RIGHT CANCELLATIVE AND LEFT AMPLE MONOIDS: QUASIVARIETIES AND PROPER COVERS

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ABSTRACT. The aim of this paper is to study certain quasivarieties of *left ample* monoids. Left ample monoids are monoids of partial one-one mappings of sets closed under the operation $\alpha \mapsto \alpha\alpha^{-1}$. The idempotents of a left ample monoid form a semilattice and have a strong influence on the structure of the monoid; however, a left ample monoid need not be inverse. Every left ample monoid has a maximum right cancellative image and a *proper cover* which is also left ample. The structure of proper left ample monoids is well understood.

Let $\mathcal{V}$ be a class of right cancellative monoids. A left ample monoid has a *proper cover over* $\mathcal{V}$ if it has a proper cover with maximum right cancellative image in $\mathcal{V}$. We show that if $\mathcal{V}$ is a quasivariety determined within right cancellative monoids by equations, then the left ample monoids having a proper cover over $\mathcal{V}$ form a quasivariety. We achieve our aim using the technique of *graph expansions* to construct proper left ample monoids from presentations of right cancellative monoids.

1. INTRODUCTION

The notion of an *expansion* from one category of monoids $\mathbf{A}$ to another $\mathbf{B}$ was introduced by Birget and Rhodes in [1]. An expansion from $\mathbf{A}$ to $\mathbf{B}$ is a functor $F : \mathbf{A} \to \mathbf{B}$ satisfying properties which ensure that the structure of any monoid MF for $M \in \mathrm{Ob}\ \mathbf{A}$ is essentially linked to that of M. Expansions were originally used to solve problems about the complexity of finite semigroups. However they have applications in quite different directions, in particular to the study of inverse monoids and related classes. Here, as elsewhere, the idea is to 'expand'

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The diagrams in this paper are drawn using Paul Taylor's commutative diagram package.

from a category **A** of monoids having relatively well understood structure via the functor F to a category **B** of monoids having possibly more complex structure. The motivation is that insight into the nature of (at least some) monoids in **B** can be obtained from known properties of **A** and the explicit description of F.

The Birget-Rhodes expansion [1], the Szendrei expansion [18] and the graph expansion of Margolis and Meakin [14], each from the category of groups to the category of inverse monoids, have been used in the study of inverse monoids and in particular, the relation of inverse monoids to groups. The Szendrei expansion and the graph expansion of right cancellative monoids are considered in [10] and the forerunner of this paper, [12]. In each case they belong to the class of left ample monoids (previously called left type A), a class which strictly contains the class of inverse monoids.

Left ample monoids may be approached via the generalisation $\mathcal{R}^*$ of Green's relation $\mathcal{R}$; elements a, b of a monoid M are $\mathcal{R}^*$-related if and only if a and b are $\mathcal{R}$-related in some overmonoid of M. Clearly $\mathcal{R} \subseteq \mathcal{R}^*$ and it is easy to see that if M is regular then $\mathcal{R} = \mathcal{R}^*$, but, in general, the inclusion is strict. A monoid M is *left adequate* if every $\mathcal{R}^*$-class of M contains an idempotent and the idempotents $E(M)$ of M form a semilattice. In this case every $\mathcal{R}^*$-class of M contains a *unique* idempotent. We denote the idempotent in the $\mathcal{R}^*$-class of a by a^+ (sometimes $a^\dagger$). A left adequate monoid M is *left ample* if $ae = (ae)^+a$ for each $a \in M$ and $e \in E(M)$. Any inverse monoid is left ample, however, the class of left ample monoids is much larger. For example, every right cancellative monoid is left ample.

It is easy to see that a monoid M with semilattice of idempotents $E(M)$ is left adequate if and only if every element a of M is *right e-cancellable* for some $e \in E(M)$. This means that $ea = a$ and for any $x, y \in M$, if $xa = ya$, then $xe = ye$; it is equivalent to the principal left ideal Ma being projective [5]. The extra restriction that M be left ample also comes from a consideration of principal left ideals; it is equivalent to insisting that $Ma \cap Me = Mae$ for all $a \in M$ and $e \in E(M)$ [6]. Left ample monoids arise naturally as monoids of one-one mappings of mathematical structures. For example, the monoid of partial one-one maps of a poset is left ample but need not be inverse. Similarly, the monoid of partial one-one continuous maps of a topological space is left ample. Indeed, with a change of signature, all left ample monoids may be obtained in this way. Regarded as algebras of type $(2, 1, 0)$, where the unary operation is given by $a \mapsto a^+$, it follows from Proposition 1.2 of [5] that left ample monoids are those left adequate monoids that are subalgebras of symmetric inverse monoids. Alternatively, left ample monoids are exactly the submonoids of symmetric inverse monoids that are closed under the operation $\alpha \mapsto \alpha\alpha^{-1}$. As algebras of type $(2, 1, 0)$, left ample monoids form a quasivariety [8, 9]: thus free left ample monoids exist. These were originally described in [9] where they are also shown to be *proper*, in the following sense.

The least right cancellative congruence σ on a left ample monoid plays a role corresponding to that of the least group congruence on an inverse monoid and, indeed, has the same description (see Section 2). We say that a left ample monoid is *proper* if $\sigma \cap \mathcal{R}^* = \iota$. In analogues of the celebrated results of McAlister for inverse monoids, proper left ample monoids are described in terms of right cancellative monoids acting on partially ordered sets [5] and further, any left ample monoid M is the image of a proper left ample monoid P under an idempotent separating morphism [5]. The monoid P is called a *proper cover* of M. An alternative characterisation of proper left ample monoids in terms of right cancellative monoids acting on categories is given in [11].

By a *monoid presentation* we mean a triple (X, f, S) where X is a set, S is a monoid and $f : X \to S$ is a function such that Xf generates S as a monoid. In [12] we used the *Cayley graph of a monoid presentation* (X, f, S) to construct a monoid, $\mathcal{M}(X, f, S)$, called a *graph expansion.* We prove in [12] that $\mathcal{M}(X, f, S)$ is a proper left ample monoid which is the initial object in a suitable category $\mathbf{PLA}(X, f, S)$ of X-generated proper left ample monoids having maximum right cancellative image S. Full definitions are given in the next section. The latter part of [12] concentrates on the larger category $\mathbf{PLA}(X)$ of X-generated proper left ample monoids, and the corresponding category $\mathbf{RC}(X)$ of X-generated right cancellative monoids. Using graph expansions we construct a functor $F^e : \mathbf{RC}(X) \to \mathbf{PLA}(X)$ and show that F^e is an expansion in the sense of Birget-Rhodes. Further, F^e is a left adjoint of $F^\sigma : \mathbf{PLA}(X) \to \mathbf{RC}(X)$, where F^σ takes an X-generated proper left ample monoid to its maximum right cancellative image.

The first object of this paper is to use the techniques developed in [12] to yield an expansion, also denoted by F^e, from the category $\mathbf{RC}$ of *all* right cancellative monoids to the category $\mathbf{PLA}$ of *all* proper left ample monoids. We would like to say that F^e is a left adjoint of a suitably defined functor $F^\sigma : \mathbf{PLA} \to \mathbf{RC}$. Unfortunately as it stands this statement is not correct. To redeem the situation we construct from $\mathbf{PLA}$ an augmented category $\mathbf{PLA^0}$ and show that F^e and F^σ may be regarded as functors between $\mathbf{RC}$ and $\mathbf{PLA^0}$ and that as such, F^e is a left adjoint of F^σ. The objects in $\mathbf{PLA^0}$ are proper left ample monoids equipped with an extra unary operation, the image of which is a transversal of the σ-classes. The morphisms between two objects in $\mathbf{PLA^0}$ are the morphisms between the objects *regarded as algebras of type* $(2,1,1,0)$. The category $\mathbf{PLA^0}$ is thus reminiscent of the category of left FA monoids and +-homomorphisms [10]; a left FA monoid being the left ample analogue of an F-inverse monoid.

The fact that inverse monoids and groups form varieties of algebras has lead to fruitful work in the theory of varieties of inverse monoids. Let $\mathcal{V}$ be a variety of groups. An inverse monoid M has a *proper cover over* $\mathcal{V}$ if M has a proper cover P (in the established sense of inverse semigroup theory), such that the maximal group image of P is in $\mathcal{V}$. The collection of inverse monoids having a proper cover over $\mathcal{V}$ form a variety of inverse monoids defined by Σ where

$$\Sigma = \{\overline{u}^2 \equiv \overline{u} : \overline{u} \equiv 1 \text{ is a law in } \mathcal{V}\}$$

(for words $\overline{u}$ over $Z \cup Z^{-1}$ for a countably infinite set Z) [15, 16]. In [14] graph expansions of groups are used to construct free objects in such varieties.

Although right cancellative monoids and left ample monoids do *not* form varieties of algebras, they do form quasivarieties. We say that a subquasivariety of the quasivariety $\mathcal{RC}$ of right cancellative monoids or the quasivariety $\mathcal{LA}$ of left ample monoids is a *q-subvariety* if it is determined (within $\mathcal{RC}$ or $\mathcal{LA}$, respectively) by equations. Let $\mathcal{V}$ be a q-subvariety of $\mathcal{RC}$. By analogy with the definition in the case for groups and inverse monoids, we say that a left ample monoid M has a *proper cover over* $\mathcal{V}$ if there is a proper cover P of M such that the maximal right cancellative image of P lies in $\mathcal{V}$. We show that

$$\widehat{\mathcal{V}} = \{M \in \mathcal{LA} : M \text{ has a proper cover over } \mathcal{V}\}$$

is a q-subvariety of $\mathcal{LA}$ and we give a set of equations determining $\widehat{\mathcal{V}}$ within $\mathcal{LA}$. Since q-subvarieties are certainly quasivarieties, free objects exist. If (X,f,S) is the canonical presentation of the free object in $\mathcal{V}$ on a given set X, then $\mathcal{M}(X,f,S)$ is the free object in $\widehat{\mathcal{V}}$. We arrive at this result by finding a presentation of $\mathcal{M}(X,f,S)$ for *any* monoid presentation (X,f,S) of a right cancellative monoid S. That is, we find a congruence $\rho = \rho_{(X,f,S)}$ on the free left ample monoid F_X on X such that $\mathcal{M}(X,f,S)$ is isomorphic to F_X/ρ. Of course, as left ample monoids do not form a variety we have to show explicitly that F_X factored by the given congruence ρ is left ample, indeed proper left ample.

In Section 2 we gather together preliminary definitions and results concerning left adequate and left ample monoids. We also define, for a given monoid presentation (X,f,S) of a right cancellative monoid S, the subcategory $\mathbf{RC}(X)$ of the category $\mathbf{RC}$ of right cancellative monoids and the subcategories $\mathbf{PLA}(X, f, S)$ and $\mathbf{PLA}(X)$ of the category $\mathbf{PLA}$ of proper left ample monoids.

In Section 3 we define the functors $F^e : \mathbf{RC} \to \mathbf{PLA}$ and $F^\sigma : \mathbf{PLA} \to \mathbf{RC}$. We show that F^e is an expansion. We define the category $\mathbf{PLA^0}$ and prove that, regarded as functors between $\mathbf{RC}$ and $\mathbf{PLA^0}$, F^e is a left adjoint of F^σ.

Section 4 concentrates on finding, given a monoid presentation (X, f, S) of a right cancellative monoid S, a congruence ρ on the free left ample monoid F_X on X such that $\mathcal{M}(X, f, S)$ is isomorphic to F_X/ρ. We use the description given in [12] of F_X as $\mathcal{M}(X, \iota, X^*)$, where (X, ι, X^*) is the canonical presentation of the free monoid on X.

Finally in Section 5 we give the promised results concerning q-subvarieties. Namely, if $\mathcal{V}$ is a q-subvariety of $\mathcal{RC}$ we show that $\widehat{\mathcal{V}}$ is a q-subvariety of $\mathcal{LA}$. Further, $\widehat{\mathcal{V}}$ is the q-subvariety determined by Σ, where

$$\Sigma = \{\overline{s}^+\overline{t} \equiv \overline{t}^+\overline{s} : \overline{s} \equiv \overline{t} \text{ is a law in } \mathcal{V}\}$$

(for words $\overline{s}, \overline{t}$ over a countably infinite set Z). We show that free objects in $\widehat{\mathcal{V}}$ are graph expansions of monoid presentations of free objects in $\mathcal{V}$. Note that this means that free objects in $\widehat{\mathcal{V}}$ are proper.

2. Preliminaries

In this section we draw together definitions and results used later in the paper. For further background in semigroup theory and universal algebra we refer the reader to [13] and [3].

We begin with the following alternative characterisation of the relation $\mathcal{R}^*$, which we use without further mention.

Lemma 2.1. [7] *Elements a, b of a monoid M are $\mathcal{R}^*$-related if and only if for all $x, y \in M$,*

$$xa = ya \text{ if and only if } xb = yb.$$

From Lemma 2.1 it is clear that $\mathcal{R}^*$ is an equivalence relation on any monoid M, indeed a left congruence.

Recall that a monoid M is *left adequate* if every $\mathcal{R}^*$-class contains an idempotent and the idempotents of M form a semilattice; the unique idempotent in the $\mathcal{R}^*$-class of a is denoted by a^+.

Lemma 2.2. [6] *Let M be a left adequate monoid. Then*
(1) $(ab)^+ = (ab^+)^+$ for all $a, b \in M$;
(2) $(ea)^+ = ea^+$ for all $a \in M$ and $e \in E(M)$;

(3) $(ab)^+ \leq a^+$ *for all* $a, b \in M$*, where* $\leq$ *is the natural partial order on* $E(M)$*.*

We consider left adequate monoids as algebras of type $(2, 1, 0)$. We emphasize that a^+ *always* denotes the idempotent in the $\mathcal{R}^*$-class of a. As pointed out in the introduction, left adequate monoids form a quasivariety of algebras. It is easy to see they are axiomatised by the set

$$\{1x = x = x1, (xy)z = x(yz), (x^+)^+ = x^+, x^+x = x,$$

$$(x^2 = x \wedge y^2 = y) \Rightarrow xy = yx, xy = zy \Rightarrow xy^+ = zy^+\}$$

of quasi-identities [8, 9].

A *left ample monoid* is a left adequate monoid M in which $ae = (ae)^+a$ for each $a \in M$ and $e \in E(M)$, that is, satisfying the additional quasi-identity

$$x^2 = x \Rightarrow yx = (yx)^+y \qquad (AL).$$

Thus left ample monoids form a subquasivariety of the quasivariety of left adequate monoids.

We regard *arbitrary* monoids as algebras of type $(2, 0)$. Now, any right cancellative monoid is a left ample monoid and later in the paper we consider right cancellative monoids S with a given set of generators. The next lemma shows that no ambiguity arises whether we regard such an S as an algebra of type $(2, 1, 0)$ or of type $(2, 0)$.

Lemma 2.3. *Let* S *be a right cancellative monoid. Then* S *is a left ample monoid. A subset* X *of* S *is a set of generators of* S *as an algebra of type* $(2, 0)$ *if and only if it is a set of generators of* S *as an algebra of type* $(2, 1, 0)$*. A subset* T *of* S *is a submonoid of* S *if and only if it is a* $(2, 1, 0)$*-subalgebra. Further, a function* ϕ *from a left ample monoid* M *to* S *is a monoid morphism, that is, a* $(2, 0)$*-morphism, if and only if it is a morphism where* S *is regarded as a left ample monoid, that is, a* $(2, 1, 0)$*-morphism.*

For a left ample monoid M, the least right cancellative congruence has the same description as that of the least group congruence on an inverse monoid.

Lemma 2.4. [5] *Let* M *be a left ample monoid and define the relation* σ *on* M *by the rule that for* $a, b \in M$, $a \;\sigma\; b$ *if and only if* $ea = eb$ *for some* $e \in E(M)$*. Then* σ *is the least right cancellative monoid congruence on* M*.*

Where there is danger of ambiguity, the relation σ on a left ample monoid M is denoted by σ_M. Note that $E(M)$ is contained within a σ-class.

We say that a left ample monoid is *proper* if $\sigma \cap \mathcal{R}^* = \iota$. For an inverse monoid, being proper is the same as being E-unitary, that is, the idempotents forming a unitary subset. A subset H of a monoid M is *unitary* if for all $a \in M$ and $h \in H$, either $ah \in H$ or $ha \in H$ implies that $a \in H$. In the general case, a proper left ample monoid M is E-unitary but the converse is not true [5]. Note that if M is E-unitary then $E(M)$ is a σ-class. The following lemma is crucial in our later arguments.

Lemma 2.5. [4] *Let M be a proper left ample monoid. If $a, b \in M$, then $a \,\sigma\, b$ if and only if $b^+a = a^+b$.*

Corollary 2.6. *Let M be a left ample monoid. Then M is proper if and only if M satifies the quasi-identity*

$$(x^2 = x \wedge y^+ = z^+ \wedge xy = xz) \Rightarrow y = z.$$

Thus proper left ample monoids are a quasivariety.

Corollary 2.7. [4] *Let M be a left adequate monoid and let N be a subalgebra of M.*

(1) The subalgebra N is a left adequate monoid and for all $a, b \in N$,

$$a \,\mathcal{R}^*\, b \text{ in } N \text{ if and only if } a \,\mathcal{R}^*\, b \text{ in } M.$$

(2) If M is left ample then so is N.

(3) If M is a proper left ample monoid then so is N and for all $a, b \in N$

$$a \,\sigma\, b \text{ in } N \text{ if and only if } a \,\sigma\, b \text{ in } M.$$

Another consequence of the above which we shall require in Section 5 is:-

Corollary 2.8. *Let I be a non-empty set indexing left adequate monoids $M_i, i \in I$. Then $\prod_{i \in I} M_i$ is left adequate and $(a_i) \,\mathcal{R}^*\, (a_i^+)$ for each $(a_i) \in \prod_{i \in I} M_i$. Hence for $(a_i), (b_i) \in \prod_{i \in I} M_i$,*

$$(a_i) \,\mathcal{R}^*\, (b_i) \text{ if and only if } a_i \,\mathcal{R}^*\, b_i \text{ for all } i \in I.$$

We recall from the introduction that a left ample monoid P is a *proper cover* of a left ample monoid M if P is proper and M is the image of P under an idempotent separating morphism.

Theorem 2.9. [5] *Every left ample monoid has a proper cover.*

It is clear that the class of right cancellative monoids forms a quasivariety. The class of right cancellative monoids together with all monoid morphisms between them is of course a category. We denote the quasivariety of right cancellative monoids by $\mathcal{RC}$ and the corresponding category by $\mathbf{RC}$. Similarly, $\mathcal{LA}$ ($\mathcal{PLA}$) denote the quasivariety of left ample (proper left ample) monoids and $\mathbf{LA}$ ($\mathbf{PLA}$) denote the corresponding categories. Recall that left ample monoids are actually $(2, 1, 0)$-algebras. Nevertheless, in view of Lemma 2.3, we may regard $\mathcal{RC}$ as a subquasivariety of $\mathcal{PLA}$ and $\mathbf{RC}$ as a full subcategory of $\mathbf{PLA}$.

In [12] we are concerned with various categories of X-generated monoids, obtained in the following manner.

Let X be a set and $\mathcal{A}$ a class of algebras of a given fixed type. Then $\mathbf{A}(X)$ is the category which has objects pairs (f, A) where $A \in \mathcal{A}, f : X \to A$ and $\langle Xf \rangle = A$; a morphism in $\mathbf{A}(X)$ from (f, A) to (g, B) is a morphism $\theta : A \to B$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "\theta"'] & & B \end{tikzcd}$$

commutes. From $\langle Xf \rangle = A$ we deduce that if such a θ exists, it must be unique; from $\langle Xg \rangle = B$ we deduce that such a θ must be onto.

For the purposes of this paper we require some results concerning full subcategories of $\mathbf{PLA}(X)$ of the form $\mathbf{PLA}(X, f, S)$ where (X, f, S) is a monoid presentation (X, f, S) of a right cancellative monoid S. An object (g, M) of $\mathbf{PLA}(X)$ is an object in $\mathbf{PLA}(X, f, S)$ if the diagram

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f"] & \\ M \arrow[r, "\sigma^\sharp"'] & S \end{tikzcd}$$

commutes, where $\sigma^\sharp$ is a morphism with kernel σ. As previously remarked, $\sigma^\sharp$ must then be the only morphism making the above diagram commute, and $\sigma^\sharp$ must be onto, so that S is the maximum right cancellative image of M. In [12] we introduce a construction of a monoid $\mathcal{M}(X, f, S) \in \text{Ob } \mathbf{PLA}(X, f, S)$ from the Cayley graph of (X, f, S), which we now detail.

For the purposes of this paper a *graph* Γ consists of two sets $V = V(\Gamma)$ (the *vertices* of Γ) and $E = E(\Gamma)$ (the *edges* of Γ), together with two maps (written on the left), $i : E \to V$ and $t : E \to V$. The maps i and t are the *initial* and *terminal* maps, respectively. We may represent $e \in E$ with $i(e) = v$ and $t(e) = v'$ by

$$\underset{v}{\bullet} \xrightarrow{\quad e \quad} \underset{v'}{\bullet}$$

A *path* from a vertex v to a vertex w is a finite sequence of edges $e_1, \ldots, e_n$ with

$$i(e_1) = v,\ t(e_1) = i(e_2),\ t(e_2) = i(e_3),\ \ldots, t(e_n) = w$$

and we write this as

$$\underset{v}{\bullet} \xrightarrow{\quad e_1 \quad} \bullet \xrightarrow{\quad e_2 \quad} \bullet \cdots\cdots\cdots \bullet \xrightarrow{\quad e_n \quad} \underset{w}{\bullet}$$

There is also an *empty path* I_v from any vertex v to itself. The graph Γ is *v-rooted*, where $v \in V$, if for all $w \in V$ there is a path from v to w. A *subgraph* Δ of Γ consists of a subset $V(\Delta)$ of $V(\Gamma)$ and a subset $E(\Delta)$ of $E(\Gamma)$ such that for any $e \in E(\Delta), i(e), t(e) \in V(\Delta)$. Clearly any path determines a subgraph; it is convenient at times to use the same notation for a path and the corresponding subgraph.

A *graph morphism* θ from a graph Γ to a graph Γ' consists of two functions, each denoted by θ, from $V(\Gamma)$ to $V(\Gamma')$ and from $E(\Gamma)$ to $E(\Gamma')$, such that for any $e \in E(\Gamma)$,

$$i(e)\theta = i(e\theta) \text{ and } t(e)\theta = t(e\theta).$$

Clearly such a θ maps subgraphs to subgraphs and paths to paths.

A monoid S *acts* on a graph Γ (on the left) if V and E are left S-sets and i and t are left S-maps, that is, $i(s \cdot e) = s \cdot i(e)$ and $t(s \cdot e) = s \cdot t(e)$ for all $s \in S$ and $e \in E$. Note that if S acts on Γ, then the action of any $s \in S$ is a graph morphism so that if Δ is a subgraph of Γ, then so is $s \cdot \Delta$.

Our interest here is in the Cayley graph $\Gamma = \Gamma(X, f, S)$ of a monoid presentation (X, f, S). Here $V(\Gamma) = S$ and

$$E(\Gamma) = \{(s, x, s(xf)) : s \in S, x \in X\}$$

where $i(s, x, s(xf)) = s$ and $t(s, x, s(xf)) = s(xf)$. We may write the edge $(s, x, s(xf))$, or the corresponding subgraph, as

$$\underset{s}{\bullet} \xrightarrow{\quad x \quad} \underset{s(xf)}{\bullet}$$

The monoid S acts on Γ where for $s \in S, v \in V, (t, x, t(xf)) \in E$ we have

$$s \cdot v = sv, s \cdot (t, x, t(xf)) = (st, x, st(xf)).$$

The *graph expansion* $\mathcal{M} = \mathcal{M}(X, f, S)$ of (X, f, S) is given by

$$\mathcal{M} = \{(\Delta, s) : \Delta \text{ is a finite 1-rooted subgraph of } \Gamma \text{ and } 1, s \in V(\Delta)\}.$$

We define a multiplication on $\mathcal{M}$ by

$$(\Delta, s)(\Sigma, t) = (\Delta \cup s \cdot \Sigma, st).$$

It is easy to check that $\mathcal{M}(X, f, S)$ is a monoid with identity $(\bullet_1, 1)$. Clearly

$$\Gamma_x = \underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{xf}{\bullet}$$

is a 1-rooted subgraph of Γ and $(\Gamma_x, xf) \in \mathcal{M}$. We define $\tau_{\mathcal{M}(X,f,S)} = \tau_{\mathcal{M}} : X \to \mathcal{M}$ by $x\tau_{\mathcal{M}} = (\Gamma_x, xf)$.

Proposition 2.10. [12] *Let (X, f, S) be a monoid presentation of a right cancellative monoid S. Then $\mathcal{M} = \mathcal{M}(X, f, S)$ is a proper left ample monoid. For any $(\Delta, s), (\Sigma, t) \in \mathcal{M}$,*
(i) $(\Delta, s) \in E(\mathcal{M})$ if and only if $s = 1$;
(ii) $(\Delta, s)^+ = (\Delta, 1)$;
(iii) $(\Delta, s)\ \mathcal{R}^*\ (\Sigma, t)$ if and only if $\Delta = \Sigma$;
(iv) $(\Delta, s)\ \sigma\ (\Sigma, t)$ if and only if $s = t$.
Further, $\mathcal{M} = \langle X\tau_{\mathcal{M}} \rangle$ and

$$\begin{tikzcd} X \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dr, "f"] & \\ \mathcal{M} \arrow[r, "\sigma^\natural"] & S \end{tikzcd}$$

commutes, where $\sigma^\natural : \mathcal{M} \to S$ is the morphism with kernel σ defined by $(\Delta, s)\sigma^\natural = s$. Thus $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in the category $\mathbf{PLA}(X, f, S)$.

Theorem 2.11. *The pair $(\tau_{\mathcal{M}}, \mathcal{M})$ is an initial object in the category* $\mathbf{PLA}(X, f, S)$.

In Section 5 we show that graph expansions can be used to construct free objects in q-subvarieties of $\mathcal{LA}$. A subclass $\mathcal{W}$ of a quasivariety $\mathcal{V}$ is a *q-subvariety* if $\mathcal{W}$ is determined within $\mathcal{V}$ by identities, that is, there is a set of identities Σ such that $\mathcal{W} = \{V \in \mathcal{V} : V \models \Sigma\}$. An alternative characterisation is given in [17] (see Section 5). We remark that, regarded as a class of algebras of type $(2, 1, 0)$, $\mathcal{RC}$ is the q-subvariety of $\mathcal{PLA}$ determined by the identity $x^+ = 1$. Since q-subvarieties are subquasivarieties, free objects in non-trivial q-subvarieties exist [3]. If F is a free object on a set X in a quasivariety $\mathcal{V}$ and $f : Y \to X$ is a bijection between a set Y and X, then we may refer to F as the *free object (in $\mathcal{V}$) on Y with canonical embedding f*.

Free left ample monoids were first described in [9]. Alternative constructions using Szendrei expansions and graph expansions were given in [10] and [12] respectively. We make use of the latter approach.

Theorem 2.12. [12] *Let X be a set and let $\iota : X \to X^*$ be the standard embedding of X into the free monoid on X. Then $\mathcal{M}(X, \iota, X^*)$ is the free left ample monoid on X with canonical embedding $\tau_{\mathcal{M}}$.*

We denote the free left ample monoid on a set X by F_X. Note that the free left ample monoid is proper.

Finally in this section on preliminaries we remark that if ρ is a congruence on an algebra A, then we denote by A/ρ the quotient algebra of A by ρ and by $\rho^\natural$ the natural morphism from A to A/ρ.

3. The category $\mathbf{PLA^0}$ and the functors F^e and F^σ

In this section we define functors $F^e : \mathbf{RC} \to \mathbf{PLA}$ and $F^\sigma : \mathbf{PLA} \to \mathbf{RC}$. We construct from $\mathbf{PLA}$ a new category $\mathbf{PLA^0}$ and show that F^e and F^σ may be regarded as functors between $\mathbf{RC}$ and $\mathbf{PLA^0}$ and as such, F^e is a left adjoint of F^σ. This is a slightly stronger result than that promised in [12]. The objects in $\mathbf{PLA^0}$ are those of $\mathbf{PLA}$ equipped with an extra unary operation; the morphisms between two objects in $\mathbf{PLA^0}$ are those of $\mathbf{PLA}$ preserving this operation.

We begin with the functor F^e. Suppose that $S \in \text{Ob } \mathbf{RC}$, that is, S is a right cancellative monoid. The triple (S, I_S, S) is certainly a monoid presentation of S, where $I_S : S \to S$ is the identity map. We put $SF^e = \mathcal{M}(S, I_S, S)$. By Proposition 2.10, $\mathcal{M}(S, I_S, S)$ is a proper left ample monoid so that F^e is a function from Ob $\mathbf{RC}$ to Ob $\mathbf{PLA}$.

Suppose now that $S, T \in \text{Ob } \mathbf{RC}$ and $\theta : S \to T$ is in $\text{Mor}_{\mathbf{RC}}(S, T)$. We first define a map $\theta' : \Gamma(S, I_S, S) \to \Gamma(T, I_T, T)$ by

$$v\theta' = v\theta$$

for any vertex v of $\Gamma(S, I_S, S)$ and

$$(s, x, sx)\theta' = (s\theta, x\theta, s\theta x\theta)$$

for any edge (s, x, sx) of $\Gamma(S, I_S, S)$. Clearly θ' is a graph morphism so that as remarked in Section 2, θ' maps subgraphs to subgraphs and paths to paths, indeed as $1\theta = 1$, θ' maps 1-rooted subgraphs to 1-rooted subgraphs. Thus we can define θF^e to be θ^e where $\theta^e : SF^e \to$

TF^e is given by

$$(\Delta, s)\theta^e = (\Delta\theta', s\theta).$$

For any subgraph Δ of $\Gamma(S, I_S, S)$ and $s \in S$,

$$(s \cdot \Delta)\theta' = s\theta \cdot \Delta\theta'.$$

Using Proposition 2.10 it is now easy to see that $\theta^e \in \mathrm{Mor}_{\mathbf{PLA}}(SF^e, TF^e)$ and that F^e defined in this manner is a functor from **RC** to **PLA**.

In fact, F^e is an *expansion* in the sense of Birget-Rhodes [1]. Regarding **RC** as a subcategory of **PLA**, we need to show that for any $S \in \mathrm{Ob}\,\mathbf{RC}$ there is an onto morphism $\eta_S \in \mathrm{Mor}_{\mathbf{PLA}}(SF^e, S)$ such that for each $\theta \in \mathrm{Mor}_{\mathbf{RC}}(S, T)$ the square

$$\begin{tikzcd}
SF^e \arrow[r, "\theta^e"] \arrow[d, "\eta_S"'] & TF^e \arrow[d, "\eta_T"] \\
S \arrow[r, "\theta"'] & T
\end{tikzcd}$$

commutes; further, if θ is onto then so also is θF^e. Defining η_S by $(\Delta, s)\eta_S = s$, it is immediate that η_S is an onto monoid morphism so that by Lemma 2.3, $\eta_S \in \mathrm{Mor}_{\mathbf{PLA}}(SF^e, S)$. For any $\theta \in \mathrm{Mor}_{\mathbf{RC}}(S, T)$ and any $(\Delta, s) \in SF^e$,

$$(\Delta, s)\theta^e\eta_T = (\Delta\theta', s\theta)\eta_T = s\theta = (\Delta, s)\eta_S\theta$$

so that the above square commutes. Suppose now that θ is onto. For any $t \in T$ we have $t = s\theta$ for some $s \in S$ so that

$$(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t) = (\underset{1}{\bullet} \xrightarrow{\quad s\theta \quad} \underset{s\theta}{\bullet}, s\theta) = (\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}, s)\theta^e.$$

Recall from Proposition 2.10 that

$$\{(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t) : t \in T\}$$

is a set of generators of TF^e, so that θ^e is onto and we have proved:-

Proposition 3.1. *The functor $F^e : \mathbf{RC} \to \mathbf{PLA}$ is an expansion.*

We now define the functor $F^\sigma : \mathbf{PLA} \to \mathbf{RC}$. The action of F^σ on objects is given by $MF^\sigma = M/\sigma$ for any $M \in \mathrm{Ob}\,\mathbf{PLA}$. By definition of σ, the monoid M/σ is right cancellative. For $\theta \in \mathrm{Mor}_{\mathbf{PLA}}(M, N)$ put $\theta F^\sigma = \theta^\sigma$ where $[m]\theta^\sigma = [m\theta]$. In view of the description of θ in Lemma 2.5, θ^σ is well defined. Clearly F^σ is a functor from **PLA** to **RC**.

We would *like* to say that F^e is a left adjoint of F^σ. Unfortunately, this is not strictly true, as the natural mapping between the relevant morphism sets (see Theorem 3.3) is not necessarily a bijection. The situation can be remedied if we augment proper left ample monoids with an extra unary operation, as we now describe.

The category $\mathbf{PLA^0}$ has as objects proper left ample monoids given an added unary operation $^\circ$ such that for any proper left ample monoid M

(i) $m\ \sigma\ m^\circ$ for all $m \in M$

and

(ii) $\{m^\circ : m \in M\}$ is a transversal of the σ-classes of M.

The morphisms of $\mathbf{PLA^0}$ are the morphisms between objects *regarded as algebras of type* $(2,1,1,0)$.

Clearly the only way a right cancellative monoid S can be made into an object of $\mathbf{PLA^0}$ is if $s^\circ = s$ for all $s \in S$. For an arbitrary proper left ample monoid there are of course many choices for $^\circ$. A left ample monoid M is *weak left FA* if every σ-class contains a maximum element under the partial ordering $\leq$, where for any $a, b \in M$, $a \leq b$ if and only if $a = eb$ for some $e \in E(M)$; a weak left FA monoid is necessarily proper [10]. The standard choice of $^\circ$ for a weak left FA monoid is to put $a^\circ = m(a)$, where $m(a)$ denotes the maximum element in the σ-class of a. The *Szendrei expansion* $\widetilde{S}^{S\mathcal{R}}$ of a right cancellative monoid S is weak left FA, indeed $\widetilde{S}^{S\mathcal{R}}$ is *left FA*, that is, $\widetilde{S}^{S\mathcal{R}}$ satisfies the identity $m(x)^+m(xy)^+ = (m(x)m(y))^+$ [10]. The relationship between $\mathbf{RC}$ and the category of left FA monoids regarded as algebras of type $(2,1,1,0)$ is studied in [10], making use of the Szendrei expansion (in fact Fountain and Gomes consider the left-right dual case throughout).

A fundamental difference between graph expansions and Szendrei expansions is that the latter are not presentation dependant. Further, for a right cancellative monoid S the expansion SF^e need not be weak left FA:-

Lemma 3.2. *If $T = \{x, y\}^*$ is the free monoid on two generators then TF^e is not weak left FA.*

Proof It is easy to see that for any monoid presentation (X, f, S) of a right cancellative monoid S and any $(\Sigma, s), (\Delta, t) \in \mathcal{M}(X, f, S)$,

$$(\Sigma, s) \leq (\Delta, t) \text{ if and only if } s = t \text{ and } \Delta \subseteq \Sigma.$$

Now consider $T = \{x,y\}^*$ and $TF^e = \mathcal{M}(T, I_T, T)$. Certainly $(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{x}{\bullet} \xrightarrow{\quad y \quad} \underset{xy}{\bullet}, xy)$ and $(\underset{1}{\bullet} \xrightarrow{\quad xy \quad} \underset{xy}{\bullet}, xy)$ are σ-related elements of TF^e. The subgraphs

$$\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{x}{\bullet} \xrightarrow{\quad y \quad} \underset{xy}{\bullet} \text{ and } \underset{1}{\bullet} \xrightarrow{\quad xy \quad} \underset{xy}{\bullet}$$

contain no common 1-rooted subgraph having vertex xy, so by the previous paragraph, their σ-class has no maximum element.

With the right choice of $^\circ$ for SF^e, the functor F^e may be regarded as a functor from **RC** to $\mathbf{PLA^0}$. For a right cancellative monoid S define $^\circ$ on SF^e by $(\Sigma, s)^\circ = (\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}, s)$. By Proposition 2.10, SF^e is then an object in $\mathbf{PLA^0}$. Suppose further that $S, T \in \text{Ob } \mathbf{RC}$ and $\theta \in \text{Mor}_{\mathbf{RC}}(S,T)$. For any $(\Sigma, s) \in SF^e$ we have

$$(\Sigma, s)^\circ \theta^e = (\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}, s)\theta^e = (\underset{1}{\bullet} \xrightarrow{\quad s\theta \quad} \underset{s\theta}{\bullet}, s\theta)$$

$$= (\Sigma\theta', s\theta)^\circ = ((\Sigma, s)\theta^e)^\circ$$

so that $\theta^e : SF^e \to TF^e$ is a $(2,1,1,0)$-morphism, that is, $\theta F^e = \theta^e \in \text{Mor}_{\mathbf{PLA^0}}(SF^e, TF^e)$. Thus F^e is a functor from **RC** to $\mathbf{PLA^0}$.

Of course, F^σ may be viewed as a functor from $\mathbf{PLA^0}$ to **RC**. We can now state the main result of this section.

Theorem 3.3. *Regarded as functors between* **RC** *and* $\mathbf{PLA^0}$, *F^e is a left adjoint of F^σ.*

Proof We must prove that for any $T \in \text{Ob } \mathbf{RC}$ and $M \in \text{Ob } \mathbf{PLA^0}$, there is a bijection

$$\alpha_{T:M} : \ \text{Mor}_{\mathbf{PLA^0}}(TF^e, M) \to \ \text{Mor}_{\mathbf{RC}}(T, MF^\sigma)$$

such that for any $T' \in \text{Ob } \mathbf{RC}$, $M' \in \text{Ob } \mathbf{PLA^0}$, $\phi \in \text{Mor}_{\mathbf{RC}}(T', T)$ and $\theta \in \text{Mor}_{\mathbf{PLA^0}}(M, M')$, the square

$$\begin{tikzcd}
\text{Mor}_{\mathbf{PLA^0}}(TF^e, M) \arrow[r, "\alpha_{T:M}"] \arrow[d, "\text{Mor }(\phi^e, \theta)"'] & \text{Mor}_{\mathbf{RC}}(T, MF^\sigma) \arrow[d, "\text{Mor }(\phi, \theta^\sigma)"] \\
\text{Mor}_{\mathbf{PLA^0}}(T'F^e, M') \arrow[r, "\alpha_{T':M'}"'] & \text{Mor}_{\mathbf{RC}}(T', M'F^\sigma)
\end{tikzcd}$$

commutes. Here

$$\text{Mor }(\phi^e,\theta):\ \text{Mor}_{\mathbf{PLA^0}}(TF^e,M)\to\ \text{Mor}_{\mathbf{PLA^0}}(T'F^e,M')$$

is given by

$$\psi\ \text{Mor }(\phi^e,\theta)=\phi^e\psi\theta$$

and

$$\text{Mor }(\phi,\theta^\sigma):\ \text{Mor}_{\mathbf{RC}}(T,MF^\sigma)\to\ \text{Mor}_{\mathbf{RC}}(T',M'F^\sigma)$$

is given by

$$\psi\ \text{Mor }(\phi,\theta^\sigma)=\phi\psi\theta^\sigma.$$

Let $T\in$ Ob $\mathbf{RC}$. In view of Proposition 2.10 we may define an isomorphism $\sigma_T:T\to TF^e/\sigma$ by

$$t\sigma_T=[(\underset{1}{\bullet}\xrightarrow{\ \ t\ \ }\underset{t}{\bullet},t)]=[(\Sigma,t)]$$

for any $(\Sigma,t)\in TF^e$. If $M\in$ Ob $\mathbf{PLA^0}$ then we define

$$\alpha_{T:M}:\ \text{Mor}_{\mathbf{PLA^0}}(TF^e,M)\to\ \text{Mor}_{\mathbf{RC}}(T,MF^\sigma)$$

by

$$\psi\alpha_{T:M}=\sigma_T\psi^\sigma.$$

We first show that the above diagram commutes. Let $\psi\in$ $\text{Mor}_{\mathbf{PLA^0}}(TF^e,M)$ and let $t'\in T'$. Then

$$t'(\psi\alpha_{T:M}\ \text{Mor }(\phi,\theta^\sigma))=t'(\phi\sigma_T\psi^\sigma\theta^\sigma)$$

$$=[(\Sigma,t'\phi)]\psi^\sigma\theta^\sigma=[(\Sigma,t'\phi)\psi\theta]$$

for any $(\Sigma,t'\phi)\in TF^e$. On the other hand,

$$t'(\psi\ \text{Mor }(\phi^e,\theta)\alpha_{T':M'})=t'(\sigma_{T'}(\phi^e\psi\theta)^\sigma)$$

$$=[(\Delta,t')](\phi^e\psi\theta)^\sigma=[(\Delta,t')\phi^e\psi\theta]=[(\Delta\phi',t'\phi)\psi\theta]$$

for any $(\Delta,t')\in T'F^e$. But for any $(\Sigma,t'\phi)\in TF^e$ and $(\Delta,t')\in T'F^e$ we have by Proposition 2.10 that $(\Sigma,t'\phi)\ \sigma\ (\Delta\phi',t'\phi)$. Lemma 2.4 then gives $(\Sigma,t'\phi)\psi\theta\ \sigma\ (\Delta\phi',t'\phi)\psi\theta$ so that $\psi\alpha_{T:M}$ Mor (ϕ,θ^σ) is the same map as ψ Mor $(\phi^e,\theta)\alpha_{T':M'}$, that is, the diagram commutes as required.

It remains to show that $\alpha_{T:M}$ is a bijection. We use results of [12] to construct an inverse $\beta_{T:M}$ of $\alpha_{T:M}$.

First consider $\gamma:M/\sigma_M\to M$ defined by $[m]\gamma=m^\circ,m\in M$. Then for any $[m]\in M/\sigma_M$,

$$[m]\gamma\sigma_M^\natural=m^\circ\sigma_M^\natural=[m^\circ]=[m]=[m]I_{M/\sigma_M}.$$

Thus the diagram

$$\begin{tikzcd} M/\sigma_M \arrow[d, "\gamma"'] \arrow[dr, "I_{M/\sigma_M}"] & \\ M \arrow[r, "\sigma_M^\natural"'] & M/\sigma_M \end{tikzcd}$$

commutes. Put $K = \langle (M/\sigma_M)\gamma \rangle$, that is, K is the $(2,1,0)$-subalgebra (or equivalently, the $(2,1,1,0)$-subalgebra) of M generated by $\{m^\circ : m \in M\}$. It follows from Corollary 2.7 that K is a proper left ample monoid; indeed, $K \in \text{Ob } \mathbf{PLA^0}$. Certainly

$$\begin{tikzcd} M/\sigma_M \arrow[d, "\gamma"'] \arrow[dr, "I_{M/\sigma_M}"] & \\ K \arrow[r, "\delta"'] & M/\sigma_M \end{tikzcd}$$

commutes, where δ is the restriction of $\sigma_M^\natural$ to K. Corollary 2.7 also gives that Ker $\delta = \sigma_K$. In the terminology of [12] this says that the pair (γ, K) is an object in the category $\mathbf{PLA}(M/\sigma_M, I_{M/\sigma_M}, M/_{\sigma_M})$. Theorem 2.11 guarantees the existence of a unique $(2,1,0)$-morphism $\pi : \mathcal{M} = \mathcal{M}(M/\sigma_M, I_{M/\sigma_M}, M/\sigma_M) \to K$ such that

$$\begin{tikzcd} & M/\sigma_M \arrow[dl, "\gamma"'] \arrow[dr, "\tau_{\mathcal{M}}"] & \\ K & & \mathcal{M} \arrow[ll, "\pi"] \end{tikzcd}$$

commutes, where $\tau_{\mathcal{M}} : M/\sigma_M \to \mathcal{M}$ is the map given by $[m]\tau_{\mathcal{M}} = (\underset{1}{\bullet}\xrightarrow{[m]}\underset{[m]}{\bullet}, [m])$. Of course $\mathcal{M} = (M/\sigma_M)F^e$; regarding $\mathcal{M}$ and K as $(2,1,1,0)$-algebras we then have

$$(\Sigma, [m])^\circ \pi = (\underset{1}{\bullet}\xrightarrow{[m]}\underset{[m]}{\bullet}, [m])\pi = [m]\tau_{\mathcal{M}}\pi = [m]\gamma = m^\circ$$

for any $(\Sigma, [m]) \in \mathcal{M}$. But as π is a $(2,1,0)$-morphism,

$$(\Sigma, [m])\pi \ \sigma \ (\Sigma, [m])^\circ \pi,$$

giving that

$$(\Sigma,[m])\pi^\circ = (\Sigma,[m])^\circ\pi^\circ = m^{\circ\circ} = m^\circ = (\Sigma,[m])^\circ\pi.$$

Thus π is a $(2,1,1,0)$-morphism.

We can now define

$$\beta_{T:M}:\ \mathrm{Mor}_{\mathbf{RC}}(T, MF^\sigma)\to\ \mathrm{Mor}_{\mathbf{PLA^0}}(TF^e, M)$$

by

$$\psi\beta_{T:M} = \psi^e\pi$$

where we regard π as a morphism from $\mathcal{M}$ to M.

Consider $\psi\in\mathrm{Mor}_{\mathbf{RC}}(T, MF^\sigma)$. From the definitions,

$$\psi\beta_{T:M}\alpha_{T:M} = \sigma_T(\psi^e\pi)^\sigma.$$

Let $t\in T$. Then

$$t\sigma_T(\psi^e\pi)^\sigma = [(\underset{1}{\bullet}\xrightarrow{\quad t\quad}\underset{t}{\bullet}, t)](\psi^e\pi)^\sigma =$$

$$= [(\underset{1}{\bullet}\xrightarrow{\quad t\quad}\underset{t}{\bullet}, t)\psi^e\pi] = [(\underset{1}{\bullet}\xrightarrow{\quad t\psi\quad}\underset{t\psi}{\bullet}, t\psi)\pi].$$

We have $t\psi=[m]$ for some $m\in M$ and so

$$t\psi\beta_{T:M}\alpha_{T:M} = [(\underset{1}{\bullet}\xrightarrow{\quad [m]\quad}\underset{[m]}{\bullet}, [m])\pi] = [[m]\tau_{\mathcal{M}}\pi]$$

$$= [[m]\gamma] = [m^\circ] = [m] = t\psi.$$

Thus $\psi\beta_{T:M}\alpha_{T:M}=\psi$ and $\beta_{T:M}\alpha_{T:M}$ is the identity map in $\mathrm{Mor}_{\mathbf{RC}}(T, MF^\sigma)$.

Finally we consider $\psi\in\mathrm{Mor}_{\mathbf{PLA^0}}(TF^e, M)$. Again from the definitions,

$$\psi\alpha_{T:M}\beta_{T:M} = (\sigma_T\psi^\sigma)^e\pi.$$

From Proposition 2.10, TF^e is generated as a $(2,1,0)$-algebra by elements of the form $(\underset{1}{\bullet}\xrightarrow{\quad t\quad}\underset{t}{\bullet}, t)$ where $t\in T$. Thus to show that $\alpha_{T:M}\beta_{T:M}$ is the identity map in $\mathrm{Mor}_{\mathbf{PLA^0}}(TF^e, M)$, it is enough to show that for any $t\in T$,

$$(\underset{1}{\bullet}\xrightarrow{\quad t\quad}\underset{t}{\bullet}, t)(\sigma_T\psi^\sigma)^e\pi = (\underset{1}{\bullet}\xrightarrow{\quad t\quad}\underset{t}{\bullet}, t)\psi.$$

Let $t\in T$. We have

$$(\underset{1}{\bullet}\xrightarrow{\quad t\quad}\underset{t}{\bullet}, t)(\sigma_T\psi^\sigma)^e\pi = (\underset{1}{\bullet}\xrightarrow{\quad [m]\quad}\underset{[m]}{\bullet}, [m])\pi$$

where $[m] = t\sigma_T\psi^\sigma = [(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t)\psi]$. Thus

$$(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t)(\sigma_T\psi^\sigma)^e\pi = [m]\tau_{\mathcal{M}}\pi = [m]\gamma$$

$$= m^\circ = (\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t)\psi^\circ = (\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t)^\circ\psi,$$

using the fact that ψ is a $(2,1,1,0)$-morphism. But

$$(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t)^\circ = (\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t),$$

giving

$$(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t)(\sigma_T\psi^\sigma)^e\pi = (\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t)\psi$$

as required.

4. A presentation of $\mathcal{M}(X, f, S)$

For the remainder of the paper we return to regarding left ample monoids as algebras of type $(2,1,0)$. Given a monoid presentation (X, f, S) of a right cancellative monoid S, the left ample monoid $\mathcal{M}(X, f, S)$ is generated by $\{x\tau_{\mathcal{M}(X,f,S)} : x \in X\}$ where $x\tau_{\mathcal{M}(X,f,S)}$ is $(\underset{1}{\bullet}\xrightarrow{\quad x \quad}\underset{xf}{\bullet}, xf)$. Let X^* be the free monoid on X and $\iota : X \to X^*$ the canonical embedding. Theorem 2.12 says that $\mathcal{M}(X, \iota, X^*)$ is the free left ample monoid on $\{x\tau_{\mathcal{M}(X,\iota,X^*)} : x \in X\}$ where $x\tau_{\mathcal{M}(X,\iota,X^*)} = (\underset{1}{\bullet}\xrightarrow{\quad x \quad}\underset{x}{\bullet}, x)$. Abbreviating our notation we write $\tau_{\mathcal{M}}$ for $\tau_{\mathcal{M}(X,f,S)}$, F_X for $\mathcal{M}(X, \iota, X^*)$ and τ for $\tau_{\mathcal{M}(X,\iota,X^*)}$. It follows from Theorem 2.12 that $\mathcal{M}(X, f, S)$ is a morphic image of F_X under a morphism θ such that $\tau\theta = \tau_{\mathcal{M}}$. The aim of this section is to give an explicit description of the kernel of θ.

Our approach is to find a congruence ρ on F_X such that the pair $(\tau\rho^\natural, F_X/\rho)$ is an initial object in the category $\mathbf{PLA}(X, f, S)$. According to Theorem 2.11, the pair $(\tau_{\mathcal{M}}, \mathcal{M}(X, f, S))$ is also an initial object in this category. Uniqueness of initial objects yields the required result.

The main problem lies in the fact that the class of proper left ample monoids is a quasivariety and quasivarieties are not closed under morphic images, not unless they are actually varieties. In view of Theorem 2.9, to see that proper left ample monoids do not form a variety, one needs only remark that there are left ample monoids which are not proper. Indeed, any inverse monoid that is not proper as an inverse monoid is not proper as a left ample monoid. For a specific example of a *right* *cancellative* monoid with a quotient that fails to be right cancellative or even left ample, take the free monoid $T = \{x, y\}^*$ on two generators and quotient T by the congruence generated by $\{(x, x^2), (y, y^2)\}$; the quotient monoid has non-commuting idempotents.

We begin with some observations concerning our construction of the free left ample monoid F_X on X. With $(X, \iota, X^*), F_X$ and τ defined as above, Proposition 2.10 gives that

$$\begin{tikzcd} X \arrow[d, "\tau"'] \arrow[dr, "\iota"] & \\ F_X \arrow[r, "\sigma^\sharp"'] & X^* \end{tikzcd}$$

commutes, where $\sigma^\sharp$ is the onto morphism with kernel the relation σ on F_X given by $(\Sigma, \overline{x})\sigma^\sharp = \overline{x}$. We lift the maps τ and ι to monoid morphisms $\overline{\tau} : X^* \to F_X$ and $\overline{\iota} = I_{X^*} : X^* \to X^*$.

Lemma 4.1. *c.f.* [9] *With $\overline{\tau}$ defined as above, the diagram*

$$\begin{tikzcd} X \arrow[d, "\overline{\tau}"'] \arrow[dr, "I_{X^*}"] & \\ F_X \arrow[r, "\sigma^\sharp"'] & X^* \end{tikzcd}$$

commutes. It follows that the submonoid *$Im\ \overline{\tau}$ of F_X is isomorphic to X^*.*

Further, if $e(\bar{x}\bar{\tau}) = g(\bar{y}\bar{\tau})$ for $e, g \in E(F_X)$ and $\overline{x}, \overline{y} \in X^*$, then $\overline{x} = \overline{y}$.

Proof The first part of the lemma is clear.

If $e(\bar{x}\bar{\tau}) = g(\bar{y}\bar{\tau})$ for $e, g \in E(F_X)$ and $\overline{x}, \overline{y} \in X^*$, then $(e(\bar{x}\bar{\tau}))\sigma^\sharp = (g(\bar{y}\bar{\tau}))\sigma^\sharp$. As $\sigma^\sharp$ is a morphism mapping all idempotents to 1 and $\overline{\tau}\sigma^\sharp = I_{X^*}$ we obtain $\overline{x} = \overline{y}$.

It follows from [9] and, more directly, from Proposition 2.10 of this paper and Lemma 4.1 of [12], that any element a of F_X can be written as $a = e(\bar{x}\bar{\tau})$ for some $e \in E(F_X)$ and $\overline{x} \in X^*$. This enables us to define the *positive part* $p(a)$ of $a \in F_X$ by $p(a) = \overline{x}$ where $a = e(\bar{x}\bar{\tau}), e \in E(F_X), \bar{x} \in X^*$. That p is a function is immediate from Lemma 4.1 above.

Lemma 4.2. *The function $p : F_X \to X^*$ is a monoid morphism.*

Proof We need only show that if $a = e(\bar{x}\bar{\tau})$ and $b = g(\bar{y}\bar{\tau})$ where $e, g \in E(F_X)$ and $\overline{x}, \overline{y} \in X^*$, then $p(a)p(b) = p(ab)$. Using condition (AL) gives

$$ab = e(\bar{x}\bar{\tau})g(\bar{y}\bar{\tau}) = e(\bar{x}\bar{\tau}g)^+\bar{x}\bar{\tau}\bar{y}\bar{\tau} = e(\bar{x}\bar{\tau}g)^+(\bar{x}\,\bar{y})\bar{\tau}$$

so that

$$p(a)p(b) = \bar{x}\,\bar{y} = p(ab)$$

as required.

Let (X, f, S) be a monoid presentation of a right cancellative monoid S. We use $\bar{f}$ to denote the extension of f to a monoid morphism from X^* to S. Let

$$H = H_{(X,f,S)} =$$

$$\{((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau}, (\bar{v}\bar{\tau})^+\bar{u}\bar{\tau}) \in F_X \times F_X : \bar{u}, \bar{v} \in X^* \text{ and } \bar{u}\bar{f} = \bar{v}\bar{f}\}$$

and let $\rho = \rho_{(X,f,S)}$ be the $(2,1,0)$-congruence on F_X generated by H.

Proposition 4.3. *With ρ defined as above, F_X/ρ is a proper left ample monoid.*

Proof Note first that if $\bar{u}, \bar{v} \in X^*$ and $\bar{u}\bar{f} = \bar{v}\bar{f}$, then using Lemma 4.1,

$$((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau})\sigma^\sharp\bar{f} = \bar{v}\bar{\tau}\sigma^\sharp\bar{f} = \bar{v}\bar{f} = \bar{u}\bar{f} = \bar{u}\bar{\tau}\sigma^\sharp\bar{f} = ((\bar{v}\bar{\tau})^+\bar{u}\bar{\tau})\sigma^\sharp\bar{f}$$

so that $H \subseteq \text{Ker}\ \sigma^\sharp\bar{f}$. As S is right cancellative, the monoid morphism $\sigma^\sharp\bar{f} : F_X \to S$ is in fact a $(2,1,0)$-morphism, so that $\rho \subseteq \text{Ker}\ \sigma^\sharp\bar{f}$. It follows that if $a, b \in F_X$ and $a\ \rho\ b$, then $p(a)\bar{f} = p(b)\bar{f}$.

We next show that every idempotent ρ-class of F_X contains an idempotent. Let $a \in F_X$ with $a\ \rho\ a^2$. Writing a as $a = e(\bar{x}\bar{\tau})$ for some $e \in E(F_X)$ and $\bar{x} \in X^*$ the above paragraph gives $p(a)\bar{f} = p(a^2)\bar{f}$. Using Lemma 4.2 we have

$$\bar{x}\bar{f} = p(a)\bar{f} = p(a^2)\bar{f} = p(a)^2\bar{f} = \bar{x}^2\bar{f} = (\bar{x}\bar{f})^2$$

and so $\bar{x}\bar{f} = 1 = 1\bar{f}$, as S is right cancellative. Thus

$$(\bar{x}\bar{\tau}, (\bar{x}\bar{\tau})^+) = ((1\bar{\tau})^+\bar{x}\bar{\tau}, (\bar{x}\bar{\tau})^+1\bar{\tau}) \in H$$

and

$$a = e(\bar{x}\bar{\tau})\ \rho\ e(\bar{x}\bar{\tau})^+ \in E(F_X)$$

as required. As the idempotents of F_X commute, it follows that so also do the idempotents of F_X/ρ.

Suppose now that $a = e(\bar{x}\bar{\tau}), b = g(\bar{y}\bar{\tau}) \in F_X$ with $a\ \mathcal{R}^*\ b$, where $e, g \in E(F_X)$ and $\bar{x}, \bar{y} \in X^*$. We claim that $a\rho\ \mathcal{R}^*\ b\rho$ in F_X/ρ. To see this, let $c = h(\bar{z}\bar{\tau}), d = k(\bar{w}\bar{\tau}) \in F_X$ with $h, k \in E(F_X)$ and $\bar{z}, \bar{w} \in X^*$, and suppose that

$$(c\rho)(a\rho) = (d\rho)(a\rho),$$

that is, $ca\ \rho\ da$. Hence

$$(p(c)p(a))\bar{f} = p(ca)\bar{f} = p(da)\bar{f} = (p(d)p(a))\bar{f}$$

and we obtain $(\bar{z}\bar{x})\bar{f} = (\bar{w}\bar{x})\bar{f}$. As S is right cancellative, $\bar{z}\bar{f} = \bar{w}\bar{f}$ and so

$$(\bar{w}\bar{\tau})^+\bar{z}\bar{\tau}\ \rho\ (\bar{z}\bar{\tau})^+(\bar{w}\bar{\tau}). \tag{1}$$

From $ca\ \rho\ da$ and the fact that ρ is a $(2,1,0)$-congruence, certainly

$$(cb)^+ = (ca)^+\ \rho\ (da)^+ = (db)^+. \tag{2}$$

Using Lemma 2.2,

$$(cb)^+ \leq c^+ = h(\bar{z}\bar{\tau})^+ \leq (\bar{z}\bar{\tau})^+$$

so that

$$(db)^+(\bar{z}\bar{\tau})^+\ \rho\ (cb)^+(\bar{z}\bar{\tau})^+ = (cb)^+\ \rho\ (db)^+.$$

Thus $(db)^+\ \rho\ (db)^+(\bar{z}\bar{\tau})^+$ and similarly, $(cb)^+\ \rho\ (cb)^+(\bar{w}\bar{\tau})^+$. Putting together (1) and (2) gives

$$(cb)^+(\bar{w}\bar{\tau})^+\bar{z}\bar{\tau}\ \rho\ (db)^+(\bar{z}\bar{\tau})^+\bar{w}\bar{\tau}$$

so that from the above remarks,

$$(cb)^+\bar{z}\bar{\tau}\ \rho\ (db)^+\bar{w}\bar{\tau},$$

hence

$$(cb)^+\bar{z}\bar{\tau}\bar{y}\bar{\tau}\ \rho\ (db)^+\bar{w}\bar{\tau}\bar{y}\bar{\tau}.$$

The left hand side equals

$$(h(\bar{z}\bar{\tau})g(\bar{y}\bar{\tau}))^+\bar{z}\bar{\tau}\bar{y}\bar{\tau} = h((\bar{z}\bar{\tau})g)^+(\bar{z}\bar{\tau}\bar{y}\bar{\tau})^+\bar{z}\bar{\tau}\bar{y}\bar{\tau} =$$

$$= h((\bar{z}\bar{\tau})g)^+\bar{z}\bar{\tau}\bar{y}\bar{\tau} = h(\bar{z}\bar{\tau})g\bar{y}\bar{\tau} = cb,$$

using (AL) and Lemma 2.2. Similarly, the expression on the right hand side equals db so that $cb\ \rho\ db$, that is,

$$c\rho b\rho = d\rho b\rho$$

in F_X/ρ. Together with the dual argument this gives that $a\rho\ \mathcal{R}^*\ b\rho$ in F_X/ρ.

The above allows us to deduce from $a\ \mathcal{R}^*\ a^+$ that $a\rho\ \mathcal{R}^*\ a^+\rho$ for any $a \in F_X$. Since $a^+\rho = (a\rho)^+$ is idempotent, we have that F_X/ρ is left adequate *and* $+$ *has its standard meaning.* From the fact that each ρ-class contains an idempotent, we deduce that F_X/ρ inherits (AL) from F_X so that F_X/ρ is left ample.

It remains to show that F_X/ρ is proper. Let $a\rho, b\rho \in F_X/\rho$ and suppose that

$$a\rho \ (\mathcal{R}^* \cap \sigma) \ b\rho$$

so that $a\rho \,\mathcal{R}^*\, b\rho$ and $ha \ \rho \ hb$ for some $h \in E(F_X)$. From $a\rho \,\mathcal{R}^*\, b\rho$ we obtain

$$a^+\rho = (a\rho)^+ = (b\rho)^+ = b^+\rho$$

so that

$$a^+ \ \rho \ b^+, \tag{3}$$

whilst from $ha \ \rho \ hb$ we obtain $p(a)\bar{f} = p(b)\bar{f}$. Thus

$$(p(b)\bar{\tau})^+ p(a)\bar{\tau} \ \rho \ (p(a)\bar{\tau})^+ p(b)\bar{\tau} \tag{4}$$

and putting together (3) and (4) a straightforward argument gives that $a \ \rho \ b$. That is, $a\rho = b\rho$ and F_X/ρ is proper.

Continuing with the notation established above, it follows from the proof of Proposition 4.3 that there is a $(2,1,0)$-morphism $\eta : F_X/\rho \to S$ given by $(a\rho)\eta = a\sigma^\sharp \bar{f}$. Clearly η is onto,

$$\begin{tikzcd}
X \arrow[d, "\tau\rho^\natural"'] \arrow[dr, "f"] & \\
F_X/\rho \arrow[r, "\eta"'] & S
\end{tikzcd}$$

commutes and $F_X/\rho = \langle X\tau\rho^\natural \rangle$.

Lemma 4.4. *The pair $(\tau\rho^\natural, F_X/\rho)$ is an object in* $\mathbf{PLA}(X, f, S)$.

Proof It remains only to show that Ker $\eta = \sigma_{F_X/\rho}$. For ease of notation we write σ_ρ for $\sigma_{F_X/\rho}$. Clearly $\sigma_\rho \subseteq$ Ker η.

Conversely, consider $a = e(\bar{x}\bar{\tau}), b = g(\bar{y}\bar{\tau}) \in F_X$ where $e, g \in E(F_X)$, $\bar{x}, \bar{y} \in X^*$ and suppose that $(a\rho)\eta = (b\rho)\eta$. By definition of η, $a\sigma^\sharp \bar{f} = b\sigma^\sharp \bar{f}$ so that as $e\sigma^\sharp = g\sigma^\sharp = 1$, we have that $\bar{x}\bar{\tau}\sigma^\sharp \bar{f} = \bar{y}\bar{\tau}\sigma^\sharp \bar{f}$. But $\bar{\tau}\sigma^\sharp = I_{X^*}$ and so $\bar{x}\bar{f} = \bar{y}\bar{f}$ and

$$(\bar{x}\bar{\tau})^+ \bar{y}\bar{\tau} \ \rho \ (\bar{y}\bar{\tau})^+ \bar{x}\bar{\tau}.$$

It is now clear that $a\rho \ \sigma_\rho \ b\rho$ in F_X/ρ so that Ker $\eta \subseteq \sigma_\rho$ and consequently, Ker $\eta = \sigma_\rho$.

Lemma 4.5. *The pair $(\tau\rho^\natural, F_X/\rho)$ is an initial object in $\mathbf{PLA}(X, f, S)$.*

Proof We need to show that if $(g, N) \in \mathrm{Ob}\ \mathbf{PLA}(X, f, S)$ then there is a morphism $\psi \in \mathrm{Mor}_{\mathbf{PLA}}((\tau\rho^\natural, F_X/\rho), (g, N))$ (recall from Section 2 that there is at most one element in any Mor set in this category). In other words, given a proper left ample monoid N and a map $g : X \to N$ with $\langle Xg\rangle = N$, such that there is a morphism $\mu : N \to S$ with Ker $\mu = \sigma_N$ and

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f"] & \\ N \arrow[r, "\mu"] & S \end{tikzcd}$$

commuting, we must show that there is a $(2, 1, 0)$-morphism $\psi : F_X/\rho \to N$ such that

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "\tau\rho^\natural"] & \\ N & & F_X/\rho \arrow[ll, "\psi"] \end{tikzcd} \qquad (*)$$

commutes.

Let (g, N) be as above. Since F_X is the free proper left ample monoid on $X\tau$ there is a $(2, 1, 0)$-morphism ϕ such that

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "\tau"] & \\ N & & F_X \arrow[ll, "\phi"] \end{tikzcd}$$

commutes. We claim that $\rho \subseteq$ Ker ϕ. To see this, first extend g to a monoid morphism $\bar{g} : X^* \to N$, so that $\bar{g}\mu = \bar{f}$ and $\bar{\tau}\phi = \bar{g}$. Let $\bar{x}, \bar{y} \in X^*$ with $\bar{x}\bar{f} = \bar{y}\bar{f}$. Then $\bar{x}\bar{g}\mu = \bar{y}\bar{g}\mu$ and since Ker $\mu = \sigma_N$ and N is proper, it follows from Lemma 2.5 that $(\bar{x}\bar{g})^+\bar{y}\bar{g} = (\bar{y}\bar{g})^+\bar{x}\bar{g}$. We deduce from $\bar{\tau}\phi = \bar{g}$ that

$$((\bar{x}\bar{\tau})^+\bar{y}\bar{\tau}, (\bar{y}\bar{\tau})^+\bar{x}\bar{\tau}) \in \text{ Ker } \phi.$$

Thus $H \subseteq$ Ker ϕ and consequently, $\rho \subseteq$ Ker ϕ.

We can now define a $(2,1,0)$-morphism $\psi : F_X/\rho \to N$ by $(a\rho)\psi = a\phi$, for any $a \in F_X$. For any $x \in X$ we have

$$x\tau\rho^\natural\psi = ((x\tau)\rho)\psi = x\tau\phi = xg$$

so that diagram (*) commutes as required.

As quoted at the beginning of this section, the pair $(\tau_{\mathcal{M}}, \mathcal{M}(X,f,S))$ is also an initial object in $\mathbf{PLA}(X,f,S)$. We thus achieve the aim of this section.

Theorem 4.6. *The proper left ample monoid F_X/ρ is isomorphic to $\mathcal{M}(X,f,S)$. Indeed, if θ is the $(2,1,0)$-morphism making the diagram*

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "\tau"] & \\ \mathcal{M}(X,f,S) & & F_X \arrow[ll, "\theta"] \end{tikzcd}$$

commute, then Ker $\theta = \rho$.

5. Proper covers over q-subvarieties of $\mathcal{RC}$

Recall from Section 2 that any left ample monoid N has a *proper cover* M. Let $\mathcal{V}$ be a subclass of $\mathcal{RC}$. We say that N has a *proper cover over* $\mathcal{V}$ if N has a proper cover M such that $M/\sigma \in \mathcal{V}$ and we put

$$\widehat{\mathcal{V}} = \{N \in \mathcal{LA} : N \text{ has a proper cover over } \mathcal{V}\}.$$

These definitions correspond to those for groups and inverse monoids given in [14]. As indicated in that paper, if $\mathcal{V}$ is a *variety* of groups then the class of inverse monoids having proper covers over $\mathcal{V}$ is a *variety* of inverse monoids and it is shown in [16] that this variety is determined by Σ where

$$\Sigma = \{\bar{u}^2 \equiv \bar{u} : \bar{u} \equiv 1 \text{ is a law in } \mathcal{V}\}.$$

Our situation is somewhat different as, unlike the case for groups, right cancellative monoids do not themselves form a variety so one cannot speak of (sub)varieties of $\mathcal{RC}$. This leads us to consider q-subvarieties.

Let $\mathcal{V}$ be a quasivariety of algebras. As in the introduction we say that a subclass $\mathcal{W}$ of $\mathcal{V}$ is a *q-subvariety* of $\mathcal{V}$ if

$$\mathcal{W} = \{V \in \mathcal{V} : V \models \Sigma\}$$

for some set Σ of *identities*. We have the following alternative characterisation.

Lemma 5.1. [17] *Let $\mathcal{W}$ be a subclass of a quasivariety $\mathcal{V}$. Then the following conditions are equivalent:*

(i) $\mathcal{W}$ is a q-subvariety of $\mathcal{V}$;

(ii) $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ for some variety $\mathcal{U}$;

(iii) $\mathcal{W}$ is closed under subalgebras, direct products and morphisms between algebras in $\mathcal{V}$.

Proof The equivalence of (ii) and (iii) is proved in [17]. The equivalence of (i) and (ii) is immediate from Birkhoff's theorem [2], c.f. [3].

It is clear from the definition that a q-subvariety is a subquasivariety so that free objects in q-subvarieties exist.

Let $\mathcal{V}$ be a q-subvariety of $\mathcal{RC}$. In this section we show that $\widehat{\mathcal{V}}$ is a q-subvariety of $\mathcal{LA}$. Further, if (X, f, S) is a monoid presentation of the free object in $\mathcal{V}$ on X, then $\mathcal{M}(X, f, S)$ is the free object in $\widehat{\mathcal{V}}$ on X. We also show that $\widehat{\mathcal{V}}$ is the q-subvariety of $\mathcal{LA}$ axiomatisied by Σ, where

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} : \bar{u} \equiv \bar{v} \text{ is a law in } \mathcal{V}\}.$$

Proposition 5.2. *Let $\mathcal{V}$ be a q-subvariety of $\mathcal{RC}$. Then $\widehat{\mathcal{V}}$ is a q-subvariety of $\mathcal{LA}$.*

Proof In view of Corollaries 2.7 and 2.8, the argument that $\widehat{\mathcal{V}}$ is closed under subalgebras and direct products is routine.

Suppose that $M \in \widehat{\mathcal{V}}$ and $\theta : M \to N$ is an onto morphism where $N \in \mathcal{LA}$. By definition of $\widehat{\mathcal{V}}$, M has a proper cover over $\mathcal{V}$, that is, there is a proper left ample monoid P and an (idempotent separating) morphism $\phi : P \to M$ such that $S = P/\sigma \in \mathcal{V}$. Put $\psi = \phi\theta : P \to N$ so that ψ and $\sigma^\natural : P \to S$ are onto morphisms.

Since $\mathcal{LA}$ is a quasivariety, $N \times S$ is a left ample monoid and from Corollary 2.8, for any $(n, s), (m, t) \in N \times S$,

$$(n, s)\ \mathcal{R}^*\ (m, t) \text{ in } N \times S \text{ if and only if } n\ \mathcal{R}^*\ m \text{ in } N.$$

Let

$$K = \{(n, s) \in N \times S : \text{ there exists } p \in P \text{ with } p\psi = n \text{ and } p\sigma^\natural = s\}.$$

It is easy to verify that K is a subalgebra of $N \times S$ so that K is left ample and by Corollary 2.7 and the remark above, for any $(n, s), (m, t) \in K$,

$$(n, s)\ \mathcal{R}^*\ (m, t) \text{ in } K \text{ if and only if } n\ \mathcal{R}^*\ m \text{ in } N.$$

Suppose that $(n, s), (m, t) \in K$ and

$$(n, s)\ (\sigma \cap \mathcal{R}^*)\ (m, t) \text{ in } K.$$

Thus $(e,1)(n,s) = (e,1)(m,t)$ for some $e \in E(N)$ and $(n,s)\ \mathcal{R}^*\ (m,t)$ in K. Let $p,q \in P$ be such that $p\psi = n, p\sigma^\natural = s, q\psi = m$ and $q\sigma^\natural = t$. As $s = t$ we have that $p\ \sigma\ q$ in P and so $p^+q = q^+p$. Applying the morphism ψ yields $n^+m = m^+n$. But from $(n,s)\ \mathcal{R}^*\ (m,t)$ in K we have that $n\ \mathcal{R}^*\ m$ in N and so $n = m$. Thus $(n,s) = (m,t)$ and K is proper. Since ψ is onto it follows that the morphism $\gamma : K \to N$ is also onto, where γ is the projection onto the first coordinate. Patently γ is idempotent separating, so that K is a proper cover of N.

Let $e \in E(N)$. As ψ is onto, there is some $p \in P$ with $e = p\psi$. Hence

$$e = e^+ = (p\psi)^+ = p^+\psi$$

and $p^+\sigma^\natural = 1$ so that $(e,1) \in E(K)$. Thus $E(K) = E(N) \times \{1\}$.

Define $\mu : K \to N/\sigma_N \times S$ by $(n,s)\mu = ([n],s)$. Clearly μ is a morphism from K to the right cancellative monoid $N/\sigma \times S$. For any $(n,s),(m,t) \in K$ we have $(n,s)\mu = (m,t)\mu$ if and only if $[n] = [m]$ and $s = t$, that is, if and only if $en = em$ for some $e \in E(N)$ and $s = t$. Since $E(K) = E(N) \times \{1\}$ this is equivalent to $(n,s)\ \sigma_K\ (m,t)$ so that $\mathrm{Ker}\ \mu = \sigma_K$ and

$$K/\sigma_K \cong \ \mathrm{Im}\ \mu \subseteq N/\sigma_N \times S.$$

As shown in Section 3 we can construct from the onto morphism $\psi : P \to N$ an onto morphism $\psi^\sigma : P/\sigma_P = S \to N/\sigma_N$. Since $N/\sigma_N \in \mathcal{RC}$ and S is in the q-subvariety $\mathcal{V}$, it follows that $N/\sigma_N \in \mathcal{V}$. As $\mathcal{V}$ is also closed under direct products and subalgebras we have that $K/\sigma_K \in \mathcal{V}$. Thus K is a proper cover of N over $\mathcal{V}$ so that $N \in \widehat{\mathcal{V}}$ and $\widehat{\mathcal{V}}$ is closed under morphisms between algebras in $\mathcal{LA}$. By Lemma 5.1, $\widehat{\mathcal{V}}$ is a q-subvariety of $\mathcal{LA}$.

Given a q-subvariety $\mathcal{V}$ of $\mathcal{RC}$ the question now arises of determining a set of identities axiomatising the q-subvariety $\widehat{\mathcal{V}}$ of $\mathcal{LA}$. At this stage it is convenient to regard $\mathcal{RC}$ as a class of algebras of type $(2,0)$ so that an *identity* in $\mathcal{RC}$ is an expression of the form $\bar{u} \equiv \bar{v}$, where $\bar{u}$ and $\bar{v}$ are words over a countably infinite set Z. Given such an identity $\bar{u} \equiv \bar{v}$ we can form the identity $\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$, which is an identity for $(2,1,0)$-algebras.

We recall that if $\mathcal{V}$ is any quasivariety of algebras and $\mathcal{V}$ is non-trivial (that is, $\mathcal{V}$ contains non-trivial algebras), then for any set X (where $X \neq \emptyset$ if there are no nullary operations) the free object in $\mathcal{V}$ on X exists. Note that if $\mathcal{V}$ is a q-subvariety of $\mathcal{RC}$ and $S \in \mathcal{V}$, then, regarded as an algebra of type $(2,1,0)$, $S \in \widehat{\mathcal{V}}$. Thus if $\mathcal{V}$ is non-trivial, so also is $\widehat{\mathcal{V}}$.

In our last two results we follow the notation $F_X, \tau, \tau_{\mathcal{M}}, \rho$ established in the previous section for a monoid presentation (X, f, S) of a right cancellative monoid S.

Theorem 5.3. *Let $\mathcal{V}$ be a non-trivial q-subvariety of $\mathcal{RC}$. Then*

$$\widehat{\mathcal{V}} = \{N \in \mathcal{LA} : N \models \Sigma\}$$

where

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} : \bar{u} \equiv \bar{v} \textit{ is a law in } \mathcal{V}\}.$$

Proof Suppose first that $N \in \widehat{\mathcal{V}}$. There is a proper left ample monoid M such that $M/\sigma \in \mathcal{V}$ and an idempotent separating onto morphism $\zeta : M \to N$. If $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{V}$ then since M is proper, Lemma 2.5 gives that $M \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$. But ζ is onto so that $N \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ and indeed $N \models \Sigma$.

Conversely, suppose that N is a left ample monoid and $N \models \Sigma$. Let X be an infinite set of generators for N, that is, there is a map $g : X \to N$ such that $\langle Xg \rangle = N$. Let S be the free object in $\mathcal{V}$ on X and let $f : X \to S$ denote the canonical embedding of X into S, so that $\langle Xf \rangle = S$ and (X, f, S) is a monoid presentation of S.

Since F_X is the free (proper) left ample monoid on X with canonical embedding τ, there is a morphism $\theta : F_X \to N$ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "g"] & \\ F_X \arrow[rr, "\theta"'] & & N \end{tikzcd}$$

commutes. We show that $\rho \subseteq \mathrm{Ker}\ \theta$.

Let $\bar{u}, \bar{v} \in X^*$ and suppose that $\bar{u}\bar{f} = \bar{v}\bar{f}$. From Theorem II.11.4 of [3] it follows that $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{V}$ so that $\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} \in \Sigma$.

Since $N \models \Sigma$ this gives that

$$(\bar{u}\bar{g})^+\bar{v}\bar{g} = (\bar{v}\bar{g})^+\bar{u}\bar{g}$$

where $\bar{g}$ denotes the lifting of g to a monoid morphism from X^* to N. But $\bar{g} = \bar{\tau}\theta$ and θ is a $(2,1,0)$-morphism, giving that

$$((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau})\theta = ((\bar{v}\bar{\tau})^+\bar{u}\bar{\tau})\theta.$$

Thus $H \subseteq \mathrm{Ker}\ \theta$ and as H generates the congruence ρ we have that $\rho \subseteq \mathrm{Ker}\ \theta$.

It follows that there is an onto morphism $\psi : F_X/\rho \to N$ such that $\rho^\natural\psi = \theta$. Proposition 4.3 and Lemma 4.4 tell us that F_X/ρ is a proper left ample monoid and F_X/ρ has maximal right cancellative image $S \in \mathcal{V}$. Certainly then $F_X/\rho \in \widehat{\mathcal{V}}$ so that by Proposition 5.2 N, being left ample and a morphic image of F_X/ρ, is also in $\widehat{\mathcal{V}}$. This completes the proof of the theorem.

In view of the preceding proof, the last result of this paper is not surprising.

Theorem 5.4. *Let $\mathcal{V}$ be a non-trivial q-subvariety of $\mathcal{RC}$. Let X be a set and (X, f, S) the canonical monoid presentation of the free object in $\mathcal{V}$ on X. Then $\mathcal{M}(X, f, S)$ is the free object in $\widehat{\mathcal{V}}$ on X with canonical embedding $\tau_{\mathcal{M}}$.*

Proof Let N be a left ample monoid in $\widehat{\mathcal{V}}$ and let $g : X \to N$ be a map. Let S be the free object in $\mathcal{V}$ on X with canonical presentation (X, f, S) and let $\theta : F_X \to N$ be the morphism making

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "g"] & \\ F_X \arrow[rr, "\theta"'] & & N \end{tikzcd}$$

commute. Exactly as in the proof of Theorem 5.3 we have that $\rho \subseteq \operatorname{Ker} \theta$ so that there is a morphism $\psi : F_X/\rho \to N$ such that $\rho^\natural \psi = \theta$.

The monoid $F_X/\rho \in \widehat{\mathcal{V}}$ and from Theorem 2.11 and Lemma 4.5, the pairs $(\tau_{\mathcal{M}}, \mathcal{M}(X, f, S))$ and $(\tau\rho^\natural, F_X/\rho)$ are both initial objects in $\mathbf{PLA}(X, f, S)$. It follows that there is an isomorphism $\phi : \mathcal{M}(X, f, S) \to F_X/\rho$ such that $\tau_{\mathcal{M}}\phi = \tau\rho^\natural$. Put $\xi = \phi\psi$ so that $\xi : \mathcal{M}(X, f, S) \to N$ is a morphism. For any $x \in X$ we have

$$x\tau_{\mathcal{M}}\xi = x\tau_{\mathcal{M}}\phi\psi = x\tau\rho^\natural\psi = x\tau\theta = xg$$

so that

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "g"] & \\ \mathcal{M}(X, f, S) \arrow[rr, "\xi"'] & & N \end{tikzcd}$$

commutes. Certainly $\mathcal{M}(X, f, S) \in \widehat{\mathcal{V}}$ and from Proposition 2.10, $X\tau_{\mathcal{M}}$ generates $\mathcal{M}(X, f, S)$. The fact that $\mathcal{V}$ and $\widehat{\mathcal{V}}$ are non-trivial yields that $\tau_{\mathcal{M}}$ is one-one. Thus $\mathcal{M}(X, f, S)$ is the free object on X in $\widehat{\mathcal{V}}$ with canonical embedding $\tau_{\mathcal{M}}$.

Let $\mathcal{V}_1$ be the trivial q-subvariety of $\mathcal{RC}$, that is, $\mathcal{V}_1$ is axiomatised by the identity $u \equiv v$. By Theorem 5.3,

$$\widehat{\mathcal{V}}_1 = \{N \in \mathcal{LA} : N \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} \text{ for all } \bar{u}, \bar{v} \in Z\}$$

for some countably infinite set Z. It is then easy to see that $\widehat{\mathcal{V}}_1$ is the q-subvariety of semilattices.

For a less trivial example let $\mathcal{V}_2$ be the q-subvariety of $\mathcal{RC}$ consisting of all commutative cancellative monoids, that is, $\mathcal{V}_2$ is axiomatised (within $\mathcal{RC}$) by the identity $uv \equiv vu$. We say that words $\bar{u}, \bar{v}$ over a countably infinite set Z *have the same content* and write $c(\bar{u}) = c(\bar{v})$, if the number of occurrences of each $z \in Z$ is the same in $\bar{u}$ as it is in $\bar{v}$. From Corollary II.11.5 of [3] we have that $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{V}_2$ if and only if $\bar{u} = \bar{v}$ in the free commutative cancellative monoid on Z, that is, if and only if $c(\bar{u}) = c(\bar{v})$. From Theorem 5.3,

$$\widehat{\mathcal{V}}_2 = \{N \in \mathcal{LA} : N \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} \text{ for all } \bar{u}, \bar{v} \in Z^* \text{ with } c(\bar{u}) = c(\bar{v})\}.$$

Further, we deduce from Theorem 5.4 that if $X \neq \emptyset$ then the free object in $\widehat{\mathcal{V}}_2$ is *not* commutative. For let (X, f, S) be the canonical monoid presentation of the free commutative cancellative monoid S on X. Then $\mathcal{M} = \mathcal{M}(X, f, S)$ is the free object in $\widehat{\mathcal{V}}_2$ on X (with canonical embedding $\tau_{\mathcal{M}}$). Let $x \in X$. Then $(\Delta, 1), (\Delta, xf) \in \mathcal{M}$ where Δ is the subgraph

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "x"] & \underset{xf}{\bullet} \end{tikzcd}$$

of $\Gamma(X, f, S)$. We know that in S the powers of xf are distinct and it follows that in $\mathcal{M}$,

$$(\Delta, 1)(\Delta, xf) = (\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "x"] & \underset{xf}{\bullet} \end{tikzcd}, xf)$$

which is distinct from

$$(\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "x"] & \underset{xf}{\bullet} \arrow[r, "x"] & \underset{(xf)^2}{\bullet} \end{tikzcd}, xf),$$

namely, the product

$$(\Delta, xf)(\Delta, 1).$$

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