma 2.1. *Let S be a left semiadequate semigroup. Then the following conditions are equivalent:*
(i) S is left adequate;
(ii) $\widetilde{\mathcal{R}} = \mathcal{R}^*$;
(iii) $a \,\mathcal{R}^*\, a^+$ for all $a \in S$.

In view of the above, in a left adequate semigroup S the $\mathcal{R}^*$-class of $a \in S$ contains the unique idempotent a^+. Again, we may regard S as an algebra of type $(2, 1)$.

It is easy to see that the relation $\mathcal{R}^*$ is always a left congruence. The same is not true of $\widetilde{\mathcal{R}}$. A left semiadequate semigroup S is *weakly left ample* if conditions (CL) and (AL) hold:

(CL) the relation $\widetilde{\mathcal{R}}$ is a left congruence;

(AL) for all $a \in S$ and $e \in E(S)$, $ae = (ae)^+a$.

A left adequate semigroup satisfying (AL) (and perforce (CL)) is called *left ample* (formerly, left type A).

Lemma 2.2. *Let S be an algebra of type $(2,1)$. Then S is weakly left ample with $a \,\widetilde{\mathcal{R}}\, a^+$ if and only if S satisfies the quasi-identities*

$$(xy)z = x(yz),\ a^+b^+ = b^+a^+,\ (a^+)^+ = a^+,\ e^2 = e \Rightarrow e = e^+,$$

$$a^+a = a,\ b^+a = a \Rightarrow b^+a^+ = a^+$$

$$a^+ = b^+ \Rightarrow (ca)^+ = (cb)^+, ab^+ = (ab^+)^+a.$$

Correspondingly, we have

Lemma 2.3. *Let S be an algebra of type $(2,1)$. Then S is left ample with $a \,\mathcal{R}^*\, a^+$ if and only if S satisfies the quasi-identities*

$$(xy)z = x(yz),\ a^+b^+ = b^+a^+,\ (a^+)^+ = a^+,\ e^2 = e \Rightarrow e = e^+,$$

$$a^+a = a,\ xa = a \Rightarrow xa^+ = a^+,\ xa = ya \Rightarrow xa^+ = ya^+,$$

$$ab^+ = (ab^+)^+a.$$

Since quasivarieties are closed under subalgebras, it follows that a subalgebra of a (weakly) left ample semigroup is (weakly) left ample. It is easy to see that in a (weakly) left ample semigroup the identities $(ab^+)^+ = (ab)^+$ and $a^+(ab)^+ = (ab)^+$ hold.

An inverse semigroup is left ample, where $a^+ = aa^{-1}$. The classes of left ample and weakly left ample semigroups are, however, much wider, as the following result makes clear. The original sources are [3] in the left ample case and [12] and [11] in the weakly left ample case. The details may be found in [10]. For any set X, we denote by $\mathcal{PT}_X$, $\mathcal{I}_X$ and E_X the semigroup of partial transformations on X, the symmetric inverse semigroup on X and the semilattice of idempotents of $\mathcal{I}_X$ respectively. Recall that E_X consists of the set of partial identity transformations of X. We define a unary operation $^+$ on $\mathcal{PT}_X$ by the rule that

$$\alpha^+ = I_{\mathrm{dom}\ \alpha}.$$

Theorem 2.4. *A semigroup S is (finite) weakly left ample if and only if it is a $(2,1)$-subalgebra of $\mathcal{PT}_X$ for some (finite) set X such that $E(S) \subseteq E_X$.*

A semigroup S is (finite) left ample if and only if it is a $(2,1)$-subalgebra of $\mathcal{I}_X$ for some (finite) set X.

We remark that for a weakly left ample subsemigroup S of $\mathcal{PT}_X$ (with the usual conventions), and any elements $\alpha, \beta \in S$,

$$\begin{array}{llrcl} \alpha \,\widetilde{\mathcal{R}}\, \beta & \text{if and only if} & \alpha^+ & = & \beta^+ \\ & \text{if and only if} & I_{\text{dom }\alpha} & = & I_{\text{dom }\beta} \\ & \text{if and only if} & \text{dom }\alpha & = & \text{dom }\beta; \end{array}$$

similarly for $\mathcal{I}_X$.

On any semigroup S we define the relation σ by the rule

$$a \,\sigma\, b \text{ if and only if } ea = eb \text{ for some } e \in E(S).$$

If S is inverse, the relation σ is the least group congruence [17]. Analogously, if S is left ample, σ is the least right cancellative monoid congruence [3], and if S is weakly left ample, σ is the least unipotent monoid congruence [5]. We make the obvious but extremely useful comment here that since a finite unipotent monoid is a group, σ is the least group congruence on a *finite* (weakly) left ample semigroup.

We say that a weakly left ample semigroup is *proper* if $\widetilde{\mathcal{R}} \cap \sigma = \iota$. In view of Lemma 2.1 a left ample semigroup is proper if and only if $\mathcal{R}^* \cap \sigma = \iota$.

Definition 2.5. *Let S be a (weakly) left ample semigroup. A* cover *of S is a (weakly) left ample semigroup $\widehat{S}$ such that there is an onto morphism from $\widehat{S}$ to S that is idempotent separating.*

The aim of this paper is to show that every finite weakly left ample semigroup has a finite proper left ample cover.

We recall here that an inverse semigroup S is proper if and only if it is E-unitary, that is, for all $a \in S$ and $e \in E(S)$,

$$ea \in E(S) \text{ or } ae \in E(S) \text{ implies that } a \in E(S).$$

However, the corresponding result is not true of left ample semigroups. In [3], Fountain gives an example of an E-unitary left ample semigroup that is not proper. His example is, however, infinite. We now give a finite one.

Here, as elsewhere, we denote a partial transformation α of a finite set $\{1, 2, \ldots, n\}$ by

$$\alpha = \begin{pmatrix} 1 & 2 & \ldots & n \\ 1\alpha & 2\alpha & \ldots & n\alpha \end{pmatrix}$$

with the convention that if $i \notin \text{dom}\,\alpha$, then we write $\times$ under i.

Example 2.6.

Let

$$S = \{\alpha, \beta, \alpha^+, \tau, \tau^+\} \subseteq \mathcal{I}_{\{1,2,3,4,5\}}$$

where

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & \times & \times & 5 & 4 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & \times & \times & 5 & 4 \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \times & \times & \times & 5 & 4 \end{pmatrix}.$$

Certainly S is closed under $^+$ and it is easy to check that S has multiplication table

	α^+	α	β	τ^+	τ
α^+	α^+	α	β	τ^+	τ
α	τ	τ^+	τ^+	τ	τ^+
β	τ	τ^+	τ^+	τ	τ^+
τ^+	τ^+	τ	τ	τ^+	τ
τ	τ	τ^+	τ^+	τ	τ^+

Since S is a $(2,1)$-subalgebra of $\mathcal{I}_{\{1,2,3,4,5\}}$, it is a left ample semigroup by Theorem 2.4. By inspection, S is E-unitary. However, S is not proper. To see this, notice that

$$\tau^+\alpha = \tau^+\beta \text{ and dom } \alpha = \text{ dom } \beta$$

so that $\alpha\,\sigma\,\beta$ by definition of σ and $\alpha\,\mathcal{R}^*\,\beta$ by the comments succeeding Theorem 2.4.

Since this paper is concerned with finite proper covers, we remark here that the standard construction of E-unitary covers does not complete our problem. For in the above example, if we adjoin an identity to S, S^1 becomes a finite left ample monoid that is E-unitary but not proper. As above, S^1/σ is a finite group and since S^1 is E-unitary, the idempotents form a σ-class. Thus the natural map $\sigma^\natural : S^1 \to S^1/\sigma$ is a (relational) morphism from S^1 to a finite group such that

$$1(\sigma^\natural)^{-1} = E(S^1).$$

The procedure to form a cover $\widehat{S^1}$ of S^1 from $\sigma^\natural$ yields

$$\{(s, s\sigma) : s \in S^1\}$$

which is clearly isomorphic to S^1 and hence not proper. Nevertheless, a much more careful choice of relational morphism and an invocation of Ash's celebrated result on inevitable graphs [1] will produce a finite proper cover. This was pointed out to the authors by Karl Auinger and forms the basis for a subsequent note [2]. The approach of this paper provides a direct construction without use of [1].

3. Weakly left ample versus left ample

In this section we begin with an example of a finite weakly left ample semigroup that is not left ample. On the other hand we then show that a finite proper weakly left ample semigroup must be left ample.

Example 3.1.

Let S be the subset of $\mathcal{PT}_{\{1,2,3\}}$ given by

$$S = \{\alpha, \beta, \alpha^+, \beta^+, \beta\alpha, \epsilon\}$$

where ϵ is the empty transformation and α, β are given by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ \times & 1 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 \\ \times & \times & 2 \end{pmatrix}.$$

Then

$$\alpha^+ = \begin{pmatrix} 1 & 2 & 3 \\ \times & 2 & 3 \end{pmatrix}, \beta^+ = \begin{pmatrix} 1 & 2 & 3 \\ \times & \times & 3 \end{pmatrix}$$

and calculating,

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 \\ \times & \times & 1 \end{pmatrix}.$$

Clearly then S is closed under $^+$. It is easy to check that the multiplication table of S is given by

	α^+	α	β^+	β	$\beta\alpha$	ϵ
α^+	α^+	α	β^+	β	$\beta\alpha$	ϵ
α	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
β^+	β^+	$\beta\alpha$	β^+	β	$\beta\alpha$	ϵ
β	β	$\beta\alpha$	ϵ	ϵ	ϵ	ϵ
$\beta\alpha$	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ

Since S is closed under multiplication, S is a $(2,1)$-subalgebra of $\mathcal{PT}_{\{1,2,3\}}$. Moreover $E(S) = S \cap E_{\{1,2,3\}}$ so that by Theorem 2.4, S is weakly left ample. Notice that $\beta\alpha = \beta^+\alpha$ but $\beta\alpha^+ \neq \beta^+\alpha^+$ so that α is not $\mathcal{R}^*$-related to α^+. By Lemma 2.1, S is not left ample.

In the above example, S has 2 generators as a $(2,1)$-subalgebra of $\mathcal{PT}_{\{1,2,3\}}$. This is the minimum possible, as we show in the next section.

Recall that a semigroup S is *E-dense* if, for each $a \in S$, there is a $b \in S$ such that $ba \in E(S)$. Notice that if $ba \in E(S)$, then $a(bab) \in E(S)$, so that the one-sidedness of the definition is only apparent.

Lemma 3.2. *Let S be a finite weakly left ample semigroup. Then S/σ is a finite group and S is E-dense.*

Proof. Since S/σ is a finite unipotent monoid, it is a group, with identity $e\sigma$ for any $e \in E(S)$. Let $a \in S$. Then $a\sigma$ has an inverse in the group S/σ, so that

$$b\sigma\, a\sigma = e\sigma$$

for some $b \in S$ and $e \in E(S)$. Thus $ba\,\sigma\, e$ and $fba = fe$ for some $f \in E(S)$. Now $E(S)$ is a semilattice so that $fe \in E(S)$ and S is E-dense. □

This simple observation enables us to show the following.

Proposition 3.3. *Let S be a finite proper weakly left ample semigroup. Then S is left ample.*

Proof. In view of Lemma 2.1 we need only show that for each $a \in S$, $a\,\mathcal{R}^*\,a^+$.

Let $a \in S$ and $x, y \in S^1$. Certainly

$$xa^+ = ya^+ \text{ implies that } xa = ya$$

since $a^+a = a$.

Suppose conversely that $xa = ya$. By (CL),

$$xa^+\,\widetilde{\mathcal{R}}\,xa = ya\,\widetilde{\mathcal{R}}\,ya^+.$$

By Lemma 3.2, S is E-dense, so that $ab \in E(S)$ for some $b \in S$. Thus

$$xa^+(ab) = (xa)b = (ya)b = (ya^+)ab.$$

Let $e = ab$. By (AL),

$$(xa^+e)^+xa^+ = xa^+e = ya^+e = (ya^+e)^+ya^+$$

whence, multiplying both sides on the left by the idempotent $(xa^+e)^+(ya^+e)^+$ we obtain $xa^+\,\sigma\,ya^+$.

Since S is proper, $xa^+ = ya^+$ as required. □

4. The one generator case

We show that every finite weakly left ample semigroup S with single generator (as a $(2,1)$-algebra) is left ample. Moreover, if such an S is E-unitary, it is proper. The reader may note that the results of this section are not needed for our main result Theorem 5.1.

For the purposes of this section it is useful to remark that the relation $\widetilde{\mathcal{R}}$ on a semigroup S is the equivalence associated with the quasi-order $\leq_{\widetilde{\mathcal{R}}}$ where for $a, b \in S$, $a \leq_{\widetilde{\mathcal{R}}} b$ if and only if

$$eb = b \text{ implies that } ea = a$$

for all $e \in E(S)$.

It is clear that in any semigroup S,

$$a \geq_{\widetilde{\mathcal{R}}} a^2 \geq_{\widetilde{\mathcal{R}}} a^3 \geq_{\widetilde{\mathcal{R}}} \dots$$

and $\geq_{\widetilde{\mathcal{R}}}$ extends the relation $\geq_{\mathcal{R}}$ when restricted to $E(S)$.

Lemma 4.1. *Let a be an element of a weakly left ample semigroup S such that a has finite index n and period r. Then*

$$a >_{\widetilde{\mathcal{R}}} a^2 >_{\widetilde{\mathcal{R}}} a^3 >_{\widetilde{\mathcal{R}}} \dots >_{\widetilde{\mathcal{R}}} a^n \,\widetilde{\mathcal{R}}\, a^{n+1}, \dots, a^{n+r-1}.$$

Proof. Since $\{a^n, a^{n+1} \dots, a^{n+r-1}\}$ is a subgroup of S, all of these elements are $\mathcal{R}$-related and so certainly $\widetilde{\mathcal{R}}$-related.

Suppose now that $a^p \,\widetilde{\mathcal{R}}\, a^{p+1}$. Since $\widetilde{\mathcal{R}}$ is a left congruence, we have $a^p \,\widetilde{\mathcal{R}}\, a^q$ for all $q \geq p$. The element a has finite index by assumption, and so there is some power a^{ps} with $ps \geq n$ which is idempotent. Since $a^p \,\widetilde{\mathcal{R}}\, a^{ps}$ we must have that $a^{ps} = (a^p)^+$. Now

$$a^p = (a^p)^+ a^p = a^{ps} a^p = a^{ps+p}$$

so that $p \geq n$, by definition of index. $\square$

Example 3.1 shows that a finite weakly left ample semigroup having two generators need not be left ample. We now show that in the monogenic case this situation cannot arise.

Theorem 4.2. *Let S be a finite weakly left ample semigroup with single generator a as an algebra of type $(2,1)$. Then S is left ample.*

Proof. In view of Lemma 2.1 we need only show that $\widetilde{\mathcal{R}} = \mathcal{R}^*$. For our purposes here it is convenient to consider the monoid S^1. It is easy to check that S^1 is a weakly left ample monoid with single generator a as an algebra of type $(2,1,0)$. This enables us to consider expressions involving $a^0 = 1$; we remark that $1^+ = 1$. Notice also that, by the comment following Lemma 2.3 for any $x \leq y$ we have that

$$(a^x)^+ \geq (a^y)^+$$

so that for any z, t we have

$$(a^z)^+ (a^t)^+ = (a^{\max\{z,t\}})^+.$$

Let

$$T = \{(a^x)^+ a^y : x, y \geq 0\}.$$

We claim that $T = S^1$. Since $1 = (a^0)^+a^0, a = a^+a \in T$, the claim will be proven if we can show that T is closed under both the binary and unary operations. With this in mind, let $(a^x)^+a^y, (a^z)^+a^t \in T$. Then

$$((a^x)^+a^y)((a^z)^+a^t) = (a^x)^+(a^y(a^z)^+)a^t = (a^x)^+((a^y(a^z)^+)^+a^y)a^t,$$

using (AL). But by (CL),

$$(a^y(a^z)^+)^+ = (a^ya^z)^+ = (a^{y+z})^+.$$

Hence

$$((a^x)^+a^y)((a^z)^+a^t) = (a^x)^+(a^{y+z})^+a^{y+t} = (a^{\max\,\{x,y+z\}})^+a^{y+t} \in T \quad (*).$$

Again, consider $(a^x)^+a^y \in T$. Then by the comments following Lemma 2.3,

$$((a^x)^+a^y)^+ = ((a^x)^+(a^y)^+)^+ = (a^x)^+(a^y)^+$$

since $E(S^1)$ is a semilattice. Hence

$$((a^x)^+a^y)^+ = (a^{\max\,\{x,y\}})^+ \in T \quad (**).$$

Notice that for $x \leq y$,

$$(a^x)^+a^y = (a^x)^+(a^y)^+a^y = (a^y)^+a^y = a^y = (a^y)^+a^y,$$

so that we can refine our description of S^1 to

$$S^1 = \{(a^x)^+a^y : x \geq y \geq 0\}.$$

Of course, if $(a^x)^+a^y \in S^1$ with $x \geq y \geq 0$, then as above we have that

$$((a^x)^+a^y)^+ = (a^{\max\,\{x,y\}})^+ = (a^x)^+.$$

We now show there is some uniqueness in the expression of elements of S. As in Lemma 4.1, we denote by n the index and by r the period of a.

Lemma 4.3. *For any $(a^x)^+a^y, (a^z)^+a^t \in S^1$ where $x \geq y \geq 0, z \geq t \geq 0$, if*

$$(a^x)^+a^y = (a^z)^+a^t$$

then $(a^x)^+ = (a^z)^+$ and either

$$x = z < n \text{ and } y = t,$$

or

$$x, z \geq n.$$

Proof. That $(a^x)^+ = (a^z)^+$ is the remark before the lemma. If $x < n$ then since $a^x \,\widetilde{\mathcal{R}}\, a^z$, we must have that $x = z$ by Lemma 4.1.

Suppose that we have $x = z < n$ and without loss of generality suppose that $y \geq t$. Let $x = t + k \leq y + k$. We have

$$(a^x)^+ a^{y+k} = (a^z)^+ a^{t+k} = (a^x)^+ a^x = a^x$$

so that

$$a^{y+k} \,\widetilde{\mathcal{R}}\, (a^{y+k})^+ = (a^x)^+ (a^{y+k})^+ = (a^x)^+ \,\widetilde{\mathcal{R}}\, a^x,$$

giving $x = y + k = t + k$ and so $y = t$. $\square$

Returning to the proof of Theorem 4.2, we show that $\widetilde{\mathcal{R}} = \mathcal{R}^*$ in S^1. Let

$$(a^x)^+ a^y, (a^z)^+ a^t, (a^u)^+ a^v \in S^1$$

where $x \geq y \geq 0, z \geq t \geq 0$ and $u \geq v \geq 0$ and suppose that

$$((a^x)^+ a^y)((a^u)^+ a^v) = ((a^z)^+ a^t)((a^u)^+ a^v).$$

As in the formula $(*)$, this gives that

$$(a^{\max\, \{x, y+u\}})^+ a^{y+v} = (a^{\max\, \{z, t+u\}})^+ a^{t+v} \quad (***).$$

We wish to show that

$$((a^x)^+ a^y)((a^u)^+ a^v)^+ = ((a^z)^+ a^t)((a^u)^+ a^v)^+,$$

that is,

$$((a^x)^+ a^y)(a^u)^+ = ((a^z)^+ a^t)(a^u)^+,$$

which in view of $(*)$ is equivalent to

$$(a^{\max\, \{x, y+u\}})^+ a^y = (a^{\max\, \{z, t+u\}})^+ a^t \quad (****).$$

Notice that

$$\max\, \{x, y+u\} \geq y + u \geq y + v$$

and

$$\max\, \{z, t+u\} \geq t + u \geq t + v$$

so that we can apply Lemma 4.3 to $(***)$ and obtain either

$$\max\, \{x, y+u\} = \max\, \{z, t+u\} < n,\ y + v = t + v$$

or

$$\max\, \{x, y+u\} \geq n \text{ and } \max\, \{z, t+u\} \geq n.$$

In the former case we have that $y = t$ so that clearly $(****)$ holds. Suppose therefore that $\max\{x, y+u\} \geq n$ and $\max\, \{z, t+u\} \geq n$.

From Lemma 4.1 we have that

$$(a^{\max\{x,y+u\}})^+ = (a^{\max\{z,t+u\}})^+ = e,$$

where e is the idempotent of the subgroup $\{a^n, \ldots, a^{n+r-1}\}$ and is the unique idempotent power of a. Notice that $E(S)$ is a chain with zero e. Pick k with $a^{v+k} = e$. From

$$ea^{y+v} = ea^{t+v}$$

we have

$$ea^{y+v}a^k = ea^{t+v}a^k$$

so that

$$ea^{y+v+k} = ea^{t+v+k}$$

giving that

$$ea^y = ea^{v+k}a^y = ea^{v+k}a^t = ea^t$$

so that $(***\ast)$ holds.

In particular, we have shown that for any $s \in S$, $s\,\mathcal{R}^*\,s^+$ in S^1. From the definition of $\mathcal{R}^*$ this gives that $s\,\mathcal{R}^*\,s^+$ in S. Invoking Lemma 2.1 completes the proof of the theorem. □

Corollary 4.4. *Let S be a finite left ample semigroup with single generator a as an algebra of type $(2,1)$. If S is E-unitary, then S is proper.*

Proof. We retain the notation of the proof of the previous theorem. Let $(a^x)^+a^y, (a^z)^+a^t \in S$ where $x \geq y \geq 0$, $z \geq t \geq 0$ (and not both x and y, and not both z and t, are zero). Suppose that

$$(a^x)^+a^y\,\widetilde{\mathcal{R}} \cap \sigma\,(a^z)^+a^t.$$

Then $(a^x)^+ = (a^z)^+$ and as e is the zero of $E(S)$, $ea^y = ea^t$. According to Lemma 4.3, either $x = z < n$, or $x, z \geq n$.

If $x, z \geq n$ then

$$(a^x)^+a^y = ea^y = ea^t = (a^z)^+a^t.$$

Suppose now that $x = z < n$ and $y \geq t$. Let $k \geq 0$ be least such that $a^{y+k} = e$. Then $ea^{t+k} = e \in E(S)$ so that as S is E-unitary, $a^{t+k} \in E(S)$. But $t + k \leq y + k$ and $y + k$ is the least u such that $a^u = e$. Thus $t + k = y + k$ and $t = y$, giving that $(a^x)^+a^y = (a^z)^+a^t$ as required.

□

5. The covering theorem

The aim of this section is to prove the following covering theorem.

Theorem 5.1. *Let S be a finite weakly left ample semigroup. Then S has a finite left ample cover.*

Proof. We proceed via a series of intermediate results. By Theorem 2.4 we may assume that S is a $(2,1)$-subalgebra of $\mathcal{PT}_X$ for a finite set X. Since S is weakly left ample, all the idempotents of S are in $E_X = \{I_Y : Y \subseteq X\}$.

We pick a set of $(2,1)$-generators

$$G = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$$

of S. It is at times useful to have the visual tool of the *graph* of G. By this we mean the directed labelled graph having vertices the elements of X and edges of the form

$$x \xrightarrow{\gamma_i} y$$

where $x\gamma_i = y$. By a *G-path* from x to y where $x, y \in X$ we mean a directed path in the graph of G from x to y such that *there is at least one edge in the graph of G with initial vertex x*. The latter condition says that x lies in the domain of at least one of the maps in G; it is easy to see that this is also equivalent to x being in the domain of at least one element in S.

The definition of G-path has the effect of excluding empty paths at vertices which do not occur in the domain of any of the maps in G. Note that the concatenation of G-paths in the usual sense is again a G-path.

Lemma 5.2. *Let $x, y \in X$. Then there is a G-path from x to y if and only if $x\alpha = y$ for some $\alpha \in S$.*

Proof. Suppose first that there is a G-path from x to y. If the path is non-empty, then clearly $x\alpha = y$ for some $\alpha \in S$, where α is a product of elements of G. If there is an empty G-path from x to y, then $x = y$ and by definition x is in the domain of some γ_i. Thus $x\gamma_i^+ = x = y$.

Conversely, suppose that $x\alpha = y$ for some $\alpha \in S$. Since S is generated by G,

$$\alpha = t(\gamma_1, \ldots, \gamma_n) = t(\gamma)$$

for some $(2,1)$-term t. Here, as elsewhere, we abbreviate $t(\gamma_1, \ldots, \gamma_n)$ by $t(\gamma)$. We proceed by induction on the complexity of t. If $\alpha = \gamma_i$ for some i, then by definition the edge

$$x \xrightarrow{\gamma_i} y$$

is in the graph of G, so there is a G-path from x to y. Suppose that $t(\gamma) = (s(\gamma))^+$ and the result is true for $s(\gamma)$. Then from $x(s(\gamma))^+ = y$ we have that $x = y$ and the empty path from x to $x = y$ is a G-path.

Finally, suppose that

$$t(\gamma) = s(\gamma)z(\gamma).$$

Then x lies in the domain of $s(\gamma)$ and $xs(\gamma)$ lies in the domain of $z(\gamma)$. By the inductive hypothesis, there is a G-path from x to $xs(\gamma)$ and a G-path from $xs(\gamma)$ to $xs(\gamma)z(\gamma) = xt(\gamma) = y$. Since the concatenation of G-paths is a G-path, we have a G-path from x to y as required. $\square$

Definition 5.3.

A *bad vee* is an ordered pair

$$\underline{V} = (\{u, v\}, \gamma_i)$$

where u and v are distinct elements of X, $\gamma_i \in G$ and $u\gamma_i = v\gamma_i$. Thus a bad vee corresponds to

$$\begin{tikzcd} u \arrow[dr, "\gamma_i"] & & v \arrow[dl, "\gamma_i"] \\ & \bullet & \end{tikzcd}$$

in the graph of G. If there are no bad vees, then S is a $(2, 1)$-subalgebra of $\mathcal{I}_X$ so that by Theorem 2.4, S is left ample. Invoking the following result finishes the proof in this case.

Theorem 5.4. [3] *Every finite left ample semigroup has a finite proper left ample cover.*

Assume that there are some bad vees. We enumerate the distinct bad vees as

$$\underline{V}_1 = (\{u_1, v_1\}, \alpha_1), \ldots, \underline{V}_k = (\{u_k, v_k\}, \alpha_k)$$

(so that each α_i is some γ_j) and put $w_i = u_i\alpha_i = v_i\alpha_i, 1 \leq i \leq k$. We remark that not all the pairs $\{u_i, v_i\}$ need be distinct, nor need be all the elements α_i.

Lemma 5.5. *For* $i \in \{1, \ldots, k\}$,

$$w_i\alpha \neq u_i, \; w_i\alpha \neq v_i$$

for any $\alpha \in S$.

Proof. Suppose that $w_i\alpha = u_i$ for some $\alpha \in S$ and $i \in \{1, \ldots, k\}$. Then

$$u_i\gamma_i\alpha = w_i\alpha = u_i$$

and

$$v_i\gamma_i\alpha = w_i\alpha = u_i.$$

Now $\beta = \gamma_i\alpha \in S$ and as S is finite, $\beta^n \in E(S)$ for some $n \in \mathbb{N}$. We have

$$u_i\beta^n = (u_i\beta)\beta^{n-1} = u_i\beta^{n-1} = \ldots = u_i$$

and also

$$v_i\beta^n = (v_i\beta)\beta^{n-1} = u_i\beta^{n-1} = \ldots = u_i,$$

contradicting the fact that β^n is the identity map in its domain. □

Notice from the above that for any $i \in \{1, \ldots, k\}$, we cannot have $w_i = u_i$ or $w_i = v_i$. For if $w_i = u_i$ say, then $w_i \in \text{dom } \alpha_i = \text{dom } (\alpha_i)^+$ and $w_i(\alpha_i)^+ = w_i = u_i$, contradicting the lemma. In terms of the G-paths, the lemma says that for any $i \in \{1, \ldots, k\}$, there is no G-path from w_i to u_i or v_i.

Lemma 5.6. *There is an $i \in \{1, \ldots, k\}$ such that*

$$w_i\alpha \notin \{u_1, v_1, \ldots, u_k, v_k\}$$

for any $\alpha \in S$.

Proof. In view of Lemma 5.2 we need to show that we can find an $i \in \{1, \ldots, k\}$ such that there is no G-path from w_i to any element of $\{u_1, v_1, \ldots, u_k, v_k\}$.

Consider w_1. By Lemma 5.5 we know there is no G-path to u_1 or v_1. If moreover there is no G-path from w_1 to any element of $\{u_2, v_2, \ldots, u_k, v_k\}$ we are done.

We proceed by finite induction. Suppose that we can find a sequence (possibly with some relabelling) of vertices $\{w_1, w_2, \ldots, w_j\}$ such that there are G-paths from w_i to $s_{i+1} \in \{u_{i+1}, v_{i+1}\}$ for each $i \in \{1, \ldots, j-1\}$. Notice that in this case there is no G-path from w_j to any element of $\{u_1, v_1, \ldots, u_j, v_j\}$. For if there were say a G-path from w_j to $t_\ell \in \{u_\ell, v_\ell\}$ where $l \in \{1, \ldots, j-1\}$, then we would have

$$\begin{tikzcd}
t_\ell \arrow[r, "\alpha_\ell"] & w_\ell \arrow[r] & s_{\ell+1} \arrow[r, "\alpha_{\ell+1}"] & w_{\ell+1} \arrow[r, dotted] & s_j \arrow[r, "\alpha_j"] & w_j \arrow[lllll, bend left]
\end{tikzcd}$$

and in particular, a G-path from w_j to s_j, contradicting Lemma 5.5.

Either there is no G-path from w_j to any element of $\{u_{j+1}, v_{j+1}, \ldots, u_k, v_k\}$ and we are done, or otherwise, (possibly with relabelling), there is a G-path from w_j to $s_{j+1} \in \{u_{j+1}, v_{j+1}\}$.

By finite induction, this process must terminate after at most k steps with a w_i such that there is no G-path from w_i to any of $\{u_{i+1}, v_{i+1}, \ldots, u_k, v_k\}$ and hence to any of $\{u_1, v_1, \ldots, u_k, v_k\}$. $\square$

In view of Lemma 5.6 we can pick a bad vee $\underline{V_i}$, such that $w_i\alpha \notin \{u_1, v_1, \ldots, u_k, v_k\}$ for any $\alpha \in S$. For ease of notation we assume that $i = 1$, $\alpha_1 = \gamma_1$ and we put $u_1 = u, v_1 = v$ and $w_1 = w$, so that our bad vee is

$$\begin{tikzcd} u \arrow[dr, "\gamma_1"] & & v \arrow[dl, "\gamma_1"] \\ & w & \end{tikzcd}$$

As noted above, u, v, w are *distinct* elements of X.

Let

$$D = \{x \in X : w\beta = x \text{ for some } \beta \in S\} \cup \{w\}.$$

Notice that

$$\{u_1, v_1, \ldots, u_k, v_k\} \cap D = \emptyset.$$

For otherwise, by choice of w we would have that $w = s_i \in \{u_i, v_i\}$ for some $i \in \{1, \ldots, k\}$. But then

$$w \in \text{ dom } \gamma_i = \text{ dom } (\gamma_i)^+$$

and

$$w(\gamma_i)^+ = w = s_i,$$

contradicting the choice of w. We also observe that if $x \in D \cap$ dom δ for any $\delta \in S$, then $x\delta \in D$. For either $x = w$ or $x = w\beta$ for some $\beta \in S$, so that $x\delta = w\delta$ or $x\delta = w\beta\delta$ and $\delta, \beta\delta \in S$.

We choose a new, finite set of symbols

$$\{\bar{x} : x \in D\}$$

in one to one correspondence with those of D. We put

$$Y = X \cup \{\bar{x} : x \in D\}$$

and define transformations $\overline{\gamma_i}$ in $\mathcal{PT}_Y$ by

$$\begin{array}{rcll} x\overline{\gamma_1} & = & x\gamma_1 & x \in X \cap \text{ dom } \gamma_1, x \neq v \\ v\overline{\gamma_1} & = & \overline{w} = \overline{v\gamma_1} & \\ \overline{x}\,\overline{\gamma_1} & = & \overline{x\gamma_1} & x \in D \cap \text{ dom } \gamma_1 \\ x\overline{\gamma_1} & & \text{undefined} & \text{otherwise} \end{array}$$

and for $i \in \{2, \ldots, n\}$,

$$\begin{array}{rcll} x\overline{\gamma_i} & = & x\gamma_i & x \in X \cap \text{ dom } \gamma_i \\ \overline{x}\,\overline{\gamma_i} & = & \overline{x\gamma_i} & x \in D \cap \text{ dom } \gamma_i \\ x\overline{\gamma_i} & & \text{undefined} & \text{otherwise.} \end{array}$$

In view of the observation above, these transformations are well defined. Let $\overline{G} = \{\overline{\gamma_i} : \gamma_i \in G\}$ and let T be the $(2,1)$-subalgebra of $\mathcal{PT}_Y$ generated by $\overline{G}$. We will show that all the idempotents of T are of the form $E_Z, Z \subseteq Y$ so that by Theorem 2.4, T is a weakly left ample semigroup. Moreover, T is a cover of S. The following lemma is then crucial in an induction process.

Lemma 5.7. *The graph of $\overline{G}$ has strictly fewer bad vees than the graph of G.*

Proof. The basic idea is that, although we have increased the number of vertices by adding those of the form $\overline{x}$ for $x \in D$, and replicated some behaviour of the original generators in G, we have carefully chosen u, v and w such that we are not replicating any bad vees.

Formally, we claim that the bad vees of $\overline{G}$ are

$$\{(\{u_i, v_i\}, \overline{\alpha_i}) : 1 \leq i \leq k, \text{ and if } \alpha_i = \gamma_1, \text{ then } u_i \neq v \text{ and } v_i \neq v\}.$$

Since we know that G has a bad vee of the form $(\{u, v\}, \gamma_1)$, $\overline{G}$ will have strictly fewer than k bad vees.

To prove our claim, we first remark that all the elements of the given set are clearly bad vees of $\overline{G}$. Conversely, suppose that $(\{a, b\}, \overline{\gamma_i})$ is a bad vee of $\overline{G}$.

There are three cases to consider. The first is $a = \overline{c}, b = \overline{d}$ for some $c, d \in D$. Here we would have

$$\overline{c\gamma_i} = \overline{c}\,\overline{\gamma_i} = a\overline{\gamma_i} = b\overline{\gamma_i} = \overline{d}\overline{\gamma_i} = \overline{d\gamma_i}$$

and so $c\gamma_i = d\gamma_i$. Thus $\{c, d\} = \{u_j, v_j\}$ for some $j \in \{1, \ldots, k\}$ with $\alpha_j = \gamma_i$, contradicting the fact that neither u_j nor v_j can lie in D.

Secondly, we consider the case where $a, b \in X$ but, if $i = 1$, then neither a nor b is v. By definition of $\overline{\gamma_i}$,

$$a\gamma_i = a\overline{\gamma_i} = b\overline{\gamma_i} = b\gamma_i$$

so that

$$(\{a,b\},\gamma_i) = (\{u_j,v_j\},\alpha_j)$$

for some $j \in \{1,\ldots,k\}$.

Finally, suppose that $a = v$, $b = \overline{d}$ for some $d \in D$ and $i = 1$. Then

$$\overline{w} = v\overline{\gamma_1} = a\overline{\gamma_1} = b\overline{\gamma_1} = \overline{d}\,\overline{\gamma_1} = \overline{d\gamma_1}$$

and so $w = d\gamma_1$. But then we would have that $v\gamma_1 = w = d\gamma_1$ and $\{v,d\} = \{u_i,v_i\}$ for some $i \in \{1,\ldots,k\}$ with $\alpha_i = \gamma_1$. But $d \in D$, contradicting the fact that neither u_i nor v_i can lie in D.

□

Looking again at the definition of $\overline{\gamma_i}$ for $\gamma_i \in G$ we have that for any $x \in X$,

$$x \in \text{dom } \overline{\gamma_i} \text{ if and only if } x \in \text{dom } \gamma_i$$

and for $x \in D$,

$$\overline{x} \in \text{dom } \overline{\gamma_i} \text{ if and only if } x \in \text{dom } \gamma_i.$$

The corresponding result is true for any $\delta \in S$. This is part of our next lemma, where we prove a number of unsurprising technicalities crucial to our main strategy. As in the proof of Lemma 5.2, if $t(x_1,\ldots,x_n)$ is a term of type $(2,1)$ in at most n variables $x_1,\ldots,x_n$, then we abbreviate $t(\gamma_1,\ldots,\gamma_n)$ and $t(\overline{\gamma_1},\ldots,\overline{\gamma_n})$ by $t(\gamma)$ and $t(\overline{\gamma})$ respectively.

Lemma 5.8. *Let $t(x_1,\ldots,x_n)$ be a term of type $(2,1)$.*

(i) For any $x \in X$,

$$x \in \textit{dom } t(\overline{\gamma}) \textit{ if and only if } x \in \textit{dom } t(\gamma).$$

(ii) For any $x \in D$,

$$\overline{x} \in \textit{dom } t(\overline{\gamma}) \textit{ if and only if } x \in \textit{dom } t(\gamma).$$

(iii) For any $x \in X\cap$ dom $t(\gamma)$,

$$xt(\overline{\gamma}) = xt(\gamma) \textit{ or } \overline{xt(\gamma)}.$$

If $xt(\overline{\gamma}) = \overline{xt(\gamma)}$, then $x\beta = v$ for some $\beta \in S$.

(iv) For any $x \in D\cap$ dom $t(\gamma)$

$$\overline{x}t(\overline{\gamma}) = \overline{xt(\gamma)}.$$

Proof. We proceed by induction on the complexity of $t(x_1,\ldots,x_n)$.

The initial step is to consider $t(\gamma) = \gamma_i$ for some $i \in \{1,\ldots,n\}$. In this case (i) and (ii) hold by comments preceding the lemma. Moreover, by the definition of $\overline{\gamma_i}$, for any $x \in X$, $x\overline{\gamma_i} = x\gamma_i$ with the exception of the case where $i = 1$ and $x = v$. In this case

$$x\overline{\gamma_i} = v\overline{\gamma_1} = \overline{v\gamma_1} = \overline{x\gamma_i}$$

and further, $x\gamma_1^+ = x = v$. Thus (iii) and clearly also (iv) hold.

Suppose inductively that $t(x_1,\ldots,x_n) = (s(x_1,\ldots x_n))^+$ and the result is true for $s(x_1,\ldots,x_n)$. By definition of $+$, for any $x \in X$,

$$x \in \text{dom } t(\overline{\gamma}) = (s(\overline{\gamma}))^+ \text{ if and only if } x \in \text{dom } s(\overline{\gamma}),$$

which by the inductive assumption is equivalent to

$$x \in \text{dom } s(\gamma) = \text{dom } (s(\gamma))^+ = \text{dom } t(\gamma),$$

so that (i) holds. Similarly, (ii) holds. For (iii), notice that if $x \in X\cap$ dom $t(\gamma)$, then

$$xt(\overline{\gamma}) = x(s(\overline{\gamma}))^+ = x = xt(\gamma).$$

Finally in this case, if $x \in D\cap$ dom $t(\gamma)$, then

$$\overline{x}t(\overline{\gamma}) = \overline{x}(s(\overline{\gamma}))^+ = \overline{x} = \overline{xt(\gamma)}$$

so that (iv) holds.

To complete our induction we suppose that

$$t(x_1,\ldots,x_n) = s(x_1,\ldots,x_n)z(x_1,\ldots,x_n)$$

and that (i) to (iv) hold for both $s(x_1,\ldots,x_n)$ and $z(x_1,\ldots,x_n)$.

Let $x \in X$. If $x \in$ dom $t(\overline{\gamma}) = s(\overline{\gamma})z(\overline{\gamma})$ then $x \in$ dom $s(\overline{\gamma})$ and $xs(\overline{\gamma}) \in$ dom $z(\overline{\gamma})$. By (i) we have $x \in$ dom $s(\gamma)$ and by (iii), $xs(\overline{\gamma}) = xs(\gamma)$ or $\overline{xs(\gamma)}$. In the former case, $xs(\gamma) \in$ dom $z(\overline{\gamma})$ so that by (i), $xs(\gamma) \in$ dom $z(\gamma)$. Thus $x \in$ dom $s(\gamma)z(\gamma) =$ dom $t(\gamma)$. In the latter, (ii) gives that $xs(\gamma) \in$ dom $z(\gamma)$ so that again, $x \in$ dom $t(\gamma)$.

Conversely, suppose that $x \in$ dom $t(\gamma)$. Then $x \in$ dom $s(\gamma)$ and $xs(\gamma) \in$ dom $z(\gamma)$. By (i), $x \in$ dom $s(\overline{\gamma})$ and $xs(\gamma) \in$ dom $z(\overline{\gamma})$. By (iii), $xs(\overline{\gamma}) = xs(\gamma)$ or $\overline{xs(\gamma)}$. In the former case, $xs(\overline{\gamma}) \in$ dom $z(\overline{\gamma})$ so that $x \in$ dom $s(\overline{\gamma})z(\overline{\gamma}) =$ dom $t(\overline{\gamma})$. In the latter, we must have that $xs(\gamma) \in D$, since $\overline{xs(\gamma)}$ exists. By (ii) and the fact that $xs(\gamma) \in D\cap$ dom $z(\gamma)$, $\overline{xs(\gamma)} \in$ dom $z(\overline{\gamma})$ so that again, $x \in$ dom $t(\overline{\gamma})$. This shows that (i) holds.

Suppose now that $x \in X\cap$ dom $t(\gamma)$. Then $x \in$ dom $t(\overline{\gamma})$ and $xt(\overline{\gamma}) = xs(\overline{\gamma})z(\overline{\gamma})$. By (ii), $xs(\overline{\gamma}) = xs(\gamma)$ or $\overline{xs(\gamma)}$. If $xs(\overline{\gamma}) = xs(\gamma)$ then as $xs(\gamma) \in$ dom $z(\overline{\gamma})$, (iii) gives that

$$xt(\overline{\gamma}) = xs(\gamma)z(\overline{\gamma}) = xs(\gamma)z(\gamma) \text{ or } \overline{xs(\gamma)z(\gamma)}$$

so that $xt(\overline{\gamma}) = xt(\gamma)$ or $\overline{xt(\gamma)}$. Moreover, if $xt(\overline{\gamma}) = \overline{xt(\gamma)}$, then (iii) also says that $xs(\gamma)\beta = v$ for some $\beta \in S$. Of course, $s(\gamma)\beta \in S$ also. On the other hand, if $xs(\overline{\gamma}) = \overline{xs(\gamma)}$, then $x\delta = v$ for some $\delta \in S$ and by (ii) and (iv), the fact that $\overline{xs(\gamma)} \in$ dom $z(\overline{\gamma})$ gives that

$$xt(\overline{\gamma}) = \overline{xs(\gamma)}z(\overline{\gamma}) = \overline{xs(\gamma)z(\gamma)} = \overline{xt(\gamma)}.$$

Thus (iii) holds.

It is more straightforward to deal with (ii) and (iv). For if $x \in D$ and $\overline{x} \in$ dom $t(\overline{\gamma})$, then $\overline{x} \in$ dom $s(\overline{\gamma})$ and $\overline{x}s(\overline{\gamma}) \in$ dom $z(\overline{\gamma})$. By the inductive hypothesis, $x \in$ dom $s(\gamma)$ and $\overline{x}s(\overline{\gamma}) = \overline{xs(\gamma)} \in$ dom$z(\overline{\gamma})$. Again by the inductive hypothesis, $xs(\gamma) \in$ dom $z(\gamma)$ and $\overline{xs(\gamma)}z(\overline{\gamma}) = \overline{xs(\gamma)z(\gamma)}$ so that $x \in$ dom $s(\gamma)z(\gamma) =$ dom $t(\gamma)$ and

$$\overline{x}t(\overline{\gamma}) = \overline{x}s(\overline{\gamma})z(\overline{\gamma}) = \overline{xs(\gamma)}z(\overline{\gamma}) = \overline{xs(\gamma)z(\gamma)} = \overline{xt(\gamma)}.$$

Finally, if $x \in D$ and $x \in$ dom $t(\gamma)$, we have $x \in$ dom $s(\gamma)$ and $xs(\gamma) \in$ dom $z(\gamma)$. By (ii), $\overline{x} \in$ dom $s(\overline{\gamma})$ and by (iv), $\overline{x}s(\overline{\gamma}) = \overline{xs(\gamma)}$. Now $xs(\gamma) \in D\cap$ dom $z(\gamma)$ so that again by (ii), $\overline{xs(\gamma)} \in$ dom $z(\overline{\gamma})$ and $\overline{x} \in$ dom $s(\overline{\gamma})z(\overline{\gamma}) =$ dom $t(\overline{\gamma})$. □

Recall that T is the $(2, 1)$-subalgebra of $\mathcal{PT}_Y$ generated by $\overline{G}$.

Corollary 5.9. *Let $\theta : T \to S$ be defined by*

$$t(\overline{\gamma})\theta = t(\gamma).$$

Then θ is an onto morphism.

Proof. Suppose that $t(\overline{\gamma}) = s(\overline{\gamma})$. By (i) of Lemma 5.8,

$$\begin{aligned} \text{dom } t(\gamma) &= X \cap \text{ dom } t(\overline{\gamma}) \\ &= X \cap \text{ dom } s(\overline{\gamma}) \\ &= \text{ dom } s(\gamma). \end{aligned}$$

For any $x \in X\cap$ dom $t(\gamma)$, (iii) of Lemma 5.8 gives that

$$xt(\overline{\gamma}) = xt(\gamma) \text{ or } \overline{xt(\gamma)}$$

and

$$xs(\overline{\gamma}) = xs(\gamma) \text{ or } \overline{xs(\gamma)}.$$

Since $X \cap \{\overline{x} : x \in D\} = \emptyset$ and $t(\overline{\gamma}) = s(\overline{\gamma})$ by assumption, we must have either

$$xt(\gamma) = xt(\overline{\gamma}) = xs(\overline{\gamma}) = xs(\gamma)$$

or

$$\overline{xt(\gamma)} = xt(\overline{\gamma}) = xs(\overline{\gamma}) = \overline{xs(\gamma)}.$$

In either case, $xt(\gamma) = xs(\gamma)$. Thus $t(\gamma) = s(\gamma)$ and θ is well defined. Clearly then θ is an onto morphism. □

Corollary 5.10. *The semigroup T is weakly left ample.*

Proof. In view of Theorem 2.4 it is enough to show that every idempotent of T is of the form I_Z for some $Z \subseteq Y$.

Let $\epsilon = t(\overline{\gamma}) \in E(T)$. The morphism θ of Corollary 5.9 must take ϵ to the idempotent $t(\gamma)$ of S. Since S is weakly left ample, $t(\gamma)$ is the identity map on its domain.

Let $\overline{x} \in$ dom ϵ. By (iv) of Lemma 5.8,

$$\overline{x}\epsilon = \overline{x}t(\overline{\gamma}) = \overline{xt(\gamma)} = \overline{x}.$$

Consider now $y \in X \cap$ dom ϵ. By (i) of Lemma 5.8, $y \in$ dom $t(\gamma)$ and by (iii) of that lemma, $yt(\overline{\gamma})$ is one of $yt(\gamma) = y$ or $\overline{yt(\gamma)} = \overline{y}$. But if $yt(\overline{\gamma}) = \overline{y}$, then $y\beta = v$ for some $\beta \in S$. However we must also have that $y \in D$ and so $y = w$ or $y = w\delta$ for some $\delta \in S$. Hence

$$v = y\beta = w\tau$$

for some $\tau \in S$, contradicting Lemma 5.5. Thus $yt(\overline{\gamma}) = y$ and

$$t(\overline{\gamma}) = I_{\text{dom } t(\overline{\gamma})}$$

as required. □

We are now in a position to complete the proof of our theorem. We have constructed a weakly left ample semigroup T and a morphism θ from T onto S. The semigroup T is patently finite, since it is a subsemigroup of $\mathcal{PT}_Y$ for a finite set Y. Moreover, T has set of generators $\overline{G}$ which has a graph with strictly fewer bad vees than those in the graph of G. To see that the morphism θ is idempotent separating, suppose that $\epsilon = t(\overline{\gamma})$ and $\nu = s(\overline{\gamma})$ are idempotents of T such that $\epsilon\theta = \nu\theta$. In view of Corollary 5.10, to show that $\epsilon = \nu$ it is enough to show that dom $\epsilon =$ dom ν. But $\epsilon\theta = \nu\theta$ gives that $t(\gamma) = s(\gamma)$ and so (i) and (ii) of Lemma 5.8 give the required result.

Proceeding by finite induction, making the simple observation that idempotent separating morphisms are closed under composition, we arrive at a weakly left ample semigroup U that is a $(2,1)$-subalgebra of $\mathcal{PT}_Z$ for a finite set Z having a set of generators with no bad vees, and an idempotent separating morphism from U onto S. Since the set of generators of U has no bad vees, U is a $(2,1)$-subalgebra of $\mathcal{I}_Z$, hence left ample by Theorem 2.4. From Theorem 5.4, since U is finite, U has a finite proper left ample cover. In view of the observation at the beginning of this paragraph, so does S. □

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