Wagner's Theory of Generalised Heaps

Christopher Hollings

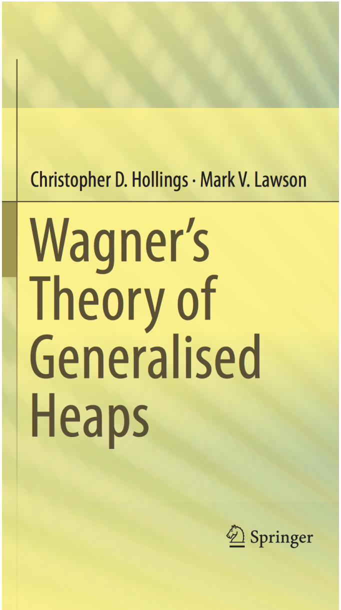
Mathematical Institute & The Queen's College
University of Oxford

York Semigroup, 17 April 2019

Mathematical Institute

Oxford Mathematics

Wagner's Theory of Generalised Heaps (Springer, 2017)



Inverse semigroups

Let S be a semigroup; $s' \in S$ is a generalised inverse for $s \in S$ if

$$ss's = s \qquad \text{and} \qquad s'ss' = s'.$$

Call S an inverse semigroup if every element has precisely one generalised inverse.

Equivalently, an inverse semigroup is a semigroup in which

1. every element has at least one generalised inverse;
2. idempotents commute with each other.

Motivation: partial bijections

A partial bijection on a set X is a bijection $A \to B$, where $A, B \subseteq X$. We compose partial transformations α, β (left $\to$ right) on the domain

$$\operatorname{dom} \alpha\beta = (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\alpha^{-1}$$

and put $x(\alpha\beta) = (x\alpha)\beta$, for $x \in \operatorname{dom} \alpha\beta$.

Let $\mathcal{I}_X$ denote the collection of all partial bijections on X, under this composition. Every $\alpha : A \to B$ in $\mathcal{I}_X$ has an inverse $\alpha^{-1} : B \to A$ in $\mathcal{I}_X$.

Theorem: $\mathcal{I}_X$ is an inverse semigroup — the symmetric inverse semigroup on X.

The Wagner–Preston representation

Every inverse semigroup can be embedded in a symmetric inverse semigroup.

Map $s \in S$ to the partial transformation $\rho_s \in \mathcal{I}_S$ with $\operatorname{dom} \rho_s = Ss^{-1}$ and $x\rho_s = xs$, for $x \in \operatorname{dom} \rho_s$.

The Erlanger Programm

Felix Klein (1872): every geometry may be regarded as the theory of invariants of a particular group of transformations.

$$\text{Groups} \longleftrightarrow \text{Geometries}$$

But there are geometries for which this approach doesn't work: for example, differential geometry.

Solution: generalise the Erlanger Programm by seeking a more general structure than that of a group.

Pseudogroups

Oswald Veblen & J. H. C. Whitehead, *The foundations of differential geometry*, CUP, 1932:

A pseudogroup Γ is a collection of partial homeomorphisms between open subsets of a topological space such that Γ is closed under composition and inverses, where we compose $\alpha, \beta \in \Gamma$ only if im $\alpha =$ dom β.

Use pseudogroups of 'regular' (i.e., one-one) partial homeomorphisms to classify 'geometric objects' ('invariants').

Abstraction of pseudogroups

Look for abstract structure corresponding to pseudogroup, just as abstract group corresponds to group of permutations.

But partially-defined operation difficult to work with, so first seek to ‘complete’ the operation in a pseudogroup . . .

Different compositions in a pseudogroup

J. A. Schouten & J. Haantjes, 'On the theory of the geometric object' *Proc. LMS* 42 (1937), 356–376:

compose partial transformations α, β only if $\operatorname{im} \alpha \subseteq \operatorname{dom} \beta$.

Stanisław Gołąb, 'Über den Begriff der "Pseudogruppe von Transformationen"', *Math. Ann.* 116 (1939), 768–780:

compose α, β if $\operatorname{im} \alpha \cap \operatorname{dom} \beta \neq \emptyset$.

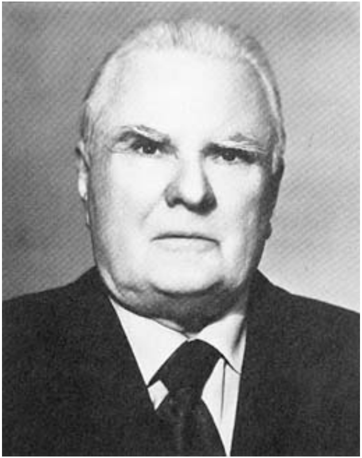
Incomplete composition

Have arrived at a composition which is almost fully defined: $\alpha\beta$ exists only if $\operatorname{im}\alpha \cap \operatorname{dom}\beta \neq \emptyset$.

It only remains to take account of the possibility of that $\operatorname{im}\alpha \cap \operatorname{dom}\beta = \emptyset$.

To modern eyes, this is easy: in this case, put $\alpha\beta = \varepsilon$, the empty transformation on X — this acts as a zero in $\mathcal{I}_X$.

Viktor Vladimirovich Wagner



Виктор Владимирович Вагнер (1908–1981).

Wagner and differential geometry

Added appendix ‘The theory of differential objects and the foundations of differential geometry’ to Russian translation of Veblen and Whitehead’s ‘The foundations of differential geometry’.

Developed rigorous theory of geometric objects, building on the work of Gołąb, et. al.

Pseudogroups of partial one-one transformations emerge as being important.

Composition of partial transformations

'К теории частичных преобразований' ('On the theory of partial transformations'), *Dokl. Akad. nauk SSSR* 84(4) (1952), 653–656:

A partial transformation α on a set X may be expressed as a binary relation:

$$\{(x, y) \in \operatorname{dom}\alpha \times \operatorname{im}\alpha : x\alpha = y\} \subseteq X \times X.$$

Then composition of partial transformations is a special case of that of binary relations:

$$x\,(\rho \circ \sigma)\,y \iff \exists z \in X \text{ such that } x\,\rho\,z \text{ and } z\,\sigma\,y.$$

Since $\emptyset \subseteq X \times X$, the empty transformation now appears naturally in the theory.

Semigroups of binary relations

Let $\mathfrak{B}(A \times A)$ be the semigroup of all binary relations on a set A.

$\mathfrak{B}(A \times A)$ is ordered by $\subset$, which is compatible with composition.

Canonical symmetric transformation $^{-1}$: $x\,\rho^{-1}\,y \iff y\,\rho\,x$.

$\mathfrak{M}(A \times A)$: collection of all partial one-one transformations on A.

$^{-1}$ and $\subset$ may be expressed in terms of composition in $\mathfrak{M}(A \times A)$:

$$\rho_2 = \rho_1^{-1} \iff \rho_1\rho_2\rho_1 = \rho_1 \text{ and } \rho_2\rho_1\rho_2 = \rho_2;$$
$$\rho_1 \subset \rho_2 \iff \exists\rho \text{ such that } \rho_1\rho\rho_1 = \rho_1,\ \rho_2\rho\rho_2 = \rho_2 \text{ and } \rho\rho_2\rho = \rho.$$

Generalised groups

'Обобщенные группы' ('Generalised groups'), *Dokl. Akad. nauk SSSR* 84(6) (1952), 1119–1122:

First page: modern definition of an inverse semigroup, here called a generalised group.

Theorem: Every symmetric semigroup of partial one-one transformations of a set forms a generalised group with respect to composition of partial transformations.

Theorem: Every generalised group may be represented as a generalised group of partial one-one transformations.

Further questions about binary relations

$\mathfrak{B}(A \times A)$ is a semigroup;
$\mathfrak{M}(A \times A)$ is an inverse semigroup;

but what about $\mathfrak{B}(A \times B)$, the collection of all binary relations between two distinct sets A and B?

or $\mathfrak{M}(A \times B)$, the collection of all injective partial mappings from A to B?

Addressed briefly in 'Тернарная алгебраическая операция в теории координатных структур' ('A ternary algebraic operation in the theory of coordinate structures'), *Dokl. Akad. nauk SSSR* 81(6) (1951), 981–984,

and then more fully in 'Теория обобщенных груд и обобщенных групп' ('Theory of generalised heaps and generalised groups'), Mat. sb. 32(74)(3) (1953), 545–632.

Ternary operations

Basic problem: we can't compose $\rho \in \mathfrak{B}(A \times B)$ with $\sigma \in \mathfrak{B}(A \times B)$ as we did before.

Easily overcome by the use of a ternary operation, specifically: for $\rho, \sigma, \tau \in \mathfrak{B}(A \times B)$:

$$[\rho\ \sigma\ \tau] = \rho \circ \sigma^{-1} \circ \tau,$$

where $\circ$ denotes the usual composition of binary relations, and $x\,\rho\,y \Leftrightarrow y\,\rho^{-1}\,x$.

Ternary precursors

Heinz Prüfer, 'Theorie der Abelschen Gruppen I: Grundeigenschaften', *Math. Z.* 20 (1924), 165–187:

introduced the ternary operation $AB^{-1}C$ onto an infinite Abelian group as a tool for studying its structure.

Reinhold Baer, 'Zur Einführung des Scharbegriffs', *J. reine angew. Math.* 163 (1929), 199–207:

extended Prüfer's ideas to the non-Abelian case.

José Isaac Corral, *Brigadas de sustituciones*, 2 vols., 1932, 1935: used ternary operations as a framework for studying transformations of a set.

Jeremiah Certaine, 'The ternary operation $(abc) = ab^{-1}c$ of a group', *Bull. Amer. Math. Soc.* 49(12) (1943), 869–877:

studied axiomatisations of the systems studied by Prüfer and Baer.

A little etymology

Baer used the term Schar (German: band, company, crowd, flock)

Translated into Russian by A. K. Sushkevich as груда (gruda: heap, pile)

Груда adopted by Wagner (hence also полугруда and обобщенная груда)

Eventually taken into English as heap (hence also semiheap and generalised heap)

B. M. Schein proposed the alternative groud

In French: amas (heap, pile)

Other terms used in English: flock, imperfect brigade, abstract coset, torsor, herd, principal homogeneous space, pregroup

Abstract ternary operations

Let K be a set. We define an abstract ternary operation $[\cdot\ \cdot\ \cdot] : K \times K \times K \to K$.

Call the operation pseudo-associative if

$$[[k_1\ k_2\ k_3]\ k_4\ k_5] = [k_1\ [k_4\ k_3\ k_2]\ k_5] = [k_1\ k_2\ [k_3\ k_4\ k_5]]\,.$$

In this case, call K a semiheap.

A semiheap forms a heap if

$$[k_1\ k_2\ k_2] = [k_2\ k_2\ k_1] = k_1.$$

$\mathfrak{B}(A \times B)$ forms a semiheap.

Generalised heaps

A semiheap K is a generalised heap if:

$$[[k\ k_1\ k_1]\ k_2\ k_2] = [[k\ k_2\ k_2]\ k_1\ k_1]\,,$$
$$[k_1\ k_1\ [k_2\ k_2\ k]] = [k_2\ k_2\ [k_1\ k_1\ k]]\,,$$
$$[k\ k\ k] = k.$$

Let $\mathfrak{K}(A \times B) \subseteq \mathfrak{B}(A \times B)$ be the collection of all partial one-one transformations from a set A to a set B.

$\mathfrak{K}(A \times B)$ forms a generalised heap.

Every abstract generalised heap may be embedded in some $\mathfrak{K}(A \times B)$.

Semigroups and semiheaps

Let S be a semigroup with involution: $(s')' = s, \quad (s_1 s_2)' = s_2' s_1'$.

Define a ternary relation $[s_1 \ s_2 \ s_3] = s_1 s_2' s_3$. S forms a semiheap under this operation.

Let K be a semiheap. $b \in K$ is a biunitary element if $\forall k \in K$

$$[k \ b \ b] = [b \ b \ k] = k.$$

For any biunitary element $b \in K$, define a binary operation and involution by

$$s_1 s_2 = [s_1 \ b \ s_2] \qquad \text{and} \qquad s' = [b \ s \ b].$$

Under these operations, K is a semigroup with involution.

Other types of semiheaps

semiheap*	$\longleftrightarrow$	semigroup
heap	$\longleftrightarrow$	group
generalised heap*	$\longleftrightarrow$	generalised group

*only in the presence of a biunitary element

In terms of binary relations

$$\text{Semiheap } \mathfrak{B}(A \times B) \quad \longleftrightarrow \quad \text{Semigroup } \mathfrak{B}(A \times A)$$

$$\text{Generalised heap } \mathfrak{K}(A \times B) \quad \longleftrightarrow \quad \text{Generalised group } \mathfrak{K}(A \times A)$$

Back to differential geometry

Let M be an n-dimensional differentiable manifold.

Such a manifold has a coordinate atlas A: a set of partial one-one transformations from M into $\mathbb{R}^n$.

Each $\kappa \in A$ represents a local system of coordinates: $\kappa(m) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are the coordinates of $m \in M$.

Again apply the following ternary operation to $\kappa, \lambda, \mu \in A$:

$$[\kappa \ \lambda \ \mu] = \kappa \circ \lambda^{-1} \circ \mu.$$

"… Wagner's work looks more relevant now than it has ever done."

Chapter 9
Generalised Heaps as Affine Structures

9.1 Introduction

The intention of this essay is not to give a blow by blow account of Wagner's paper (translated as Chapter 8) since, as Christopher Hollings states in Chapter 1, Wagner writes lucidly and so there is little that needs explaining. Instead, I shall show how subsequent developments in mathematics shed light on what he was doing and at the same time suggest further research. In reading this essay, it may help the reader to keep in mind the following analogy which is also mathematically precise: the relationship between generalised heaps and inverse semigroups is analogous to the relationship between affine spaces and vector spaces.

The full appreciation of Wagner's paper was hindered by no less than three obstacles: the Cold War acted as a barrier to the free exchange of information between East and West; Russian was, and still is, treated as being outside the canon of European languages [14]; and finally, if these were not enough, the fact that the paper deals primarily with ternary operations seems to put it beyond the pale. Algebraists tend to think in terms of binary and unary operations with operations of larger arity not usually being encountered on a day-to-day basis. There are exceptions, however. It was another Soviet mathematician, A.I. Mal'cev (1909–1967), who discovered that the general algebras which are congruence-permutable are precisely those which possess what is now known as a *Mal'cev term*. Such terms are, in particular, ternary operations. Thus groups are congruence-permutable having the operation $(x, y, z) \mapsto xy^{-1}z$ as their Mal'cev term [4]. This operation and its generalisations play an important rôle in Wagner's paper. That ternary operations have come to be more mainstream is, ironically enough, due to developments in geometry.

Our deepening understanding of geometry has been a major force in the development of mathematics and Wagner's paper should be viewed in this light.

C.D. Hollings, M.V. Lawson, *Wagner's Theory of Generalised Heaps*,
DOI 10.1007/978-3-319-63621-4_9