GRAPH EXPANSIONS OF UNIPOTENT MONOIDS

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Abstract

Margolis and Meakin use the Cayley graph of a group presentation to construct E-unitary inverse monoids [11]. This is the technique we refer to as *graph expansion*. In this paper we consider graph expansions of *unipotent* monoids, where a monoid is unipotent if it contains a unique idempotent. The monoids arising in this way are E-unitary and belong to the quasivariety of *weakly left ample* monoids. We give a number of examples of such monoids. We show that the least unipotent congruence on a weakly left ample monoid is given by the same formula as that for the least group congruence on an inverse monoid and we investigate the notion of *proper* for weakly left ample monoids.

Using graph expansions we construct a functor F^e from the category $\mathbf{U}$ of unipotent monoids to the category $\mathbf{PWLA}$ of proper weakly left ample monoids. The functor F^e is an *expansion* in the sense of Birget and Rhodes [2]. If we equip proper weakly left ample monoids with an extra unary operation and denote the corresponding category by $\mathbf{PWLA^0}$ then, regarded as a functor $\mathbf{U} \to \mathbf{PWLA^0}$, F^e is a left adjoint of the functor $F^\sigma : \mathbf{PWLA^0} \to \mathbf{U}$ that takes a proper weakly left ample monoid to its greatest unipotent image.

Our main result uses the covering theorem of [8] to construct free weakly left ample monoids.

1 Introduction

This paper uses *graph expansions* to study a class of monoids called *weakly left ample*. We arrive at this class by considering the relation $\widetilde{\mathcal{R}}$, defined on a monoid M by the rule that $a \,\widetilde{\mathcal{R}}\, b$ if and only if a and b have the same set of idempotent left identities [12].

A monoid M is *left semiadequate* if every $\widetilde{\mathcal{R}}$-class contains an idempotent and $E(M)$ is a semilattice. It is easy to see that in a left semiadequate monoid the idempotent in the $\widetilde{\mathcal{R}}$-class of $a \in M$ is unique; we denote this element by a^+.

We say that a left semiadequate monoid M is *weakly left ample* if $\widetilde{\mathcal{R}}$ is a left congruence and

$$ae \;=\; (ae)^+a \qquad \text{(AL)}$$

for all $a \in M$ and $e \in E(M)$.

The class of weakly left ample monoids contains all inverse monoids but much more besides. For example, any unipotent monoid S is weakly left ample, as is any Bruck-Reilly extension $BR(S, \theta)$, for any endomorphism θ of S. Let T be a monoid such that $E(T)$ is a semilattice. It is easy to check that if T is a semilattice of unipotent monoids, then it is a *strong* semilattice of unipotent monoids, and $E(T)$ is central in T; it is then clear that T is weakly left ample.

For a further example of a weakly left ample monoid we consider a submonoid N of the monoid M of partial endomorphisms of $\mathbf{F}$, where $\mathbf{F}$ is a free right S-set over a unipotent monoid S on a non-empty set I. The S-set $\mathbf{F}$ is the disjoint union $\bigcup_{i \in I} x_iS$ where $\{x_i : i \in I\}$ is a set in one-one correspondence with I. The monoid S acts on the right of $\mathbf{F}$ in the obvious way. The monoid M of partial endomorphisms of $\mathbf{F}$ consists of all those maps $\alpha : \mathbf{G} \to \mathbf{F}$, where $\mathbf{G}$ is an S-subset of $\mathbf{F}$, such that $(ys)\alpha = y\alpha s$ for all $y \in \mathbf{G}$ and $s \in S$. Let N be the subset of M given by

$$N = \{\emptyset, I\} \cup \{\alpha \in M : \text{ dom } \alpha = x_iS, \text{ im } \alpha \subseteq x_jS \text{ for some } i, j \in I\}$$

where $\emptyset$ is the empty map and I the identity map on $\mathbf{F}$. The set N is closed under partial composition and so is a submonoid of M. The proof of the following straightforward lemma is left to the reader.

Lemma 1.1 *With notation as above, the submonoid N of M is weakly left ample. Moreover, N is isomorphic to the Brandt semigroup with adjoined identity $B^1(S, I)$.*

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We mention that while the examples above satisfy the left-right dual property of being weakly *right* ample, this is not always the case; see for example Proposition 3.6 of [6].

Results for E-unitary, that is, proper, inverse monoids have been successfully extended to wider classes, in particular, to E-dense monoids [5]. This paper may be viewed as another with the same goal. Here we do not insist that our monoids be E-dense, but rather that the relation $\widetilde{\mathcal{R}}$ behaves remotely like the relation $\mathcal{R}$. On the other hand we impose the condition that they are proper, which in this context is a stronger condition than that of being E-unitary.

This paper concentrates on *graph expansions*; the graph expansion of a unipotent monoid is proper weakly left ample. The basic philosophy of this paper and also of [7] and [8], is that weakly left example monoids are related to (and determined by) unipotent monoids in a way analogous to that in which inverse monoids are related to groups.

In Section 2 we show that the least unipotent congruence σ on a weakly left ample monoid M is given by the rule that

$$a \ \sigma \ b \text{ if and only if } ea = eb \text{ for some } e \in E(M),$$

that is, σ has the same description as the least group congruence on an inverse monoid. Analagously to the concept for inverse monoids, we say that a weakly left ample monoid is *proper* if $\sigma \cap \widetilde{\mathcal{R}} = \iota$.

By a *monoid presentation* we mean a triple (X, f, S) where X is a set, S is a monoid and $f : X \to S$ is a function such that Xf generates S as a monoid. As shown in [9] one can use the *Cayley graph of a monoid presentation* to construct monoids which we call graph expansions. In Section 3 we show that, given a monoid presentation (X, f, S), the corresponding graph expansion $\mathcal{M}(X, f, S)$ is weakly left ample if and only if S is unipotent. In this case, $\mathcal{M}(X, f, S)$ is proper and, in terminology introduced in Section 4, $\mathcal{M}(X, f, S)$ is the initial object in a suitable category $\mathbf{PLA}(X, f, S)$ of X-generated proper weakly left ample monoids having maximum unipotent image S. This latter result is analogous to Theorem 2.2 of [11], in which Margolis and Meakin show that the corresponding category of X-generated proper (E-unitary) inverse monoids with maximum group image G has an initial object, constructed from the Cayley graph of the *group* presentation of G with set of generators X; we remark that Ash gives an alternative construction of the initial object in [1].

In Section 4 we define the categories $\mathbf{U}$ and $\mathbf{U}(X)$ (where X is a set) of unipotent monoids and of X-generated unipotent monoids. We introduce the categories $\mathbf{PWLA^0}$ and $\mathbf{PWLA}(X)$ of proper weakly left ample monoids. To define the former, we equip proper weakly left ample monoids with an extra unary operation; the latter is the category of X-generated proper weakly left ample monoids. We use graph expansions to construct functors

$F^e : \mathbf{U} \to \mathbf{PWLA^0}$ and $F^e_X : \mathbf{U}(X) \to \mathbf{PWLA}(X)$. The functors F^e and F^e_X are expansions and are left adjoints of functors $F^\sigma : \mathbf{PWLA^0} \to \mathbf{U}$ and $F^\sigma_X : \mathbf{PWLA}(X) \to \mathbf{U}(X)$, respectively, which in each case take a proper weakly left ample monoid to its maximum unipotent image.

In the final section we show that if $\iota : X \to X^*$ is the natural embedding, then $\mathcal{M}(X, \iota, X^*)$ is the free weakly left ample monoid on (a set in one to one correspondence with) X.

Throughout the paper we consider weakly left ample monoids as *algebras of type* $(2,1,0)$ where the unary operation is $a \mapsto a^+$. We also consider unipotent monoids and monoid morphisms. As remarked earlier, a unipotent monoid is weakly left ample. The potential ambiguity never arises, in view of the following lemma.

Lemma 1.2 *Let S be a unipotent monoid. A subset X of S is a set of generators of S as an algebra of type $(2,0)$ if and only if it is a set of generators of S as an algebra of type $(2,1,0)$. A subset T of S is a submonoid of S if and only if it is a $(2,1,0)$-subalgebra. Further, a function ϕ from a weakly left ample monoid M to S is a monoid morphism, that is, a $(2,0)$-morphism, if and only if it is a morphism where S is regarded as a weakly left ample monoid, that is, a $(2,1,0)$-morphism.*

The proof of Lemma 1.2 is exactly analogous to that of Lemma 2.3 of [10]. Indeed, once some basics are established, the proofs of many of our results follow those in [9] or [10]. Whenever this is the case we omit the argument.

Finally in this introduction we remark that for any monoid M, it is easy to see that if $a \in M$ and $e \in E(M)$, then $a \,\widetilde{\mathcal{R}}\, e$ if and only if $ea = a$ and for any $e \in E(M)$

$$fa = a \text{ implies } fe = e.$$

Thus if $E(M)$ is a semilattice, $a \,\widetilde{\mathcal{R}}\, e$ if and only if e is the minimum element in the set of idempotent left identities of a (see [12]).

2 The least unipotent congruence σ

We follow standard terminology by referring to a congruence ρ on a monoid M as a Π congruence, where Π is a property defined for monoids, if M/ρ has property Π.

For any monoid M the relation $\overline{\sigma}$ is defined by the rule that

$$a \,\overline{\sigma}\, b \text{ if and only if } ea = eb \text{ for some } e \in E(M).$$

Of course, $\overline{\sigma}$ will not, in general, be a congruence. However, $\overline{\sigma}$ has the property that it is contained in every unipotent congruence on M and hence in σ, where

σ is the *least unipotent congruence* on M. If M is an inverse monoid, then σ is in fact the least *group congruence* on M and it is well known [13] that $\sigma = \overline{\sigma}$. Analogously, if M is a weakly left ample monoid and $\widetilde{\mathcal{R}} = \mathcal{R}^*$, then σ is the least *right cancellative* congruence on M and again, $\sigma = \overline{\sigma}$ [3]. Recall that the relation $\mathcal{R}^*$ is defined by the rule that elements a, b of M are $\mathcal{R}^*$-related if and only if they are $\mathcal{R}$-related in a monoid containing M. A weakly left ample monoid in which $\widetilde{\mathcal{R}} = \mathcal{R}^*$ is called *left ample* (formerly, left type A).

In our first result we show that for $\sigma = \overline{\sigma}$ it is enough that M be weakly left ample.

Lemma 2.1 *Let M be a weakly left ample monoid. Then $\sigma = \overline{\sigma}$. That is, the least unipotent congruence σ on M is given by the rule that*

$$a \; \sigma \; b \text{ if and only if } ea = eb \text{ for some } e \in E(M).$$

Proof In view of the remarks preceding the lemma, it is enough to show that the relation $\overline{\sigma}$ is a congruence and $M/\overline{\sigma}$ is unipotent.

Since M is a monoid with *semilattice* of idempotents $E(M)$, it is easy to see that $\overline{\sigma}$ is an equivalence. Clearly $\overline{\sigma}$ is right compatible. If $a, b, c \in M$ and $a \; \overline{\sigma} \; b$, then $ea = eb$ for some $e \in E(M)$ and so $cea = ceb$. Using condition (AL) we have $(ce)^+ca = (ce)^+cb$, so that $ca \; \overline{\sigma} \; cb$ and $\overline{\sigma}$ is left compatible. Thus $\overline{\sigma}$ is a congruence on M.

Certainly all idempotents of M lie in the same $\overline{\sigma}$-class as 1. Suppose now that $a \in M$ and $a\overline{\sigma}$ is idempotent in $M/\overline{\sigma}$. Hence $a \; \overline{\sigma} \; a^2$ so that $ea = ea^2$ for some $e \in E(M)$. Consider the element eae. Using (AL) we have

$$(eae)^2 = e(ae)ae = e((ae)^+a)ae$$

and so

$$(eae)^2 = (ae)^+ea^2e = (ae)^+eae = e(ae)^+ae = eae.$$

Thus $eae \in E(M)$. Notice also that from $eae = e(ae)^+a$ we have that $g(eae) = ga$ where $g = eae(ae)^+ \in E(M)$. Hence

$$a \; \overline{\sigma} \; eae \; \overline{\sigma} \; 1$$

and $M/\overline{\sigma}$ is unipotent, as required.

We recall from the introduction that a weakly left ample monoid is *proper* if $\widetilde{\mathcal{R}} \cap \sigma = \iota$. As $\sigma = \iota$ on any unipotent monoid S, certainly S is proper. On the other hand any Brandt extension $B^1(S, I)$ cannot be proper, as $\overline{\sigma}$ is universal on any semigroup with zero.

As previously remarked, any Bruck-Reilly extension $BR(S, \theta)$ of S is also weakly left ample. It is not hard to show that $BR(S, \theta)$ is proper if and only if θ is one-one. If T is a monoid and a semilattice $E(T)$ of unipotent monoids

$T_e, e \in E(T)$, then again T is weakly left ample. As mentioned in the previous section, T is a strong semilattice of the monoids T_e. An easy argument shows that T is proper if and only if the connecting homomorphisms are one-one. Further examples of proper weakly left ample monoids emerge in the course of this paper and [8]. In particular, *free* weakly left ample monoids are proper.

For an inverse monoid, being proper is the same as being E-unitary. In the general case a proper weakly left ample monoid M is E-unitary but the converse is not true [3]. Note that if M is E-unitary then $E(M)$ is a σ-class.

The proof of the next lemma is analogous to that in the left ample case.

Lemma 2.2 *[9] Let M be a proper weakly left ample monoid. If $a, b \in M$, then $a \,\sigma\, b$ if and only if $b^+a = a^+b$.*

The results of this section are essentially for *semigroups* (see [8]). However, for the remainder of the paper the requirement that we deal with *monoids* is more crucial.

3 Graph expansions

We begin this section by recalling from [9] the construction of the graph expansion $\mathcal{M}(X, f, S)$ of a monoid presentation (X, f, S).

For the purposes of this paper a *graph* Γ consists of two sets $V = V(\Gamma)$ (the *vertices* of Γ) and $E = E(\Gamma)$ (the *edges* of Γ), together with two maps (written on the left), $i : E \to V$ and $t : E \to V$. The maps i and t are the *initial* and *terminal* maps, respectively. We may represent $e \in E$ with $i(e) = v$ and $t(e) = v'$ by

$$\underset{v}{\bullet} \xrightarrow{\;\;e\;\;} \underset{v'}{\bullet}$$

A *path* from a vertex v to a vertex w is a finite sequence of edges $e_1, \ldots, e_n$ with

$$i(e_1) = v,\ t(e_1) = i(e_2),\ t(e_2) = i(e_3),\ \ldots, t(e_n) = w$$

and we write this as

$$\underset{v}{\bullet} \xrightarrow{\;\;e_1\;\;} \bullet \xrightarrow{\;\;e_2\;\;} \bullet \cdots\cdots \bullet \xrightarrow{\;\;e_n\;\;} \underset{w}{\bullet}$$

There is also an *empty path* I_v from any vertex v to itself. A graph Γ is *v-rooted*, where $v \in V$, if for all $w \in V$ there is a path from v to w. A *subgraph* Δ of Γ consists of a subset $V(\Delta)$ of $V(\Gamma)$ and a subset $E(\Delta)$ of $E(\Gamma)$ such that for any $e \in E(\Delta), i(e), t(e) \in V(\Delta)$. Clearly any path determines a subgraph; it is convenient at times to use the same notation for a path and the corresponding subgraph.

A *graph morphism* θ from a graph Γ to a graph Γ' consists of two functions, each denoted by θ, from $V(\Gamma)$ to $V(\Gamma')$ and from $E(\Gamma)$ to $E(\Gamma')$, such that for any $e \in E(\Gamma)$,

$$i(e)\theta = i(e\theta) \text{ and } t(e)\theta = t(e\theta).$$

Clearly such a θ maps subgraphs to subgraphs and paths to paths.

A monoid S *acts* on a graph Γ (on the left) if V and E are left S-sets and i and t are left S-maps, that is, $i(se) = si(e)$ and $t(se) = st(e)$ for all $s \in S$ and $e \in E$. Note that if S acts on Γ, then the action of any $s \in S$ is a graph morphism so that if Δ is a subgraph of Γ, then so is $s\Delta$.

Our interest here is in the Cayley graph $\Gamma = \Gamma(X, f, S)$ of a monoid presentation (X, f, S). Here $V(\Gamma) = S$ and

$$E(\Gamma) = \{(s, x, s(xf)) : s \in S, x \in X\}$$

where $i(s, x, s(xf)) = s$ and $t(s, x, s(xf)) = s(xf)$. We may write the edge $(s, x, s(xf))$, or the corresponding subgraph, as

$$\underset{s}{\bullet} \xrightarrow{\quad x \quad} \underset{s(xf)}{\bullet}$$

The monoid S acts on Γ where for $s \in S, v \in V, (t, x, t(xf)) \in E$ we have

$$s.v = sv, \quad s.(t, x, t(xf)) = (st, x, st(xf)).$$

The *graph expansion* $\mathcal{M} = \mathcal{M}(X, f, S)$ of (X, f, S) is given by

$$\mathcal{M} = \{(\Delta, s) : \Delta \text{ is a finite 1-rooted subgraph of } \Gamma \text{ and } 1, s \in V(\Delta)\}.$$

We define a multiplication on $\mathcal{M}$ by

$$(\Delta, s)(\Sigma, t) = (\Delta \cup s\Sigma, st).$$

The following is easy to check.

Lemma 3.1 *With $\mathcal{M} = \mathcal{M}(X, f, S)$ and multiplication as above, $\mathcal{M}$ is a monoid with identity $(\bullet_1, 1)$.*

In [11] Margolis and Meakin use an analogous construction to study proper (i.e. E-unitary) inverse monoids. In [9] it is shown that $\mathcal{M}(X, f, S)$ is left ample, indeed proper left ample, if and only if S is right cancellative. Right cancellative monoids are, of course, unipotent; the converse is far from true.

Proposition 3.2 *Let (X, f, S) be a monoid presentation. Then the following conditions are equivalent:*

(i) S is unipotent;

(ii) $\mathcal{M}(X, f, S)$ is weakly left ample;

(iii) $\widetilde{\mathcal{R}}$ is a left congruence on $\mathcal{M}(X, f, S)$ and every $\widetilde{\mathcal{R}}$-class contains an idempotent.

Proof Suppose first that S is unipotent. If Δ is a finite 1-rooted subgraph of Γ, then $(\Delta, 1) \in \mathcal{M} = \mathcal{M}(X, f, S)$ and clearly $(\Delta, 1)$ is idempotent. Moreover any idempotent of $\mathcal{M}$ must have this form. It is then easy to see that $E(\mathcal{M})$ is a semilattice. If $(\Delta, s) \in \mathcal{M}$ then $(\Delta, s)\widetilde{\mathcal{R}}(\Delta, 1)$ so that $\mathcal{M}$ is left semiadequate and $(\Delta, s)^+ = (\Delta, 1)$. It follows that for any $(\Delta, s), (\Sigma, t) \in \mathcal{M}$, $(\Delta, s)\ \widetilde{\mathcal{R}}\ (\Sigma, t)$ if and only if $\Delta = \Sigma$. It is then easy to check that $\widetilde{\mathcal{R}}$ is a left congruence. That condition (AL) holds is exactly as in Proposition 3.3 of [10].

That (ii) implies (iii) is by definition. It remains to prove that (iii) implies (i). Suppose then that (iii) holds and that S is *not* unipotent.

Choose $e \in E(S)\backslash\{1\}$. Since $\langle Xf \rangle = S$, there exist $x_1, x_2, ..., x_n \in X, n \geq 1$ such that $e = (x_1 f)(x_2 f)...(x_n f)$. We may assume that n is minimum, so that the subgraph Δ of Γ

$$1 \xrightarrow{x_1} x_1 f \xrightarrow{x_2} (x_1 f)(x_2 f) \cdots\cdots \bullet \xrightarrow{x_n} (x_1 f)\dots(x_n f) = e$$

has no loops.

Let t be any vertex of Δ, so that $(\Delta, t) \in \mathcal{M}$. We claim that $(\Delta, t)\ \widetilde{\mathcal{R}}\ (\Delta, 1)$.

We know that $(\Delta, 1)$ is idempotent and clearly $(\Delta, 1)(\Delta, t) = (\Delta, t)$. Suppose now that (Σ, f) is idempotent and $(\Sigma, f)(\Delta, t) = (\Delta, t)$. Since (Σ, f) is idempotent, we have that $\Sigma = \Sigma \cup f\Sigma$ so that $f\Sigma \subseteq \Sigma$, and from the fact that (Σ, f) is a left identity for (Δ, t) we have that $\Sigma \subseteq \Delta$. If $f \neq 1$, then as f is a vertex of Σ, Σ must contain a non-trivial path from 1 to f and as $f\Sigma \subseteq \Sigma$, Σ must contain a loop with vertex f. But then Δ contains a loop. From this contradiction we deduce that $f = 1$. Then

$$(\Sigma, 1)(\Delta, 1) = (\Sigma \cup \Delta, 1) = (\Delta, 1)$$

so that by the remarks at the end of the introduction, $(\Delta, t)\ \widetilde{\mathcal{R}}\ (\Delta, 1)$. Since t was any vertex of Δ, we certainly have that $(\Delta, e)\ \widetilde{\mathcal{R}}\ (\Delta, 1)$.

Put $\Theta = \Delta \cup e\Delta$. Then Θ is a finite 1-rooted subgraph of Γ and has vertices including 1 and e. Moreover, $e\Theta \subseteq \Theta$ so that both $(\Theta, 1)$ and (Θ, e) are idempotents of $\mathcal{M}$. Since by assumption $\widetilde{\mathcal{R}}$ is a left congruence,

$$(\Theta, 1)(\Delta, e)\ \widetilde{\mathcal{R}}\ (\Theta, 1)(\Delta, 1),$$

that is, $(\Theta, e)\ \widetilde{\mathcal{R}}\ (\Theta, 1)$. But (Θ, e) is a left identity for its $\widetilde{\mathcal{R}}$-class and so $(\Theta, e)(\Theta, 1) = (\Theta, 1)$ and $e = 1$, a contradiction. Thus S is unipotent.

The remainder of the proofs of the results in this section is exactly as in [9].

Proposition 3.3 *Let (X, f, S) be a monoid presentation of a unipotent monoid S. Then $\mathcal{M} = \mathcal{M}(X, f, S)$ is a proper weakly left ample monoid. Further,*

for any $(\Delta, s), (\Sigma, t) \in \mathcal{M}$,
(i) $(\Delta, s) \in E(\mathcal{M})$ *if and only if* $s = 1$;
(ii) $(\Delta, s)^+ = (\Delta, 1)$;
(iii) $(\Delta, s) \, \widetilde{\mathcal{R}} \, (\Sigma, t)$ *if and only if* $\Delta = \Sigma$;
(iv) $(\Delta, s) \, \sigma \, (\Sigma, t)$ *if and only if* $s = t$.

For any monoid presentation (X, f, S) it is clear that

$$\Gamma_x = \underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{xf}{\bullet}$$

is a 1-rooted subgraph of Γ and $(\Gamma_x, xf) \in \mathcal{M}$. We define $\tau_{\mathcal{M}} : X \to \mathcal{M}$ by $x\tau_{\mathcal{M}} = (\Gamma_x, xf)$.

Proposition 3.4 *Let* (X, f, S) *be a monoid presentation of a unipotent monoid* S. *Then* $\mathcal{M} =< X\tau_{\mathcal{M}} >$. *Further, defining* $\sigma^\sharp_{\mathcal{M}} : \mathcal{M} \to S$ *by* $(\Delta, s)\sigma^\sharp_{\mathcal{M}} = s$ *we have that* $\sigma^\sharp$ *is an onto morphism with kernel* $\sigma_{\mathcal{M}}$ *and the following diagram commutes*

$$\begin{tikzcd} X \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dr, "f"] & \\ \mathcal{M} \arrow[r, "\sigma^\sharp_{\mathcal{M}}"'] & S \end{tikzcd}$$

In terminology defined in Section 4 the next result says that if S is unipotent, then the pair $(\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S))$ is an initial object in the category $\mathbf{PWLA}(X, f, S)$.

Proposition 3.5 *Let* (X, f, S) *be a monoid presentation of a unipotent monoid* S. *Suppose that* N *is a proper weakly left ample monoid and* $N = \langle Xh \rangle$ *for some function* $h : X \to N$. *In addition suppose that*

$$\begin{tikzcd} X \arrow[d, "h"'] \arrow[dr, "f"] & \\ N \arrow[r, "\sigma^\sharp_N"'] & S \end{tikzcd}$$

commutes, where $\sigma^\sharp_N : N \to S$ *is a morphism with kernel* σ_N. *Then there is a morphism* $\theta : \mathcal{M} = \mathcal{M}(X, f, S) \to N$ *such that*

$$\begin{tikzcd}
 & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "h"] & \\
\mathcal{M} \arrow[rr, "\theta"'] & & N
\end{tikzcd}$$

commutes.

4 The categories $\mathbf{PWLA}(X)$ and $\mathbf{PWLA^0}$

If $\mathcal{A}$ is a class of algebras of fixed type then we denote by $\mathbf{A}$ the corresponding category of algebras and morphisms. Thus, if $\mathcal{U}$ denotes the class of unipotent monoids and $\mathcal{PWLA}$ the class of proper weakly left ample monoids (regarded as algebras of type $(2, 1, 0)$), then the corresponding categories are $\mathbf{U}$ and $\mathbf{PWLA}$. In view of Lemma 1.2, $\mathbf{U}$ may be regarded as a full subcategory of $\mathbf{PWLA}$.

Using the techinque of graph expansions we construct a functor $F^e : \mathbf{U} \to \mathbf{PWLA}$. Suppose that $S \in \text{Ob } \mathbf{U}$, that is, S is a unipotent monoid. The triple (S, I_S, S) is certainly a monoid presentation of S, where $I_S : S \to S$ is the identity map. We put $SF^e = \mathcal{M}(S, I_S, S)$. By Proposition 3.3, $\mathcal{M}(S, I_S, S)$ is a proper weakly left ample monoid so that F^e is a function from $\text{Ob } \mathbf{U}$ to $\text{Ob } \mathbf{PWLA}$.

Suppose now that $S, T \in \text{Ob } \mathbf{U}$ and $\theta : S \to T$ is in $\text{Mor}_{\mathbf{U}}(S, T)$. We first define a map $\theta' : \Gamma(S, I_S, S) \to \Gamma(T, I_T, T)$ by

$$v\theta' = v\theta$$

for any vertex v of $\Gamma(S, I_S, S)$ and

$$(s, x, sx)\theta' = (s\theta, x\theta, s\theta x\theta)$$

for any edge (s, x, sx) of $\Gamma(S, I_S, S)$. Clearly θ' is a graph morphism so that as remarked in Section 3, θ' maps subgraphs to subgraphs and paths to paths, indeed as $1\theta = 1$, θ' maps 1-rooted subgraphs to 1-rooted subgraphs. Thus we can define θF^e to be θ^e where $\theta^e : SF^e \to TF^e$ is given by

$$(\Delta, s)\theta^e = (\Delta\theta', s\theta).$$

For any subgraph Δ of $\Gamma(S, I_S, S)$ and $s \in S$,

$$(s \cdot \Delta)\theta' = s\theta \cdot \Delta\theta'.$$

Using Proposition 3.3 it is now easy to see that

$$\theta^e \in \operatorname{Mor}_{\mathbf{PWLA}}(SF^e, TF^e)$$

and that F^e defined in this manner is a functor from $\mathbf{U}$ to $\mathbf{PWLA}$.

In fact, F^e is an *expansion* in the sense of Birget-Rhodes [2]. Regarding $\mathbf{U}$ as a subcategory of $\mathbf{PWLA}$, we need to show that for any $S \in \operatorname{Ob} \mathbf{U}$ there is an onto morphism $\eta_S \in \operatorname{Mor}_{\mathbf{PWLA}}(SF^e, S)$ such that for each $\theta \in \operatorname{Mor}_{\mathbf{U}}(S, T)$ the square

$$\begin{tikzcd} SF^e \arrow[r, "\theta^e"] \arrow[d, "\eta_S"'] & TF^e \arrow[d, "\eta_T"] \\ S \arrow[r, "\theta"'] & T \end{tikzcd}$$

commutes; further, if θ is onto then so also is θF^e. Defining η_S by $(\Delta, s)\eta_S = s$, it is immediate that η_S is an onto monoid morphism so that by Lemma 1.2, $\eta_S \in \operatorname{Mor}_{\mathbf{PWLA}}(SF^e, S)$. For any $\theta \in \operatorname{Mor}_{\mathbf{U}}(S, T)$ and any $(\Delta, s) \in SF^e$,

$$(\Delta, s)\theta^e\eta_T = (\Delta\theta', s\theta)\eta_T = s\theta = (\Delta, s)\eta_S\theta$$

so that the above square commutes. Suppose now that θ is onto. For any $t \in T$ we have $t = s\theta$ for some $s \in S$ so that

$$(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t) = (\underset{1}{\bullet} \xrightarrow{\quad s\theta \quad} \underset{s\theta}{\bullet}, s\theta) = (\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}, s)\theta^e.$$

Recall from Proposition 3.4 that

$$\{(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t) : t \in T\}$$

is a set of generators of TF^e, so that θ^e is onto and we have proved:

Proposition 4.1 *The functor $F^e : \mathbf{U} \to \mathbf{PWLA}$ is an expansion.*

We now define in the obvious way a functor $F^\sigma : \mathbf{PWLA} \to \mathbf{U}$. The action of F^σ on objects is given by $MF^\sigma = M/\sigma$ for any $M \in \operatorname{Ob} \mathbf{PWLA}$. By definition of σ, the monoid M/σ is unipotent. For $\theta \in \operatorname{Mor}_{\mathbf{PWLA}}(M, N)$ put $\theta F^\sigma = \theta^\sigma$ where $[m]\theta^\sigma = [m\theta]$. In view of the description of θ in Lemma 2.1, θ^σ is well defined. Clearly F^σ is a functor from $\mathbf{PWLA}$ to $\mathbf{U}$.

We would *like* to say that F^e is a left adjoint of F^σ. Unfortunately, this is not strictly true and we now present two approaches to remedying this situation. The first is analogous to that in [11] for X-generated proper inverse monoids and that in [9] for X-generated proper left ample monoids; the idea is to fix a set of generators for the monoids under consideration. The second parallels the alternative approach in [10] for proper left ample monoids, where the trick is to regard proper left ample monoids as possessing an extra unary operation. The category constructed in this way includes the category $\mathbf{F}$ of weakly left FA monoids and $(2,1,1,0)$-morphisms, defined in the forerunner [7] to this paper. The category $\mathbf{F}$ arose from *Szendrei expansions* of unipotent monoids.

Let X be a set and $\mathcal{A}$ a class of algebras of a given fixed type. Then $\mathbf{A}(X)$ is the category which has objects pairs (f, A) where $A \in \mathcal{A}, f : X \to A$ and $\langle Xf \rangle = A$; a morphism in $\mathbf{A}(X)$ from (f, A) to (g, B) is an algebra morphism $\theta : A \to B$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "\theta"'] & & B \end{tikzcd}$$

commutes. As remarked in [9], if such a θ exists, it is unique and must be onto. It is easy to check that $\mathbf{A}(X)$ is a category. Again using Lemma 1.2 we may, if we wish, regard $\mathbf{U}(X)$ as a full subcategory of $\mathbf{PWLA}(X)$.

Let (X, f, S) be a monoid presentation of a unipotent monoid S. The full subcategory $\mathbf{PWLA}(X, f, S)$ of $\mathbf{PWLA}(X)$ has as objects those $(g, M) \in$ Ob $\mathbf{PWLA}(X)$ such that the diagram

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f"] & \\ M \arrow[r, "\sigma_M^\sharp"'] & S \end{tikzcd}$$

commutes, where $\sigma_M^\sharp$ is a morphism with kernel σ_M. As previously remarked, $\sigma_M^\sharp$ must be onto, so that S is the maximum unipotent image of M. As σ_S is the identity relation on S it is clear that $(f, S) \in$ Ob $\mathbf{PWLA}(X, f, S)$. If $(g, M) \in$ Ob $\mathbf{PWLA}(X, f, S)$, then $\sigma_M^\sharp : M \to S$ is the unique morphism in $\mathrm{Mor}_{\mathbf{PWLA}(X)}((g, M), (f, S))$, so that (f, S) is a *terminal object* in $\mathbf{PWLA}(X, f, S)$. Proposition 3.5 translates as the following.

Theorem 4.2 *Let (X, f, S) be a monoid presentation of a unipotent monoid*

S. *Then the pair* $(\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S))$ *is an* initial object *in the category* $\mathbf{PWLA}(X,f,S)$.

Following the approach of [9] we define a functor $F^e_X : \mathbf{U}(X) \to \mathbf{PWLA}(X)$. Suppose that (f,S) is an object in $\mathbf{U}(X)$. From Proposition 3.4 we know that $(\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S))$ is an object in $\mathbf{PWLA}(X)$. We put

$$(f,S)F^e_X = (\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S)).$$

If (g,T) is another object in $\mathbf{U}(X)$ and $\theta \in \mathrm{Mor}_{\mathbf{U}(X)}((f,S),(g,T))$, then we define a map, denoted by θ'', from $\Gamma(X,f,S)$ to $\Gamma(X,g,T)$ by the obvious action on vertices and action on edges given by

$$(s,x,s(xf))\theta'' = (s\theta, x, s\theta xg).$$

Then θ'' is a graph morphism and

$$\theta^e_X : \mathcal{M}(X,f,S) \to \mathcal{M}(X,g,T)$$

defined by

$$(\Delta, s)\theta^e_X = (\Delta\theta'', s\theta)$$

is a $(2,1,0)$-morphism such that

$$\theta^e_X \in \mathrm{Mor}_{\mathbf{PWLA}(X)}((\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S)), (\tau_{\mathcal{M}(X,g,T)}, \mathcal{M}(X,g,T)).$$

We now put $\theta F^e_X = \theta^e_X$. Then $F^e_X : \mathbf{U}(X) \to \mathbf{PWLA}(X)$ is a functor. Indeed, exactly as in [9], we have

Proposition 4.3 *The functor* $F^e_X : \mathbf{U}(X) \to \mathbf{PWLA}(X)$ *is an expansion.*

We now construct a functor $F^\sigma_X : \mathbf{PWLA}(X) \to \mathbf{U}(X)$. The action of F^σ_X on objects is given by

$$(f,M)F^\sigma_X = (f\sigma^\natural_M, M/\sigma_M)$$

where $\sigma^\natural_M : M \to M/\sigma_M$ is the natural morphism. Suppose now that (f,M) and (g,N) are objects in $\mathbf{PWLA}(X)$ and $\theta \in \mathrm{Mor}_{\mathbf{PWLA}(X)}((f,M),(g,N))$. Define $\theta^e_X : M/\sigma_M \to N/\sigma_n$ by $[m]\theta^e_X = [m\theta]$. Then θ^e_X is well defined and indeed $\theta^e_X \in \mathrm{Mor}_{\mathbf{U}(X)}((f\sigma^\natural_M, M/\sigma_M),(g\sigma^\natural_N, N/\sigma_N))$ and we put $\theta F^\sigma_X = \theta^\sigma_X$. Clearly F^σ_X is a functor from $\mathbf{PWLA}(X)$ to $\mathbf{U}(X)$. As in [9] we have the desired result.

Theorem 4.4 *The functor* F^e_X *is a left adjoint of the functor* F^σ_X.

An alternative approach, introduced in [10] for the special case of right cancellative and proper left ample monoids, does not involve specific generating sets and allows us to consider the entire category $\mathbf{U}$ and 'almost' the entire category $\mathbf{PWLA}$. However we lose by being forced to consider proper weakly left ample monoids as algebras of type $(2, 1, 1, 0)$, as follows.

The category $\mathbf{PWLA^0}$ has as objects proper weakly left ample monoids given an added unary operation $^\circ$ such that for any proper weakly left ample monoid M

(i) $m \, \sigma \, m^\circ$ for all $m \in M$

and

(ii) $\{m^\circ : m \in M\}$ is a transversal of the σ-classes of M.

The morphisms of $\mathbf{PWLA^0}$ are the morphisms between objects *regarded as algebras of type* $(2, 1, 1, 0)$.

As σ is trivial on a unipotent monoid S, the only way S can be made into an object of $\mathbf{PWLA^0}$ is if $s^\circ = s$ for all $s \in S$. For an arbitrary proper weakly left ample monoid there are of course many choices for $^\circ$.

With the right choice of $^\circ$ for SF^e, the functor F^e may be regarded as a functor from $\mathbf{U}$ to $\mathbf{PWLA^0}$. For a unipotent monoid S define $^\circ$ on SF^e by

$$(\Sigma, s)^\circ = (\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}\, , s).$$

As in [10], F^e is then a functor from $\mathbf{U}$ to $\mathbf{PWLA^0}$. Of course, F^σ may be viewed as a functor from $\mathbf{PWLA^0}$ to $\mathbf{U}$. For the proof of the last result of this section, see [10].

Theorem 4.5 *Regarded as functors between* $\mathbf{U}$ *and* $\mathbf{PWLA^0}$*,* F^e *is a left adjoint of* F^σ*.*

5 Free weakly left ample monoids

The class $\mathcal{WLA}$ of weakly left ample monoids, although it fails to be a variety, is a quasivariety of algebras of type $(2, 1, 0)$. By definition it is axiomatised by the set

$$\begin{gathered}\{1x = 1 = x1, \quad (xy)z = x(zy), \\ (x^2 = x \wedge y^2 = y) \Rightarrow xy = yx, \\ (x^+)^2 = x^+, \;\; x^+x = x, \;\; (x^2 = x \wedge xy = y) \Rightarrow xy^+ = y^+, \\ x^+ = y^+ \Rightarrow (zx)^+ = (zy)^+, \;\; x^2 = x \Rightarrow yx = (yx)^+y\}\end{gathered}$$

of quasi-identities. Thus free weakly left ample monoids exist. The aim of this final section is to show that graph expansions can be used to construct the free objects in this quasivariety.

Let N be a subalgebra of a weakly left ample monoid M. Since $\mathcal{WLA}$ is a quasivariety and a^+ *always* denotes the idempotent in the $\widetilde{\mathcal{R}}$-class of a, it follows that for all $a, b \in N$

$$a \,\widetilde{\mathcal{R}}\, b \text{ in } N \text{ if and only if } a \,\widetilde{\mathcal{R}}\, b \text{ in } M.$$

The class $\mathcal{PWLA}$ of proper weakly left ample monoids is the subquasivariety of $\mathcal{WLA}$ determined by the quasi-identity

$$(x^+ = y^+ \wedge z^2 = z \wedge zx = zy) \Rightarrow x = y.$$

We emphasize that consequently, a subalgebra N of a proper weakly left ample monoid M is proper. Moreover from Lemma 2.2,

$$\sigma_M \cap (N \times N) = \sigma_N.$$

At this stage we require a result from a predecessor of this paper, [8].

Theorem 5.1 *[8] Let S be a weakly left ample monoid. Then S has a proper weakly left ample cover. That is, there is a proper weakly left ample monoid P and an idempotent separating morphism ϕ from P onto S.*

Theorem 5.2 *Let X be a set and let $\iota : X \to X^*$ be the canonical embedding. Let $\mathcal{M} = \mathcal{M}(X, \iota, X^*)$. Then $\tau_{\mathcal{M}} : X \to \mathcal{M}$ is an embedding and $\mathcal{M}$ is the free weakly left ample monoid on $X\tau_{\mathcal{M}}$.*

Proof Certainly $\tau_{\mathcal{M}} : X \to \mathcal{M}$ is an embedding.

Suppose that M is a weakly left ample monoid and $g : X \to M$ is a function. By Theorem 5.1, there is a proper weakly left ample monoid P and an onto morphism $\phi : P \to M$. For each $x \in X$ choose $p_x \in P$ such that $p_x\phi = xg$. Let $h : X \to P$ be given by $xh = p_x$ and let $Q = \langle Xh \rangle$. By the remarks at the beginning of this section, Q is a proper weakly left ample monoid.

Put $S = Q/\sigma_Q$ so that S is unipotent and $S = Q\sigma_Q^\natural = \langle Xh \rangle \sigma_Q^\natural = \langle Xh\sigma_Q^\natural \rangle$. Thus $(X, h\sigma_Q^\natural, S)$ is a monoid presentation and we let $\mathcal{N}$ be the proper weakly left ample monoid given by $\mathcal{N} = \mathcal{M}(X, h\sigma_Q^\natural, S)$.

Denote by θ the extension of $h\sigma_Q^\natural : X \to S$ to a morphism $\theta : X^* \to S$. We have that $(\iota, X^*), (h\sigma_Q^\natural, S) \in \text{Ob } \mathbf{U}(X)$ and

$$\begin{tikzcd} & X \arrow[dl, "\iota"'] \arrow[dr, "h\sigma_Q^\natural"] & \\ X^* \arrow[rr, "\theta"'] & & S \end{tikzcd}$$

commutes, so that $\theta \in \mathrm{Mor}_{\mathbf{U}(X)}((\iota, X^*), (h\sigma_Q^\natural, S))$. From Section 4, $\theta_X^e \in \mathrm{Mor}_{\mathbf{PWLA}(X)}((\tau_{\mathcal{M}}, \mathcal{M}), (\tau_{\mathcal{N}}, \mathcal{N}))$ so that $\theta_X^e : \mathcal{M} \to \mathcal{N}$ is a morphism and

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "\tau_{\mathcal{N}}"] & \\ \mathcal{M} \arrow[rr, "\theta_X^e"'] & & \mathcal{N} \end{tikzcd}$$

commutes. Notice now that (h, Q) is an object in $\mathbf{PWLA}(X, h\sigma_Q^\natural, S)$ and by Theorem 4.2 $\mathcal{N}$ is the initial object in this category. Thus there is a morphism $\psi : \mathcal{N} \to Q$ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{N}}"'] \arrow[dr, "h"] & \\ \mathcal{N} \arrow[rr, "\psi"'] & & Q \end{tikzcd}$$

commutes. Regarding $\theta_X^e\psi$ as a morphism from $\mathcal{M}$ to P we have that $\theta_X^e\psi\phi : \mathcal{M} \to M$ is a morphism and for any $x \in X$,

$$x\tau_{\mathcal{M}}\theta_X^e\psi\phi = x\tau_{\mathcal{N}}\psi\phi = xh\phi = p_x\phi = xg$$

so that

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "g"] & \\ \mathcal{M} \arrow[rr, "\theta_X^e\psi\phi"'] & & M \end{tikzcd}$$

commutes as required.

Remark that since $\mathcal{M} = \mathcal{M}(X, \iota, X^*)$ is a graph expansion of a right cancellative monoid, it follows from Proposition 3.3 of [9] that $\mathcal{M}$ is left ample. Indeed, $\mathcal{M}$ is the free left ample monoid on $X\tau_{\mathcal{M}}$.

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