FUNDAMENTAL SEMIGROUPS HAVING A BAND OF IDEMPOTENTS

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Abstract. The construction by Hall of a fundamental orthodox semigroup W_B from a band B provides an important tool in the study of orthodox semigroups. We present here a semigroup S_B that plays the role of W_B for a class of semigroups having a band of idempotents B. Specifically, the semigroups we consider are weakly B-abundant and satisfy the congruence condition (C). Any orthodox semigroup S with $E(S) = B$ lies in our class. On the other hand, if a semigroup S lies in our class, then S is Ehresmann if and only if B is a semilattice.

The Hall semigroup W_B is a subsemigroup of S_B, as are the (weakly) idempotent connected semigroups V_B and U_B. We show how the structure of S_B can be used to extract information relating to arbitrary weakly B-abundant semigroups with (C).

1. Introduction

One of the significant early approaches to the structure theory of regular semigroups was via *fundamental* semigroups, that is, regular semigroups having no non-trivial idempotent separating congruences. Munn showed that an inverse semigroup S with semilattice of idempotents E is fundamental if and only if it is isomorphic to a full subsemigroup of T_E, where T_E is the inverse semigroup of isomorphisms between principal ideals of E. Further, if S is an inverse semigroup with semilattice of idempotents E, then there exists a homomorphism $\phi : S \to T_E$ whose kernel is μ, the maximum idempotent separating congruence on S [14].

The founding work of Munn has been generalised in several directions. Dropping the condition of commutativity of idempotents but retaining the property that idempotents form a subsemigroup, that is,

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a *band*, leads to the study of *orthodox* semigroups. The *Hall semigroup* W_B of a band B is an orthodox semigroup with band of idempotents isomorphic to B and properties analogous to those described above for T_E [9]. Whereas T_E (E a semilattice) consists of partial isomorphisms of E, W_B (B band) is a subsemigroup of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$. Here $\mathcal{OP}(X)$ denotes the semigroup of order preserving selfmaps of a partially ordered set X, and a * denotes the dual of a semigroup. A pair of maps $(\alpha, \beta) \in \mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$ lies in W_B if and only if α and β are connected in a specific way via an isomorphism between principal ideals of B. If B is a semilattice, then $B/\mathcal{L} \cong B \cong B/\mathcal{R}$; this is essentially the reason why *pairs* of maps are not required in the construction of the Munn semigroup. Hall and Nambooripad extended this approach still further to the case of regular semigroups in [10] and [15] respectively.

In another direction one can weaken the condition of regularity. Here we consider *weakly U-abundant* semigroups, where U is a subset of idempotents of a semigroup. Such semigroups, also referred to as U-semiabundant semigroups, arise independently from a number of sources. They appear in the work of de Barros [1], in that of Ehresmann on certain small ordered categories [2] and in that of El-Qallali [3]. A systematic study of such semigroups was initiated by Lawson, who establishes in [12] the connection between Ehresmann's work and weakly E-abundant semigroups, where E is a semilattice. The restriction semigroups of [13] are a class of weakly E-abundant semigroups where E is again a semilattice, as are the two-sided versions of the twisted C-semigroups of [11].

A semigroup S with subset of idempotents U is *weakly U-abundant* if the classes of the equivalence relations $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ contain idempotents of U. The relations $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ are defined in the next section; $\mathcal{L} \subseteq \widetilde{\mathcal{L}}_U$ and $\mathcal{R} \subseteq \widetilde{\mathcal{R}}_U$, with equality if S is regular and $U = E(S)$. We remark that $\widetilde{\mathcal{L}}_U$ ($\widetilde{\mathcal{R}}_U$) need not be right (left) congruences; if they are we say that S satisfies the *congruence condition* (C) (with respect to U). We denote by $\widetilde{\mathcal{H}}_U$ the relation $\widetilde{\mathcal{L}}_U \cap \widetilde{\mathcal{R}}_U$ and say that S is *U-fundamental* if the greatest congruence μ_U contained in $\widetilde{\mathcal{H}}_U$ is the identity ι; certainly μ_U separates the idempotents of U. It is straightforward to show that for any semigroup S with $U \subseteq E(S)$, S/μ_U is $\overline{U}$-fundamental [5] where $\overline{U}$ is the image of U under the natural morphism associated with μ_U. Moreover, S is weakly U-abundant (with (C)) if and only if S/μ_U is weakly $\overline{U}$-abundant (with (C)) [5]. This is where, in our study, the concept of being weakly abundant is more useful than that of being abundant; if S is abundant then S/μ need not be [3]. If $U = E(S)$

we drop the subscript U from $\widetilde{\mathcal{L}}_U, \widetilde{\mathcal{R}}_U, \widetilde{\mathcal{H}}_U$, and μ_U and refer to *weakly abundant* and *fundamental* semigroups.

In the case of classes of weakly E-abundant semigroups where E is a semilattice, a theory analogous to that of Munn has been developed in [6], [7] and in the most general case in [8], where a fundamental weakly E-abundant semigroup C_E with (C) is constructed. We remark that, unlike T_E, the semigroup C_E is constructed from *pairs* of maps, but for a rather different reason than in the orthodox case: here it is due to the fact there is no isomorphism of E in the background. Also, for a technical reason, we need to work with E^1 rather than E. It is worth commenting that although it is straightforward to show that for a semilattice E and a weakly E-abundant semigroup S with (C), S/μ_E is embeddable into $\mathcal{OP}(E^1) \times \mathcal{OP}^*(E^1)$, the difficulty lies entirely in finding a maximum fundamental weakly E-abundant subsemigroup of $\mathcal{OP}(E^1) \times \mathcal{OP}^*(E^1)$.

In this article we extend the work of [8] to the case of a band B of idempotents; we make the convention that B will *always* denote a band. Abundant semigroups in this class satisfying the idempotent connected condition (IC) have been considered by El-Qallali and Fountain in [4]. Weakly B-abundant semigroups with (C) and an idempotent connected condition, again called (IC), or the rather weaker but related condition (WIC), are the topic of [5]. Conditions of this type give some control over the position of idempotents in products of elements of the semigroup, reminiscent of that in a regular semigroup. Indeed the construction from B of the fundamental semigroups U_B and V_B in [5] are strongly influenced by Hall's construction of W_B.

The aim of this paper is to remove the idempotent connected condition from the results of [4, 5]. We stress that to do so we need a completely fresh approach. From a band B we construct a weakly B-abundant subsemigroup of $\mathcal{OP}(B^1/\mathcal{L}) \times \mathcal{OP}^*(B^1/\mathcal{R})$, satisfying (C), calling this semigroup S_B. The semigroup S_B is B-fundamental, and is universal in the sense that any B-fundamental weakly B-abundant semigroup with (C) is a subsemigroup of S_B. Consequently, the fundamental semigroup U_B of [5], having (C) and (WIC), is embeddable into S_B. We note that in general $U_B \neq S_B$ and whereas $E(U_B) = B$, S_B may contain idempotents *not* in B. Further, if B is a semilattice, then S_B is isomorphic to the semigroup C_B of [8].

The structure of the paper is as follows. In Section 2 we give the necesssary preliminaries. In Section 3 we build and investigate the semigroup S_B. Our next task is to show in Section 4 that if E is a semilattice, then the semigroup S_E is isomorphic to the fundamental

E-abundant semigroup C_E with (C), developed in [8]. In Section 5 we discuss the weak idempotent connected condition (WIC) and find a natural embedding of U_B into S_B. The final section is concerned with examples; we develop a series of lemmas to test for membership in S_B, which could be used in implementing a programme to calculate S_B for finite bands B. We use our techniques to calculate S_B directly, for some bands B of small finite cardinality, and thus demonstrate examples of semigroups that distinguish between the various classes under consideration in this paper.

2. Preliminaries

This section gathers together some basic definitions and elementary observations concerning weakly U-abundant semigroups.

Let S be a semigroup with subset of idempotents U. The relation $\widetilde{\mathcal{L}}_U$ on S is defined by the rule that for any $a, b \in S$, $a \,\widetilde{\mathcal{L}}_U\, b$ if and only if for all $e \in U$,

$$ae = a \text{ if and only if } be = b.$$

Clearly $\widetilde{\mathcal{L}}_U$ is an equivalence relation. It is easy to see that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$, and $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}_U$, if S is regular and $U = E(S)$. A useful observation is that if $a \in S$ and $e \in U$, then $a \,\widetilde{\mathcal{L}}_U\, e$ if and only if $ae = a$ and for any $f \in U$, $af = f$ implies that $ef = e$. Consequently, for $e, f \in U$, $e \,\widetilde{\mathcal{L}}_U\, f$ if and only if $e \,\mathcal{L}\, f$. The relation $\widetilde{\mathcal{R}}_U$ is the left-right dual of $\widetilde{\mathcal{L}}_U$.

A semigroup S is *abundant* (*U-abundant*, where $U \subseteq E(S)$) if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contains an idempotent (of U). Our convention is that any definition obtained by replacing $\mathcal{L}^*$ or $\mathcal{R}^*$ by $\widetilde{\mathcal{L}}$ or $\widetilde{\mathcal{R}}$ is qualified by the adjective 'weakly'. Thus a semigroup S is *weakly U-abundant* if every $\widetilde{\mathcal{L}}_U$-class and every $\widetilde{\mathcal{R}}_U$-class contain an idempotent of U. If S is such a semigroup and $a \in S$, then we commonly denote idempotents in the $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class of a by a^* and a^+ respectively. Beware however, that there may not be a unique choice for a^* or a^+. The following lemma is immediate.

Lemma 2.1. *Let S be a weakly U-abundant semigroup. Then for any $a, b \in S$,*

$$(ab)^* \leq_{\mathcal{L}} b^* \text{ and } (ab)^+ \leq_{\mathcal{R}} b^+.$$

Regular semigroups are clearly weakly abundant but the latter class is much wider. It is worth remarking that any monoid M for which $E(M)$ is a finite semilattice, and so, in particular, any unipotent monoid, is weakly abundant. Further examples abound; we refer the reader to the articles cited in the Introduction. We present new examples arising from our current work at the end of this paper.

Morphic images of regular and inverse semigroups are regular and inverse respectively. The same is not true even for abundant semigroups with semilattice of idempotents [6]. However, it follows from Lemma 2.3 and Proposition 2.4 of [5] that the property of being weakly U-abundant, and condition (C), are both preserved when quotienting by μ_U. This is essentially because the natural map associated with μ_U is strongly admissible (with respect to U), where a morphism θ from a weakly U-abundant semigroup S is *strongly admissible* if for any $a, b \in S$,

$$a \,\widetilde{\mathcal{L}}_U\, b \text{ if and only if } a\theta \,\widetilde{\mathcal{L}}_{U\theta}\, b\theta$$

and dually

$$a \,\widetilde{\mathcal{R}}_U\, b \text{ if and only if } a\theta \,\widetilde{\mathcal{R}}_{U\theta}\, b\theta.$$

We denote the image of U under the natural morphism $\mu_U^\natural$ by $\overline{U}$.

Proposition 2.2. [5] *Let S be a semigroup and let $U \subseteq E(S)$. The morphism $\mu_U^\natural$ is strongly admissible and restricts to an injection on U, and S/μ_U is $\overline{U}$-fundamental.*

If S is weakly U-abundant, then S/μ_U is weakly $\overline{U}$-abundant; if S satisfies (C), then so does S/μ_U.

The remainder of this paper concentrates on weakly B-abundant semigroups where B is a band. We make the convention that throughout this article B *will always denote a band*; further, if B is a band of idempotents of a semigroup S, then unless stated otherwise, Green's relations will always be those on B. Let S be such a semigroup; for any $a \in S$ we define

$$\alpha_a : B^1/\mathcal{L} \to B/\mathcal{L} \text{ and } \beta_a : B^1/\mathcal{R} \to B/\mathcal{R}$$

by

$$L_e\alpha_a = L_{(ea)^*} \text{ and } R_e\beta_a = R_{(ae)^+}.$$

We warn the reader, that for a technical reason, we need to consider B^1 rather than B. In the case where S has condition (WIC) (defined in Section 5) B suffices; we have extended the domain of the maps α_a and β_a appearing in [5] from $B/\mathcal{L}$ and $B/\mathcal{R}$, but kept the same notation. It follows from earlier comments that α_a and β_a are well defined. We note that for any $e \in B$,

$$(\alpha_e, \beta_e) = (\rho_e, \lambda_e)$$

where for any $x \in B^1$,

$$L_x\rho_e = L_{xe},\ R_x\lambda_e = R_{ex}.$$

The band B^1 admits the quasi-orders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ associated with $\mathcal{L}$ and $\mathcal{R}$. We consider $B^1/\mathcal{L}$ and $B^1/\mathcal{R}$ as partially ordered sets under the induced orderings. For ease of notation we denote by $\mathcal{O}^1(B)$ the set

$$\{(\alpha,\beta)\in \mathcal{OP}(B^1/\mathcal{L})\times \mathcal{OP}^*(B^1/\mathcal{R}):\ \mathrm{Im}\ \alpha\subseteq B/\mathcal{L},\ \mathrm{Im}\ \beta\subseteq B/\mathcal{R}\}.$$

Clearly $\mathcal{O}^1(B)$ is a subsemigroup of $\mathcal{OP}(B^1/\mathcal{L})\times \mathcal{OP}^*(B^1/\mathcal{R})$, but, unless B has an identity, it cannot be a submonoid.

We omit the proof of the following lemma, as it closely follows that of Lemma 3.1 of [5].

Lemma 2.3. *Let S be a weakly B-abundant semigroup where B is a band. For any $a\in S$, $\alpha_a\in \mathcal{OP}(B^1/\mathcal{L})$ and $\beta_a\in \mathcal{OP}(B^1/\mathcal{R})$. Let $\theta: S\to \mathcal{O}^1(B)$ be given by*

$$a\theta=(\alpha_a,\beta_a).$$

If condition (C) holds, then θ is a strongly admissible morphism with kernel μ_B. Moreover, putting $\overline{B}=\{(\rho_e,\lambda_e): e\in B\}$, we have that $\theta|_B: B\to \overline{B}$ is an isomorphism.

3. The semigroup S_B

Our aim is to construct from a band B a semigroup S_B that is B-fundamental weakly B-abundant with (C), and is such that any fundamental weakly B-abundant semigroup with (C) is a subsemigroup of S_B. Consequently, the fundamental weakly B-abundant semigroup U_B of [5], which is the canonical such semigroup that satisfies (C) and (WIC), is embeddable into S_B. This embedding may be strict, as we show in Section 6. In the construction of U_B, the weak idempotent connected condition (WIC) allows us to connect $\alpha\in \mathcal{OP}(B/\mathcal{L})$ and $\beta\in \mathcal{OP}^*(B/\mathcal{R})$ via a relation between principal ideals of a band B that is analogous to the isomorphisms appearing in Hall's contruction of W_B. Without (WIC) we need a new approach to conditions connecting α and β.

Before introducing S_B, a word on notation: for a set X, an equivalence κ on X and $\gamma: X/\kappa\to X/\kappa$, the relation $\overline{\gamma}$ is defined by

$$\overline{\gamma}=\{(x,y)\in X\times X: y\in [x]\gamma\};$$

we denote by $x\overline{\gamma}$ a typical element y such that $(x,y)\in\overline{\gamma}$. We put

$$S_B=\{(\alpha,\beta)\in \mathcal{O}^1(B):\ \text{for all } x\in B^1,\ x\overline{\alpha}\in L_x\alpha \text{ and } x\overline{\beta}\in R_x\beta,$$

$$\beta\lambda_x=\lambda_{x\overline{\alpha}}\beta\lambda_x \text{ and } \alpha\rho_x=\rho_{x\overline{\beta}}\alpha\rho_x\}.$$

Lemma 3.1. *The set S_B is a subsemigroup of $\mathcal{O}^1(B)$ containing the band $\overline{B}$.*

Proof. Let $(\alpha, \beta), (\gamma, \delta) \in S_B$. Then for any $x \in B^1$,

$$\begin{array}{rcll} \delta\beta\lambda_x & = & \delta\lambda_u\beta\lambda_x & \text{for all } u \in L_x\alpha \\ & = & \lambda_w\delta\lambda_u\beta\lambda_x & \text{for all } u \in L_x\alpha, w \in L_u\gamma = L_x\alpha\gamma \\ & = & \lambda_w\delta\beta\lambda_x & \text{for all } w \in L_x\alpha\gamma. \end{array}$$

The dual argument completes the proof that S_B is a subsemigroup.

To show that $\overline{B} \subseteq S_B$, let $e \in B, x \in B^1$ and $y \in L_x\rho_e = L_{xe}$. Then for any $z \in B^1$,

$$R_z\lambda_y\lambda_e\lambda_x = R_{xeyz} = R_{xez} = R_z\lambda_e\lambda_x.$$

Thus $\lambda_y\lambda_e\lambda_x = \lambda_e\lambda_x$; a similar argument verifies the second condition that (ρ_e, λ_e) must satisfy to lie in S_B. $\square$

The two subsequent results show that S_B is the fundamental semigroup for which we seek.

Theorem 3.2. *The semigroup S_B is weakly $\overline{B}$-abundant with (C) and is $\overline{B}$-fundamental.*

Proof. Let $(\alpha, \beta) \in S_B$; choose any $u \in L_1\alpha$ and $v \in R_1\beta$. We claim that

$$(\alpha, \beta)\, \widetilde{\mathcal{L}}_{\overline{B}}\, (\rho_u, \lambda_u), \quad (\alpha, \beta)\, \widetilde{\mathcal{R}}_{\overline{B}}\, (\rho_v, \lambda_v).$$

We prove the result for $\widetilde{\mathcal{L}}_{\overline{B}}$, that for $\widetilde{\mathcal{R}}_{\overline{B}}$ being dual. Notice first that if $u, u' \in L_1\alpha$, then $u \,\mathcal{L}\, u' \in B$ so that as $e \to (\rho_e, \lambda_e)$ is an isomorphism from B to $\overline{B}$, we have that $(\rho_u, \lambda_u)\, \mathcal{L}\, (\rho_{u'}, \lambda_{u'})$ in $\overline{B}$ and hence in S_B. Thus $(\rho_u, \lambda_u)\, \widetilde{\mathcal{L}}_{\overline{B}}\, (\rho_{u'}, \lambda_{u'})$.

We have that

$$(\alpha, \beta)(\rho_u, \lambda_u) = (\alpha\rho_u, \lambda_u\beta).$$

For any $x \in B^1, L_x\alpha \leq L_1\alpha = L_u$, since α is order preserving. Putting $L_x\alpha = L_y$, we have that

$$L_x\alpha\rho_u = L_y\rho_u = L_{yu} = L_y = L_x\alpha,$$

so that $\alpha\rho_u = \alpha$. On the other hand, using one of the defining conditions for membership of S_B,

$$\beta = \beta\lambda_1 = \lambda_u\beta\lambda_1 = \lambda_u\beta,$$

giving that $(\alpha, \beta)(\rho_u, \lambda_u) = (\alpha, \beta)$.

Suppose now that $(\alpha, \beta)(\rho_e, \lambda_e) = (\alpha, \beta)$ for some $e \in B$. Then $\alpha\rho_e = \alpha$ so that

$$L_u = L_1\alpha = L_1\alpha\rho_e = L_u\rho_e = L_{ue},$$

giving that $u\,\mathcal{L}\,ue$. Thus $u = u(ue) = ue$ and from the isomorphism between B and $\overline{B}$ we have that $(\rho_u, \lambda_u)(\rho_e, \lambda_e) = (\rho_u, \lambda_u)$. Hence $(\alpha, \beta)\,\widetilde{\mathcal{L}}_{\overline{B}}\,(\rho_u, \lambda_u)$ and dually, $(\alpha, \beta)\,\widetilde{\mathcal{R}}_{\overline{B}}\,(\rho_v, \lambda_v)$. Therefore S_B is weakly $\overline{B}$-abundant.

Observe now that if $(\alpha, \beta), (\gamma, \delta) \in S_B$, $u \in L_1\alpha$ and $u' \in L_1\gamma$, then

$$\begin{aligned}(\alpha, \beta)\,\widetilde{\mathcal{L}}_{\overline{B}}\,(\gamma, \delta) &\Leftrightarrow (\rho_u, \lambda_u)\,\widetilde{\mathcal{L}}_{\overline{B}}\,(\rho_{u'}, \lambda_{u'})\\ &\Leftrightarrow (\rho_u, \lambda_u)\,\mathcal{L}\,(\rho_{u'}, \lambda_{u'}) \quad \text{in } \overline{B}\\ &\Leftrightarrow u\,\mathcal{L}\,u'\\ &\Leftrightarrow L_1\alpha = L_1\gamma.\end{aligned}$$

Dually,

$$(\alpha, \beta)\,\widetilde{\mathcal{R}}_{\overline{B}}\,(\gamma, \delta) \Leftrightarrow R_1\beta = R_1\delta.$$

It is now easy to see that (C) holds.

Suppose now that

$$(\alpha, \beta)\,\mu_{\overline{B}}\,(\gamma, \delta).$$

Certainly

$$(\alpha, \beta)\,\widetilde{\mathcal{H}}_{\overline{B}}\,(\gamma, \delta)$$

so that $L_1\alpha = L_1\gamma$ and $R_1\beta = R_1\delta$. Further, since $\mu_{\overline{B}}$ is a congruence contained in $\widetilde{\mathcal{H}}_{\overline{B}}$, for any $e \in B$,

$$(\rho_e, \lambda_e)(\alpha, \beta)\,\widetilde{\mathcal{H}}_{\overline{B}}\,(\rho_e, \lambda_e)(\gamma, \delta),$$

so that

$$L_1\rho_e\alpha = L_1\rho_e\gamma$$

and consequently, $L_e\alpha = L_e\gamma$, giving that $\alpha = \gamma$. Similarly,

$$(\alpha, \beta)(\rho_e, \lambda_e)\,\widetilde{\mathcal{H}}_{\overline{B}}\,(\gamma, \delta)(\rho_e, \lambda_e)$$

so that $R_1\lambda_e\beta = R_1\lambda_e\delta$, giving $R_e\beta = R_e\delta$. We deduce that $(\alpha, \beta) = (\gamma, \delta)$, and so S_B is $\overline{B}$-fundamental as required. $\square$

Theorem 3.3. *Let S be a weakly B-abundant semigroup with (C). Then $\theta : S \to S_B$ given by*

$$a\theta = (\alpha_a, \beta_a)$$

where for all $x \in B^1$, $L_x\alpha_a = L_{(xa)^*}$ and $R_x\beta_a = R_{(ax)^+}$, is a strongly admissible morphism with kernel μ_B. Moreover, $\theta|_B : B \to \overline{B}$ is an isomorphism.

Proof. In view of Lemma 2.3, it remains only to show that the image of θ is contained in S_B.

Let $a \in S$, $x \in B^1$ and let $u \in L_x\alpha_a = L_{(xa)^*}$. For any $y \in B^1$ and for any choice of $(ay)^+$, we have that

$$x(ay)^+\,\widetilde{\mathcal{R}}_B\,xay = xauy\,\widetilde{\mathcal{R}}_B\,x(auy)^+$$

for any choice of $(auy)^+$. Hence

$$R_y\beta_a\lambda_x = R_{(ay)^+}\lambda_x = R_{x(ay)^+} = R_{x(auy)^+} = R_y\lambda_u\beta_a\lambda_x.$$

Thus $\beta_a\lambda_x = \lambda_u\beta_a\lambda_x$. Putting this together with the dual argument we have shown that

$$a\theta = (\alpha_a, \beta_a) \in S_B.$$

$\square$

Corollary 3.4. *Let S be a weakly B-abundant semigroup with (C). If S is fundamental, then the morphism θ given in Theorem 3.3 is an embedding.*

4. A semilattice of idempotents

Weakly E-abundant semigroups with (C), where E is a semilattice, are the topic of [8]. In that article we construct a weakly E-abundant E-fundamental semigroup C_E, and show that any weakly E-abundant semigroup with (C) embeds into C_E. Clearly then we have both that S_E is embedded into C_E and C_E is embedded into S_E. We now show that, in fact, S_E and C_E are isomorphic.

The semigroup C_E is the subset of $\mathcal{OP}(E^1) \times \mathcal{OP}^*(E^1)$ defined by

$$C_E = \{(\alpha,\beta) \in \mathcal{OP}(E^1) \times \mathcal{OP}^*(E^1) : \text{ Im } \alpha, \text{ Im } \beta \subseteq E \text{ and } \forall x \in E^1,$$
$$\tau_{x\alpha} \leq \beta\tau_x\alpha \text{ and } \tau_{x\beta} \leq \alpha\tau_x\beta\},$$

where for $x \in E$, $\tau : E^1 \to E$ is the order preserving map given by $e\tau_x = ex = xe$ and for $\gamma, \delta \in \mathcal{OP}(E^1)$, $\gamma \leq \delta$ means that $y\gamma \leq y\delta$ for all $y \in E^1$.

Proposition 4.1. *Let E be a semilattice. Then S_E is isomorphic to C_E.*

Proof. Green's relations are all trivial on the semilattice E^1; we identify E^1 with $E^1/\mathcal{L}$ and $E^1/\mathcal{R}$ and order preserving maps from $E^1/\mathcal{L}$ to $E/\mathcal{L}$ and from $E^1/\mathcal{R}$ to $E/\mathcal{R}$ with the corresponding order preserving maps from E^1 to E. Under this identification, both ρ_x and λ_x are identified with τ_x, and we may regard both C_E and S_E as subsemigroups of $\mathcal{OP}(E^1) \times \mathcal{OP}^*(E^1)$.

With this rephrasing, the definition of S_E simplifies to

$$S_E = \{(\alpha,\beta) \in \mathcal{OP}(E^1) \times \mathcal{OP}^*(E^1) : \text{ Im } \alpha, \text{ Im } \beta \subseteq E \text{ and } \forall x \in E^1,$$
$$\beta\tau_x = \tau_{x\alpha}\beta\tau_x \text{ and } \alpha\tau_x = \tau_{x\beta}\alpha\tau_x\}.$$

Let $(\alpha,\beta) \in C_E$. Then for any $x \in E^1$,

$$\tau_{x\alpha} \leq \beta\tau_x\alpha \text{ and } \tau_{x\beta} \leq \alpha\tau_x\beta.$$

For any $y \in E^1$ certainly $y\,(x\beta) \leq y$ and so

$$y\tau_{x\beta}\alpha\tau_x \leq y\alpha\tau_x.$$

In addition we have that

$$y(\alpha\tau_x) = (y\alpha)\,x = x\tau_{y\alpha} \leq x\beta\tau_y\alpha = ((x\beta)y)\alpha = y(\tau_{x\beta}\alpha).$$

Since

$$y\,\alpha\tau_x \leq y\,\tau_{x\beta}\alpha$$

and τ_x is idempotent, we have that

$$y\,\alpha\tau_x \leq y\,\tau_{x\beta}\alpha\tau_x$$

and so $\alpha\tau_x = \tau_{x\beta}\alpha\tau_x$. Together with the dual argument, we have shown that $(\alpha, \beta) \in S_E$.

Conversely, suppose we are given $(\alpha, \beta) \in S_E$. Then for any $x, y \in E^1$,

$$\begin{aligned} y\tau_{x\alpha} &= y\,(x\alpha) = x\alpha\tau_y = x\tau_{y\beta}\alpha\tau_y = (x\,(y\beta))\alpha\tau_y \\ &= y\beta\tau_x\alpha\tau_y = (y\beta\tau_x\alpha)\,y \leq y\beta\tau_x\alpha \end{aligned}$$

so that $\tau_{x\alpha} \leq \beta\tau_x\alpha$. Again with the dual, we obtain that $(\alpha, \beta) \in C(E)$ and so $S_E = C_E$ as required.

□

5. The idempotent connected case

In [5] El Qallali, Fountain and the second author construct a fundamental weakly abundant semigroup U_B having band of idempotents B, satisfying (C) and the *weak idempotent connected* condition (WIC). From Theorem 3 it follows that U_B is embedded into S_B. The aim of this section is to give a very natural way of achieving this; we can 'almost' regard U_B as a subsemigroup of S_B. To this end we recall from [5] the definition of (WIC), designed to give us some control over the position of idempotents in products (reminiscent of that in a regular semigroup) and the construction of U_B.

For an element e of B we denote by $\langle e\rangle$ the principal order ideal generated by e; so that

$$\langle e\rangle = \{x \in B : x \leq e\} = \{x \in B : ex = xe = x\}.$$

Let S be a weakly B-abundant semigroup where B is a band. We say that S satisfies the *weak idempotent connected* condition (WIC) (with respect to B) if for any $a \in S$ and some a^*, a^+, if $x \in \langle a^+\rangle$ then there exists $y \in B$ with $xa = ay$; dually, if $z \in \langle a^*\rangle$ then there exists $t \in B$ with $ta = az$. As remarked in [5], we can replace 'some' in (WIC) by 'any', and assume that $y \in \langle a^\circ\rangle$ and $t \in \langle a^\dagger\rangle$ for any $a^\circ, a^\dagger \in B$ with $a^\circ\,\widetilde{\mathcal{L}}_B\,a\,\widetilde{\mathcal{R}}_B\,a^\dagger$. Further details can be found in [16] and [5]; we content ourselves here with commenting that a regular semigroup has (WIC), as does an idempotent connected abundant semigroup [5].

The semigroup U_B is a subsemigroup of $\mathcal{OP}(B) \times \mathcal{OP}^*(B)$, defined via the notion of connecting relations between order ideals of B.

Let $e, f \in B$ and let $I^{e,f} \subseteq \langle e \rangle \times \langle f \rangle$. We say that $I^{e,f}$ is *connecting* if $I^{e,f}$ is a subsemigroup of $\langle e \rangle \times \langle f \rangle$ and for every $(x, x'), (y, y') \in I^{e,f}$ we have that

$$x \leq_{\mathcal{L}} y \text{ implies that } x' \leq_{\mathcal{L}} y'$$

and

$$x' \leq_{\mathcal{R}} y' \text{ implies that } x \leq_{\mathcal{R}} y.$$

The relation $I^{e,f} \subseteq \langle e \rangle \times \langle f \rangle$ is said to be *full* if both projection maps are both onto.

The following lemma will be needed in the proof of Proposition 5.3.

Lemma 5.1. *(Lemma 4.1,* [5]*) Let $I^{e,f}$ be connecting. Then for any $(x, y), (z, t) \in I^{e,f}$,*

$$x \leq_{\mathcal{D}} z \text{ if and only if } y \leq_{\mathcal{D}} t.$$

We use full connecting relations to define elements of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$; for brevity we denote this semigroup by $\mathcal{O}(B)$. Let $I^{e,f}$ be full connecting; we begin by defining partial maps $I^{e,f}_\ell$ of $B/\mathcal{L}$ and $I^{e,f}_r$ of $B/\mathcal{R}$ by setting

$$L_x I^{e,f}_\ell = L_y \text{ where } (x, y) \in I^{e,f}$$

and

$$R_y I^{e,f}_r = R_x \text{ for } (x, y) \in I^{e,f}.$$

The fact that $I^{e,f}$ is full connecting gives immediately that $I^{e,f}_\ell$ and $I^{e,f}_r$ have domains $\{L_x : x \leq e\}$ and $\{R_y : y \leq f\}$, respectively, and that they are well defined and order preserving on these domains.

Let $e \in B$; to distinguish the order preserving map ρ_e of $B^1/\mathcal{L}$ induced by right multiplication by e, from that of $B/\mathcal{L}$, we denote the latter by ρ'_e. Similarly, $\lambda'_e : B/\mathcal{R} \to B/\mathcal{R}$ is given by $R_x \lambda'_e = R_{ex}$.

Consider now the element $\rho'_e \in \mathcal{OP}(B/\mathcal{L})$; the image of ρ'_e is $\{L_{xe} : x \in B\}$. Since $exe \,\mathcal{L}\, xe$, we have that the image of ρ'_e is $\{L_x : x \leq e\}$, that is, the image of ρ'_e is the domain of $I^{e,f}_\ell$. Thus we may compose the order preserving maps ρ'_e and $I^{e,f}_\ell$ to obtain an element of $\mathcal{OP}(B/\mathcal{L})$. Similarly, $\lambda'_f I^{e,f}_r \in \mathcal{OP}^*(B/\mathcal{R})$. We have shown that

$$U_B = \{(\rho'_e I^{e,f}_\ell, \lambda'_f I^{e,f}_r) : e, f \in B, I^{e,f} \subseteq \langle e \rangle \times \langle f \rangle \text{ is full connecting}\}$$

is a subset of $\mathcal{O}^1(B)$.

Theorem 5.2. *(Lemma 4.3, Theorem 4.4, [5]) The set U_B is a subsemigroup of $\mathcal{O}(B)$ with band of idempotents*

$$B^* = \{(\rho'_e, \lambda'_e) : e \in B\}$$

isomorphic to B. Further, U_B is fundamental, weakly abundant and has (C) and (WIC).

We would like to say that U_B is a subsemigroup of S_B; this is not quite true, since for $(\alpha, \beta) \in U_B$, the domain of α is $B/\mathcal{L}$, rather than $B^1/\mathcal{L}$; similarly for β. However, the natural extension of domain gives the result we seek.

We first note that from the proof of Theorem 4.4 of [5], for any $(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f}) \in U_B$,

$$(\rho'_f, \lambda'_f)\, \widetilde{\mathcal{L}}\, (\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f})\, \widetilde{\mathcal{R}}\, (\rho'_e, \lambda'_e).$$

Consequently, if

$$(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f}) = (\rho'_g J_\ell^{g,h}, \lambda'_h J_r^{g,h})$$

then we have that $e\,\mathcal{R}\,g$ and $f\,\mathcal{L}\,h$.

Consider $(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f}) \in U_B$; we put $(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f})\kappa = (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})$. We remark that we have extended the domain of $\rho'_e I_\ell^{e,f}$ from $B/\mathcal{L}$ to $B^1/\mathcal{L}$, in such a way that

$$L_1 \rho_e I_\ell^{e,f} = L_e \rho_e I_\ell^{e,f} = L_e \rho'_e I_\ell^{e,f},$$

and similarly

$$R_1 \lambda_f I_r^{e,f} = R_f \lambda'_f I_r^{e,f}.$$

We observe after the proof of Lemma 4.2 in [5] that if $I^{e,f} \subseteq \langle e \rangle \times \langle f \rangle$ is full and connecting, then

$$(e, x) \in I^{e,f} \text{ if and only if } x = f$$

and dually,

$$(x, f) \in I^{e,f} \text{ if and only if } x = e.$$

So if $(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f}) = (\rho'_g J_\ell^{g,h}, \lambda'_h J_r^{g,h})$ then

$$L_1 \rho_e I_\ell^{e,f} = L_e \rho'_e I_\ell^{e,f} = L_e I_\ell^{e,f} = L_f = L_h = \ldots = L_1 \rho_g J_\ell^{g,h}$$

and so $\rho_e I_\ell^{e,f} = \rho_g J_\ell^{g,h}$. Similarly, $\lambda_f I_r^{e,f} = \lambda_h J_r^{g,h}$. Hence κ is well defined.

Proposition 5.3. *With κ defined as above, $\kappa : U_B \to S_B$ is an embedding.*

Proof. We must first show that the image of κ is contained in S_B. Suppose that $(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f}) \in U_B$; we must argue that $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \in S_B$.

Let $x \in B^1$; we have that

$$L_x \rho_e I_\ell^{e,f} = L_{exe} I_\ell^{e,f} = L_u$$

where $(exe, u) \in I^{e,f}$. Let $y \in L_u$, so that $y \,\mathcal{L}\, u \leq f$ and

$$u \,\mathcal{L}\, y = yfy \,\mathcal{L}\, fy = fyf,$$

whence $fyf = fyfu$. Certainly $fyf \in \langle f \rangle$, so that $(w, fyf) \in I^{e,f}$ for some $w \in \langle e \rangle$, since $I^{e,f}$ is full. Notice that $u \,\mathcal{D}\, fyf$ so that from Lemma 5.1, $xe \,\mathcal{L}\, exe \,\mathcal{D}\, w$ and $xewxe = xe$.

Let $z \in B^1$ and pick $(v, fzf) \in I^{e,f}$. As $I^{e,f}$ is a semigroup,

$$(wexev, fyffzf) = (wexev, fyfufzf) = (w, fyf)(exe, u)(v, fzf) \in I^{e,f}.$$

Calculating, we have

$$\begin{array}{rcl}
R_z \lambda_y \lambda_f I_r^{e,f} \lambda_x & = & R_{fyz} I_r^{e,f} \lambda_x \\
& = & R_{fyzf} I_r^{e,f} \lambda_x \\
& = & R_{fyffzf} I_r^{e,f} \lambda_x \\
& = & R_{wexev} \lambda_x \\
& = & R_{xwexev} \\
& = & R_{xewxev} \quad \text{since } w \in \langle e \rangle \\
& = & R_{xev} \\
& = & R_{xv} \quad \text{since } v \in \langle e \rangle \\
& & \\
& = & R_v \lambda_x \\
& = & R_{fzf} I_r^{e,f} \lambda_x \\
& = & R_z \lambda_f I_r^{e,f} \lambda_x .
\end{array}$$

We have shown that

$$\lambda_y \lambda_f I_r^{e,f} \lambda_x = \lambda_f I_r^{e,f} \lambda_x;$$

together with the dual argument, we have that $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \in S_B$, as required.

It is clear from the definition that κ is one to one, and $(\rho'_e, \lambda'_e)\kappa = (\rho_e, \lambda_e)$, so that $\kappa : B^* \to \overline{B}$ is a bijection.

To see that κ is a morphism, let $(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f}), (\rho'_g J_\ell^{g,h}, \lambda'_h J_r^{g,h}) \in U_B$; as in [5] we have that

$$(\rho'_e I_\ell^{e,f}, \lambda'_f I_r^{e,f})(\rho'_g J_\ell^{g,h}, \lambda'_h J_r^{g,h}) = (\rho'_z K_\ell^{z,w}, \lambda'_w K_r^{z,w})$$

where $(z, fgf) \in I^{e,f}, (gfg, w) \in J^{g,h}$. Then

$$\begin{array}{rcl}((\rho'_e I^{e,f}_\ell, \lambda'_f I^{e,f}_r)(\rho'_g J^{g,h}_\ell, \lambda'_h J^{g,h}_r))\kappa & = & (\rho'_z K^{z,w}_\ell, \lambda'_w K^{z,w}_r)\kappa \\ & = & (\rho_z K^{z,w}_\ell, \lambda_w K^{z,w}_r),\end{array}$$

whereas

$$\begin{array}{rcl}(\rho'_e I^{e,f}_\ell, \lambda'_f I^{e,f}_r)\kappa(\rho'_g J^{g,h}_\ell, \lambda'_h J^{g,h}_r)\kappa & = & (\rho_e I^{e,f}_\ell, \lambda_f I^{e,f}_r)(\rho_g J^{g,h}_\ell, \lambda_h J^{g,h}_r) \\ & = & (\rho_e I^{e,f}_\ell \rho_g J^{g,h}_\ell, \lambda_h J^{g,h}_r \lambda_f I^{e,f}_r) \\ & = & (\rho_e I^{e,f}_\ell \rho'_g J^{g,h}_\ell, \lambda_h J^{g,h}_r \lambda'_f I^{e,f}_r)\end{array}$$

Now

$$\begin{array}{rclcl}L_1 \rho_e I^{e,f}_\ell \rho'_g J^{g,h}_\ell & = & L_e(\rho'_e I^{e,f}_\ell \rho'_g J^{g,h}_\ell) & = & L_e \rho'_z K^{z,w}_\ell = \\ L_{zez} K^{z,w}_\ell & = & L_z K^{z,w}_\ell & = & L_1 \rho_z K^{z,w}_\ell\end{array}$$

and clearly, for any $t \in B$,

$$L_t \rho_e I^{e,f}_\ell \rho'_g J^{g,h}_\ell = L_t(\rho'_e I^{e,f}_\ell \rho'_g J^{g,h}_\ell) = L_t \rho'_z K^{z,w}_\ell = L_t \rho_z K^{z,w}_\ell.$$

With the dual, we have shown that κ is a morphism, and hence completed the proof of the proposition.

$\square$

6. Examples

In this final section we examine some semigroups S_B for bands B of small finite cardinality. Where B is rectangular, S_B does not differ from the fundamental weakly B-abundant semigroup U_B with (C) and (WIC), constructed in [5]: more than this, S_B is equal to the Hall semigroup W_B. By considering a two-element right zero semigroup with an identity adjoined, we find a regular S_B distinct from U_B (and hence from W_B). On the other hand, if we adjoin both an identity and a zero to a two-element right zero semigroup, we have an example of a four element band for which $S_B \neq U_B$ and S_B is not regular.

We begin by considering the case of a non-trivial rectangular band B. For any $\alpha \in \mathcal{OP}(B^1/\mathcal{L})$, and for any $a \in B$, since it is certainly true that $L_a \leq L_1$ we must have that $L_a\alpha \leq L_1\alpha$, and hence that $L_a\alpha = L_1\alpha$. Thus if $L_1\alpha = L_e$, we have that $\alpha = \rho_e$. Dually, any $\beta \in \mathcal{OP}(B^1/\mathcal{R})$ is of the form λ_f for some $f \in B$. Notice that for any $e, f \in B$, we have that $(\rho_e, \lambda_f) = (\rho_{fe}, \lambda_{fe})$, whence we deduce that

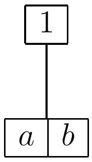
$$B \cong \overline{B} = \mathcal{O}^1(B) = S_B.$$

Certainly then S_B is regular, moreover, in view Example 6.1 of [5], we deduce the following.

Proposition 6.1. *Let B be a rectangular band. Then*

$$W_B = V_B = U_B \cong S_B \cong B.$$

The situation already becomes more interesting if we consider the three element band $C = \{1, a, b\}$ having $\mathcal{D}$-class structure as depicted below:



For comparison with S_B we first consider the fundamental idempotent connected semigroups on C and $D = C^0$.

Lemma 6.2. *For $C = \{1, a, b\}$ as above, $W_C = V_C = U_C$ is a four element (regular) semigroup.*

Proof. We examine U_C: recall from Section 5 that

$$U_C = \{(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) : e, f \in C, I^{e,f} \subseteq \langle e\rangle \times \langle f\rangle \text{ is full connecting}\},$$

where we may now drop the $'$ from ρ'_e, λ'_f, since $C = C^1$. Note that by Lemma 4.1 of [5], any $I^{e,f}$ must induce an order isomorphism on the $\mathcal{D}$-classes of C. So the only possibilities for full connecting relations $I^{e,f}$ are where $e = f$, $e = a, f = b$, or $e = b, f = a$. It is also helpful to remark that ρ_a and ρ_b are constant maps with image L_a and L_b, respectively, and that $\lambda_a = \lambda_b$ is constant with image R_a.

The only full connecting relation in $\langle a\rangle \times \langle a\rangle$ is $\iota^{a,a} = \{(a, a)\}$, and it is easy to see that $(\rho_a \iota_\ell^{a,a}, \lambda_a \iota_r^{a,a}) = (\rho_a, \lambda_a)$; dually for b. Similarly, the only full connecting relation contained in $\langle a\rangle \times \langle b\rangle$ is $I^{a,b} = \{(a, b)\}$ and we again find that $(\rho_a I_\ell^{a,b}, \lambda_b I_r^{a,b}) = (\rho_b, \lambda_b) \in \overline{C}$; dually for the case where a and b are interchanged.

We are left with examining full connecting relations contained in

$$\langle 1\rangle \times \langle 1\rangle = C \times C.$$

One will certainly be the equality relation, which will give rise to the element $(\rho_1, \lambda_1) \in \overline{C}$. If $I^{1,1}$ is a full connecting distinct from equality, then we must have that (a, b) or $(b, a) \in I^{1,1}$. If $(a, b) \in I^{1,1}$, then as $I^{1,1}$ is connecting, we *cannot* have that $(a, a) \in I^{1,1}$, since $a \,\mathcal{L}\, a$ but b is *not* $\mathcal{L}$-related to a. But then, as $I^{1,1}$ is full, we must have that $(b, a) \in I^{1,1}$, and the same reasoning gives $(b, b) \notin I^{1,1}$. We deduce that the only possibility for $I^{1,1}$ is

$$I^{1,1} = \{(1, 1), (a, b), (b, a)\};$$

by inspection this is full connecting. An easy check shows that

$$(\rho_1 I_\ell^{1,1}, \lambda_1 I_r^{1,1}) = \left(\begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_b & L_a \end{pmatrix}, \begin{pmatrix} R_1 & R_a \\ R_1 & R_a \end{pmatrix} \right) = (\gamma, \lambda_1),$$

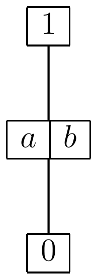
say, where $\gamma^2 = \rho_1$. For ease of notation, expressing $(\rho_e, \lambda_e) \in \overline{C}$ as e, and (γ, λ_1) as c, we have that $U_C = \{1, a, b, c\}$ and has table

	1	c	a	b
1	1	c	a	b
c	c	1	a	b
a	a	b	a	b
b	b	a	a	b

As $I^{1,1}$ is the graph of an isomorphism, it follows that $c \in W_C$ and hence that $W_C = U_C$; as V_C is intermediate between W_C and U_C, we deduce that $U_C = V_C$ also.

□

It is easy to check that for any band B, $W_{B^0} = W_B^0, V_{B^0} = V_B^0$ and $U_{B^0} = U_B^0$, so that if D is the four element band $\{1, a, b, 0\}$ with $\mathcal{D}$-class structure given by



then, in view of Lemma 6.2, $W_D = V_D = U_D$ is a five element regular semigroup. We show that by contrast, $|S_C| = 23$ and $|S_D| = 154$. To achieve this aim, we now present a series of lemmas that will enable us to check for a general band B when a pair $(\alpha, \beta) \in \mathcal{O}^1(B)$ lies in S_B, before applying these lemmas to the specific bands C and D as above.

We begin by remarking that if B is a band with identity, $\mathcal{O}^1(B) = \mathcal{O}(B)$ is a monoid with identity (ρ_1, λ_1); since $(\rho_1, \lambda_1) \in \overline{B}$, it is immediate that $(\rho_1, \lambda_1) \in S_B$, so that S_B is a monoid. We take the opportunity to stress that $B = B^1$ if and only if B has an identity.

We turn our attention to zeros: if B is a band with zero, then for any $\alpha \in \mathcal{OP}(B^1/\mathcal{L})$ and $\beta \in \mathcal{OP}(B^1/\mathcal{R})$ is is clear that for any $0\overline{\alpha} \in L_0\alpha$ and $0\overline{\beta} \in R_0\beta$,

$$\beta\lambda_0 = \lambda_{0\overline{\alpha}}\beta\lambda_0 \text{ and } \alpha\rho_0 = \rho_{0\overline{\beta}}\alpha\rho_0.$$

Thus, in checking whether a pair (α, β) satisfies the membership condition for S_B, we need only consider non-zero $x \in B^1$.

For a fixed $\mathcal{L}$-class L and fixed $\mathcal{R}$-class R of B, denote by c_L and d_R the constant maps in $\mathcal{OP}(B^1/\mathcal{L})$ and $\mathcal{OP}(B^1/\mathcal{R})$ with images L and

R, respectively. The proof of the next lemma is immediate from the definition of S_B; it serves to demonstrate that for an arbitrary band B, the band $\overline{B}$ is a strict subset of $E(S_B)$.

Lemma 6.3. *Any pair of the form (c_L, d_R) lies in S_B.*

Lemma 6.4. *For any $\alpha \in \mathcal{OP}(B^1/\mathcal{L})$, $(\alpha, d_R) \in S_B$ if and only if $\alpha = \rho_u\alpha$ for all $u \in R$. Dually, for any $\beta \in \mathcal{OP}(B^1/\mathcal{R})$, $(c_L, \beta) \in S_B$ if and only if $\beta = \lambda_v\beta$ for all $v \in L$.*

Proof. Clearly $d_R\lambda_x = \lambda_{x\overline{\alpha}}d_R\lambda_x$ for any $x \in B^1$ and $x\overline{\alpha} \in L_x\alpha$. Thus $(\alpha, d_R) \in S_B$ if and only if $\alpha\rho_x = \rho_{x\overline{d_R}}\alpha\rho_x$ for any $x \in B^1$ and $x\overline{d_R} \in R_xd_R = R$. But this is saying that $\alpha\rho_x = \rho_u\alpha\rho_x$ for all $x \in B^1$ and $u \in R$, which is clearly equivalent to $\alpha = \rho_u\alpha$ for all $u \in R$.

$\square$

We make the notational convention that if B has an identity 1, then c_1 denotes the constant map c_{L_1} and d_1 denotes the constant map d_{R_1}. The next result is immediate from Lemma 6.4.

Corollary 6.5. *If B has an identity, then for any $\alpha \in \mathcal{OP}(B/\mathcal{L})$ and $\beta \in \mathcal{OP}(B/\mathcal{R})$ we have that $(\alpha, d_1), (c_1, \beta) \in S_B$.*

On the other hand, if B has a zero, then certainly ρ_0, λ_0 are constants, so that again calling upon Lemma 6.4, we deduce:

Lemma 6.6. *Let B be a band with zero. Then for any $\alpha \in \mathcal{OP}(B^1/\mathcal{L}), \beta \in \mathcal{OP}(B^1/\mathcal{R})$,*

$$(\alpha, \lambda_0) \in S_B \text{ if and only if } \alpha \text{ is constant}$$

and dually,

$$(\rho_0, \beta) \in S_B \text{ if and only if } \beta \text{ is constant.}$$

We now give a condition for a band $B = B^1$ such that (ρ_1, β) or (α, λ_1) lie in S_B.

Lemma 6.7. *Let B be a band with identity. Then for any $\alpha \in \mathcal{OP}(B/\mathcal{L})$, $(\alpha, \lambda_1) \in S_B$ if and only if $\alpha\rho_x = \rho_u\alpha\rho_x$ for all $x \in B$ and $u \in R_x$, and $\lambda_x = \lambda_{xv}$ for all $v \in L_x\alpha$.*

Proof. By the membership condition for S_B, we have that $(\alpha, \lambda_1) \in S_B$ if and only if for any $x \in B$, $v \in L_x\alpha$ and $u \in R_x\lambda_1 = R_x$,

$$\lambda_1\lambda_x = \lambda_v\lambda_1\lambda_x, \text{ and } \alpha\rho_x = \rho_u\alpha\rho_x,$$

that is, if and only if

$$\lambda_x = \lambda_{xv} \text{ and } \alpha\rho_x = \rho_u\alpha\rho_x,$$

as required. $\square$

The dual of Lemma 6.7 says that for a band B with identity, $(\rho_1, \beta) \in S_B$ if and only if for all $x \in B, u \in L_x$ and $v \in R_x\beta$,

$$\beta\lambda_x = \lambda_u\beta\lambda_x \text{ and } \rho_x = \rho_{vx}.$$

In the case where $B = T^1$ where T is rectangular, Lemma 6.7 simplifies, as we now show.

Lemma 6.8. *Let $B = T^1$, where T is a non-trivial rectangular band. Then for any $\alpha \in \mathcal{OP}(B/\mathcal{L}), \beta \in \mathcal{OP}(B/\mathcal{R})$, we have that*

$$(\alpha, \lambda_1) \in S_B \textit{ if and only if } L_1\alpha = L_1$$

and dually

$$(\rho_1, \beta) \in S_B \textit{ if and only if } R_1\beta = R_1.$$

Proof. For any $x \in T$ it is clear that $\alpha\rho_x = \rho_u\alpha\rho_x$ for any $u \in R_x$, and $\lambda_x = \lambda_{xv}$ for any $v \in L_x\alpha$. Thus in view of Lemma 6.7, $(\alpha, \lambda_1) \in S_B$ if and only if $\alpha\rho_1 = \rho_u\alpha\rho_1$ for all $u \in R_1 = \{1\}$, and $\lambda_1 = \lambda_{1v}$ for all $v \in L_1\alpha$. The first condition always holds. The second is equivalent to λ_v being the identity for all $v \in L_1\alpha$. This is true if and only if $v = 1$, that is, $L_1\alpha = L_1$.

□

Lemma 6.4 also simplifies in the case $B = T^1$, where T is rectangular.

Lemma 6.9. *Let $B = T^1$, where T is a non-trivial rectangular band. Then for any $\mathcal{R}$-class R of T and $\alpha \in \mathcal{OP}(B/\mathcal{L})$,*

$$(\alpha, d_R) \in S_B \textit{ if and only if } \alpha \textit{ is constant}$$

and dually, for any $\mathcal{L}$-class L of T and any $\beta \in \mathcal{OP}(B/\mathcal{R})$

$$(c_L, \beta) \in S_B \textit{ if and only if } \beta \textit{ is constant.}$$

Turning our attention now to the case where $B = T_0^1$, for a non-trivial rectangular band T, we consider the analogue of Lemma 6.9.

Lemma 6.10. *Let $B = T_0^1$ where T is a non-trivial rectangular band. Then for any $\mathcal{R}$-class R of T, and $\alpha \in \mathcal{OP}(B/\mathcal{L})$,*

$$(\alpha, d_R) \in S_B \textit{ if and only if } L_1\alpha = L_v\alpha \textit{ for all } v \in T,$$

and dually, for any $\mathcal{L}$-class L of T, and $\beta \in \mathcal{OP}(B/\mathcal{R})$,

$$(c_L, \beta) \in S_B \textit{ if and only if } R_1\beta = R_v\beta \textit{ for all } v \in T.$$

Proof. Let α, R be as given. By Lemma 6.4,

$$\begin{array}{rcll} (\alpha, d_R) \in S_B & \Leftrightarrow & \alpha = \rho_u \alpha & \forall u \in R \\ & \Leftrightarrow & L_1\alpha = L_u\alpha, L_f\alpha = L_{fu}\alpha & \forall u \in R, \forall f \in T \\ & & \text{and } L_0\alpha = L_{0u}\alpha & \\ & \Leftrightarrow & L_1\alpha = L_v\alpha & \forall v \in T. \end{array}$$

□

We now turn out attention to pairs of the form (ρ_e, λ_f) where $e, f \in B$. We pause to remark that for any $e, f \in B$, $\rho_e = \rho_f$ if and only if $xe \,\mathcal{L}\, xf$ for all $x \in B^1$. Thus if $\rho_e = \rho_f$, then $e \,\mathcal{L}\, f$, and if $\mathcal{L}$ is a congruence on B, then $\rho_e = \rho_f$ if and only if $e \,\mathcal{L}\, f$. Dually, if $\lambda_e = \lambda_f$ then certainly $e \,\mathcal{R}\, f$, and if $\mathcal{R}$ is a congruence, then $\lambda_e = \lambda_f$ if and only if $e \,\mathcal{R}\, f$. In both bands C and D, $\mathcal{L}$ and $\mathcal{R}$ are congruences, since $\mathcal{L}$ is equality and $\mathcal{R}$ is $\mathcal{D} = \mathcal{J}$, the latter always being a congruence for a band.

Lemma 6.11. *For any $e, f \in B$, if $(\rho_e, \lambda_f) \in S_B$, then $e \,\mathcal{D}\, f$. If $\mathcal{L}$ and $\mathcal{R}$ are both congruences on B, then $(\rho_e, \lambda_f) \in S_B$ if and only if $e \,\mathcal{D}\, f$.*

Proof. Suppose that $(\rho_e, \lambda_f) \in S_B$. Then for any $x \in B^1$ and $u \in L_x\rho_e = L_{xe}$, we have that

$$\lambda_f \lambda_x = \lambda_u \lambda_f \lambda_x$$

and so $\lambda_{xf} = \lambda_{xfu}$. Taking $x = f$ and $u = fe$ we have that $\lambda_f = \lambda_{fe}$ so that by the comments preceding the lemma, $f \,\mathcal{R}\, fe$. It follows that $f \leq_{\mathcal{J}} e$; together with the dual argument we obtain $e \,\mathcal{D}\, f$.

Now assume that $\mathcal{L}$ and $\mathcal{R}$ are both congruences on B, and suppose that $e \,\mathcal{D}\, f$. Let $y \in B^1$. Then for any $x \in B^1$ and $u \in L_x\rho_e = L_{xe}$, we have that

$$R_y \lambda_u \lambda_f \lambda_x = R_{xfuy}.$$

Now $xf \,\mathcal{D}\, xe \,\mathcal{L}\, u$ so that $xf \,\mathcal{R}\, xfu$ and as $\mathcal{R}$ is a congruence on B, $xfy \,\mathcal{R}\, xfuy$. Hence

$$R_y \lambda_f \lambda_x = R_{xfy} = R_{xfuy} = R_y \lambda_u \lambda_f \lambda_x,$$

so that $\lambda_f \lambda_x = \lambda_u \lambda_f \lambda_x$. Together with the dual argument we obtain that $(\rho_e, \lambda_f) \in S_B$.

□

We present one more lemma for membership of special pairs in S_B, before turning our attention to the specific cases of S_C and S_D.

Lemma 6.12. *Let $\mathcal{L}$ be a congruence on a band B. Then for any $\alpha \in \mathcal{OP}(B^1/\mathcal{L})$ and $\mathcal{R}$-class R of B,*

$$(\rho_e, d_R) \in S_B \text{ if and only if } e \leq_{\mathcal{J}} u, u \in R.$$

Dually, if $\mathcal{R}$ is a congruence on B, then for any $\beta \in \mathcal{OP}(B^1/\mathcal{R})$ and $\mathcal{L}$-class L of B,

$$(c_L, \lambda_f) \in S_B \text{ if and only if } f \leq_{\mathcal{J}} u, u \in L.$$

Proof. Let α and R be as given. Using Lemma 6.4,

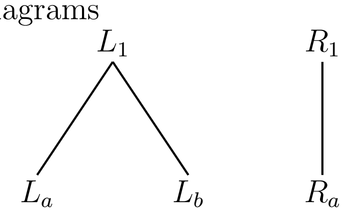
$$\begin{array}{rcll} (\rho_e, d_R) \in S_B & \Leftrightarrow & \rho_e = \rho_u \rho_e & \forall u \in R \\ & \Leftrightarrow & \rho_e = \rho_{ue} & \forall u \in R \\ & \Leftrightarrow & e \,\mathcal{L}\, ue & \forall u \in R \\ & \Leftrightarrow & e \leq_{\mathcal{J}} u & \forall u \in R. \end{array}$$

The dual proof may be used if $\mathcal{R}$ is a congruence.

□

The preceding sequence of lemmas enables us to determine precisely the elements of S_C and S_D. We present the argument for S_C, leaving the details for S_D to the interested reader (perhaps with the aid of a suitable programme).

To determine S_C, we first need to find $\mathcal{O}^1(C)$; to do this, we need to list the elements of $\mathcal{OP}(C/\mathcal{L})$ and $\mathcal{OP}(C/\mathcal{R})$. Noting that $C/\mathcal{L}$ and $C/\mathcal{R}$ have Hasse diagrams



inspection yields that

$$\mathcal{OP}(C/\mathcal{L}) = \{\rho_1, \rho_a, \rho_b, c_1, \alpha, \beta, \gamma, \delta, \epsilon, c_{1,a}, c_{1,b}\}$$

where

$$\alpha = \begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_1 & L_a \end{pmatrix} \quad \beta = \begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_b & L_1 \end{pmatrix} \quad \gamma = \begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_b & L_a \end{pmatrix}$$
$$\delta = \begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_1 & L_b \end{pmatrix} \quad \epsilon = \begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_a & L_1 \end{pmatrix} \quad c_{1,a} = \begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_a & L_a \end{pmatrix}$$
$$c_{1,b} = \begin{pmatrix} L_1 & L_a & L_b \\ L_1 & L_b & L_b \end{pmatrix},$$

and so $|\mathcal{OP}(C/\mathcal{L})| = 11$. On the other hand,

$$\mathcal{OP}(C/\mathcal{R}) = \{\lambda_1, \lambda_a, d_1\}.$$

Thus $|\mathcal{O}^1(C)| = 33$; we use the lemmas developed to select the elements of $\mathcal{O}^1(C)$ lying in S_C. By Lemma 6.8, a pair $(\nu, \lambda_1) \in S_C$ if and only if $L_1\nu = L_1$. Thus all pairs $(\nu, \lambda_1), \nu \in \mathcal{OP}(C/\mathcal{L})$ lie in S_C with the exception of $\nu = \rho_a$ and $\nu = \rho_b$. On the other hand, Corollary 6.5 tells us that all pairs (ν, d_1) lie in S_C. Considering now the pairs (ν, λ_a), noticing that $\lambda_a = d_{R_a}$, we call upon Lemma 6.9. For a pair (ν, λ_a) to be in S_C we must have that ν is constant, thus the possibilities for $(\nu, \lambda_a) \in S_C$ are $(c_1, \lambda_a), (\rho_a, \lambda_a)$ and (ρ_b, λ_a). We deduce that S_C has 23 elements.

Corollary 6.13. *For the band C, the weakly $\overline{C}$-abundant semigroup S_C with (C) does not have (WIC).*

Proof. If S_C were to have (WIC), then, as it is fundamental, it would be embeddable into U_C by Theorem 4.5 of [5]. But we have argued that $|U_C| = 4$.

$\square$

The semigroup S_C is rather curious. We note that the non-idempotent elements are:

$$(\alpha, \lambda_1), (\beta, \lambda_1), (\gamma, \lambda_1), (\alpha, d_1), (\beta, d_1), (\gamma, d_1).$$

It is easy to check that α and β are mutually inverse, and that γ is self-inverse. Consequently, the non-idempotent elements of S_C are all regular, and so certainly S_C is regular. Nevertheless, $\mathcal{L} \neq \widetilde{\mathcal{L}}_{\overline{C}}$ in S_C and $\mathcal{R} \neq \widetilde{\mathcal{R}}_{\overline{C}}$. For, as remarked in the proof of Theorem 3.2, for any $(\alpha, \beta), (\gamma, \delta) \in S_B$ we have that

$$(\alpha, \beta) \, \widetilde{\mathcal{L}}_{\overline{C}} \, (\gamma, \delta) \Leftrightarrow L_1\alpha = L_1\gamma.$$

Thus $(c_{1,a}, \lambda_1) \, \widetilde{\mathcal{L}}_{\overline{C}} \, (c_{1,b}, \lambda_1)$. Since $(c_{1,a}, \lambda_1)(c_{1,b}, \lambda_1) = (c_{1,b}, \lambda_1)$ and $(c_{1,b}, \lambda_1)$ is idempotent, it is clear that $(c_{1,a}, \lambda_1)$ is not $\mathcal{L}$-related to $(c_{1,b}, \lambda_1)$. Similarly one can argue that $\mathcal{R} \neq \widetilde{\mathcal{R}}_{\overline{C}}$.

Finally we consider S_D; an analysis along the lines of that for S_C, but rather lengthier, and drawing heavily upon the technical lemmas developed in this section, gives that $|S_D| = 154$. However, S_D is not regular. With the help of the dual of Lemma 6.7 it is easy to check that $(\rho_1, \nu) \in S_B$, where

$$\nu = \begin{pmatrix} R_1 & R_a & R_0 \\ R_1 & R_1 & R_a \end{pmatrix}.$$

Suppose now that $(\alpha, \beta) \in S_B$ and

$$(\rho_1, \nu)(\alpha, \beta)(\rho_1, \nu) = (\rho_1, \nu).$$

Clearly we must have that $\alpha = \rho_1$. If $R_a\beta = R_1$ or R_a, then

$$R_0\nu\beta\nu = R_a\beta\nu \in \{R_1, R_a\}\nu = R_1;$$

but $R_0\nu\beta\nu = R_0\nu = R_a$, a contradiction. Thus $R_a\beta = R_0$. From Lemma 6.6, as ρ_1 is not constant we cannot have that $\beta = \lambda_0$ and it follows that $\beta = \begin{pmatrix} R_1 & R_a & R_0 \\ R_t & R_0 & R_0 \end{pmatrix}$, where $t = 1$ or $t = a$. But then, again using the dual of Lemma 6.7, we have that $(\rho_1, \beta) \notin S_D$.

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