Idempotent bounded C–semigroups

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Abstract

A semigroup with zero is *idempotent bounded (IB)* if it is the 0-direct union of idempotent generated principal left ideals and the 0-direct union of idempotent generated principal right ideals. Notable examples are completely 0-simple semigroups and the wider class of primitive abundant semigroups. Significant to the structure of these semigroups is that they are all categorical at zero. In this paper we describe IB semigroups that are categorical at zero in terms of *double blocked Rees matrix semigroups*. This generalises Fountain's characterisation of primitive abundant semigroups via blocked Rees matrix semigroups [F], which in turn yields the Rees theorem for completely 0-simple semigroups.

1 Introduction

A semigroup with zero is *left (right) idempotent bounded*, abbreviated to *LIB* or *RIB*, if it is the 0-direct union of idempotent generated principal left (right) ideals and *idempotent bounded* or *IB* if it is both left and right idempotent bounded. A semigroup S with zero is *categorical at zero* or is a C-*semigroup* if for all $a, b, c \in S$, if $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$. This paper is the second of two studying C-semigroups. In the first, [FG], we concentrate on those that are LIB; here we deal with those that are IB. Examples of IB C-semigroups are plentious - most notable are the completely 0-simple semigroups, which are of course characterised as regular Rees matrix semigroups over groups. Our aim is to find a generalization of the Rees theorem which will enable us to describe arbitrary IB C-semigroups.

Theorem 5.2 of [FG] is a structure theorem for LIB C-semigroups in terms of what we call left blocked Rees matrix semigroups. The construction involved may loosely be thought of as a 'one-sided' version of that of Rees. Here we introduce a new concept, that of a double blocked Rees matrix semigroup. These are perhaps truer generalizations of Rees matrix semigroups, having a construction that has no left or right bias. They are the semigroups we need to describe IB C-semigroups; to show this we make use of Theorem 5.2 of [FG].

In Section 2 we gather together a number of preliminary definitions and results. In particular, we briefly recall the construction of left blocked Rees matrix semigroups. Section 3 concentrates on examples illustrating the wide range of behaviour IB C-semigroups can display.

Double blocked Rees matrix semigroups are described in Section 4. We give sufficient conditions for such a semigroup to be IB - by construction they are C-semigroups. A double blocked Rees matrix semigroup satisfying these conditions is called an IB-DB Rees matrix semigroup. Theorem 4.5 determines the conditions under which two IB-DB Rees matrix semigroups are isomorphic. In the following section we show that any IB C-semigroup is isomorphic to an IB-DB Rees matrix semigroup.

Finally in Section 6 we consider a number of special cases. In particular we show how the structure theorem for primitive abundant semigroups given in [F] can be easily deduced from Theorem 5.1. As pointed out in [F], a further specialisation yields the Rees theorem for completely 0-simple semigroups.

2 Preliminaries

Where possible we use standard notation and terminology, as found in [H]. In particular, for any semigroup S we denote by $E(S)$ the set of idempotents of S. Throughout we consider only semigroups with zero for our interest lies in developing structure theorems for C-semigroups; given any semigroup S without zero, then S^0 is clearly a C-semigroup and the structure of S can be deduced from that of S^0. For any subset U of a semigroup S the set of non-zero elements of U is denoted by U^*.

Underlying our approach are the relations $\sim_r$ and $\sim_\ell$, defined on a semigroup S as follows. For any $a, b \in S$,

$$a \sim_r b \text{ if and only if } a = b = 0 \text{ or } aS^1 \cap bS^1 \neq 0$$

and

$$a \sim_\ell b \text{ if and only if } a = b = 0 \text{ or } S^1 a \cap S^1 b \neq 0.$$

The relations $\sim_r$ and $\sim_\ell$ are more usually denoted by ρ and λ respectively. However, for the purposes of this paper it is convenient to reserve the symbols ρ and λ for other uses. Clearly $\sim_r$ and $\sim_\ell$ are reflexive and symmetric, but in general they are not transitive. Denoting the transitive closure of a relation θ by θ^t, we put $\sim_r^t = \mathcal{R}'$ and $\sim_\ell^t = \mathcal{L}'$.

Let S be a RIB semigroup. By definition, $S = \bigcup^0 \{e_i S : i \in I, e_i \in E(S)\}$ where $\bigcup^0$ denotes 0-direct union. It is easy to see that if $a, b \in S^*$ then $a\mathcal{R}'b$ if and only if $a, b \in e_i S$ for some $i \in I$. Thus the $\mathcal{R}'$-classes of S are $\{0\}$ and $\{(e_i S)^* : i \in I\}$. The non-zero $\mathcal{R}'$-class containing e_i, that is, $(e_i S)^*$, will be denoted by R_i'. If $e \in E(S)$ then e is an $\mathcal{R}'$*-generator* if $e\mathcal{R}e_i$ for some $i \in I$. Dually, a LIB semigroup S may be written as $S = \bigcup^0 \{S f_\lambda : \lambda \in \Lambda, f_\lambda \in E(S)\}$ and corresponding comments apply. In particular, $e \in E(S)$ is an $\mathcal{L}'$*-generator* if $e\mathcal{L}f_\lambda$ for some $\lambda \in \Lambda$. If S is an IB C-semigroup, then as remarked after Lemma 2.3 of [FG], for any $i \in I$ and $\lambda \in \Lambda$, either $L_\lambda' R_i' = 0$ or $L_\lambda' R_i' \subseteq S^*$.

A non-empty set M is a *partial semigroup* if there is a partial binary operation on M such that for all $a, b, c \in M$, $(ab)c$ is defined if and only if $a(bc)$ is, and if $(ab)c$ is defined, then $(ab)c = a(bc)$. On a partial semigroup M the relation $\sim_r$ is defined by

$$a \sim_r b \text{ if and only if } aM^1 \cap bM^1 \neq \emptyset$$

for $a, b \in M$. As before we put $\mathcal{R}' = \sim_r^t$. The relations $\sim_\ell$ and $\mathcal{L}'$ are defined dually.

We end this section with a description of left blocked Rees matrix semigroups, which were introduced in [FG].

Consider the following data:-three non-empty sets I, Λ, Γ, a partition $P(\Lambda) = \{\Lambda_\gamma; \gamma \in \Gamma\}$ of Λ indexed by Γ, a function $s : \Gamma \to I$ and a set $\{M_{i\alpha}; i \in I, \alpha \in \Gamma\}$ of pairwise-disjoint sets.

Let $M = \bigcup\{M_{i\alpha} : i \in I, \alpha \in \Gamma\}$ and for $i \in I$ put $U_i = \bigcup\{M_{i\alpha} : \alpha \in \Gamma\}$. We suppose there is a partial binary operation on M such that xy is defined if and only if $x \in M_{i\alpha}$ and $y \in M_{s(\alpha)\beta}$ for some i, α, β and then, if defined, $xy \in M_{i\beta}$. Further, we assume that under this multiplication M is a partial semigroup.

Let 0 be a symbol not in M and let P be a $\Lambda \times I$ matrix such that for any $(\lambda, i) \in \Lambda \times I$, $p_{\lambda i} = 0$ or $p_{\lambda i}$ is a function $p_{\lambda i} : U_i \to U_{s(\alpha)}$, where $\lambda \in \Lambda_\alpha$. In the latter case, $p_{\lambda i}|_{M_{i\beta}} : M_{i\beta} \to M_{s(\alpha)\beta}$ for all $\beta \in \Gamma$ where $M_{i\beta} \neq \emptyset$; also, for any $\beta, \gamma \in \Gamma$, $x \in M_{i\beta}$ and $y \in M_{s(\beta)\gamma}$, we have $p_{\lambda i}(x)y = p_{\lambda i}(xy)$.

Let $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$

$$= \{(i, u, \lambda) : i \in I, \lambda \in \Lambda, u \in M_{i\alpha} \text{ where } \lambda \in \Lambda_\alpha\} \cup \{0\}.$$

Define a product on $\mathcal{LM}^0$ by

$$(i, u, \lambda)(j, v, \mu) = \begin{cases} (i, up_{\lambda j}(v), \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

$$0x = x0 = 0 \text{ for all } x \in \mathcal{LM}^0.$$

Proposition 4.1 of [FG] gives that $\mathcal{LM}^0$ is a C-semigroup. We say that $\mathcal{LM}^0$ is a *left blocked Rees matrix semigroup*, which is abbreviated to *LB Rees matrix semigroup*. We define here four conditions on such semigroups.

LIB(i) For $\alpha \in \Gamma, M_{s(\alpha)\alpha}$ contains an idempotent g_α such that $xg_\alpha = x$ for all $i \in I$ and for all $x \in M_{i\alpha}$.

LIB(ii) For each $\alpha \in \Gamma$ and $\lambda \in \Lambda_\alpha$ there exists $i \in I$ with $p_{\lambda i} \neq 0$ and $u \in M_{i\alpha}$ with $p_{\lambda i}(u) = g_\alpha$.

LIB(iii) The $\mathcal{R}'$-classes of M are $\{U_i : i \in I\}$.

LIB(iv) Given $\alpha, \beta \in \Gamma$ and $a \in M_{s(\alpha)\beta}, b \in M_{s(\beta)\alpha}$, where $aba = a, g_\alpha ab = g_\alpha$ and $g_\beta ba = g_\beta$, then $\alpha = \beta$.

Theorem 2.1 *[FG] If $\mathcal{LM}^0$ is a LB Rees matrix semigroup satisfying LIB(i),(ii),(iii) and (iv), then $\mathcal{LM}^0$ is a LIB C-semigroup. Conversely, any LIB C-semigroup is isomorphic to a LB Rees matrix semigroup satisfying LIB(i),(ii),(iii) and (iv).*

In view of this there is no danger of ambiguity in defining a *LIB-LB Rees matrix semigroup* to be a LB Rees matrix semigroup satisfying LIB(i), (ii), (iii) and (iv). For convenience we also quote here the following result from [FG].

Proposition 2.2 *[FG] Let $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$ be a LIB-LB Rees matrix semigroup. Then*
(i) an element (i, a, λ) is idempotent if and only if $p_{\lambda i} \neq 0$ and $ap_{\lambda i}(a) = a$;
(ii) the elements $(i, a, \lambda), (j, b, \mu)$ are $\mathcal{R}'(\mathcal{L}')$-related if and only if $i = j (\lambda = \mu)$;
(iii) an element (i, u, λ) where $\lambda \in \Lambda_\alpha$ is an $\mathcal{L}'$-generator if and only if $g_\alpha p_{\lambda i}(u) = g_\alpha$;
(iv) $\mathcal{L}'$-generators $(i, a, \lambda), (j, b, \mu)$ where $\lambda \in \Lambda_\alpha, \mu \in \Lambda_\beta$ are $\mathcal{D}$-related if and only if $\alpha = \beta$.

3 Examples

We begin by remarking that any monoid M with zero is IB in a trivial way so that M is an IB C-semigroup exactly when it is categorical at zero. Of course, our theory is not designed to tell us anything about such examples.

It is clear that a semilattice with zero which is a 0-direct union of finite chains is an IB C-semigroup. In general, however, the structure of IB semilattices is rather complicated. Nevertheless, it is perhaps worth pointing out that IB Clifford semigroups can be described using such semilattices. Let Y be a semilattice with zero ζ and let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. For S to have a zero we must assume that G_ζ is trivial. We leave the easy proof of the following result to the reader.

Proposition 3.1 *The Clifford semigroup S is an IB semigroup (respectively IB C-semigroup) if and only if Y is an IB semilattice (respectively IB C-semilattice).*

Now for each element i of an index set I let M_i be a monoid with zero but without zero divisors. Let S be the semigroup formed by taking the 0-direct union of the M_i and let

$$T = \{\alpha \in \text{ End } S : 0\alpha = 0 \text{ and } S\alpha \subseteq M_i \text{ for some } i\}.$$

Let $\alpha \in T$ and suppose $S\alpha \subseteq M_i$ and $S\alpha \neq 0$. Then $m\alpha \neq 0$ for an element m in some M_j. For $k \in I, k \neq j$ we must have $M_k\alpha = 0$ for if $m' \in M_k$, then

$(m'\alpha)(m\alpha) = (m'm)\alpha = 0\alpha = 0$ and since M_i has no zero divisors we have $m'\alpha = 0$.

For each i we define $\epsilon_i : S \to S$ by putting $x\epsilon_i = x$ if x is in M_i and $x\epsilon_i = 0$ otherwise. Clearly, ϵ_i is an idempotent member of T. It is easy to see that T is the 0-direct union of the left ideals $T\epsilon_i$ so that T is LIB. Further, since for a non-zero element α of T there is a unique j in I for which $M_j\alpha \neq 0$, we have $\epsilon_j\alpha = \alpha$ and hence $T = \bigcup_{j\in I} \epsilon_j T$. If β is in both $\epsilon_j T$ and $\epsilon_k T$ for some j, k with $j \neq k$, then clearly $\beta = 0$ so that T is RIB. Thus T is an example of an IB semigroup.

For appropriate choices of the monoids M_i we observe that T is also a C-semigroup. For example, when each M_i is a 0-simple monoid suppose that α, β, γ are members of T such that $\alpha\beta\gamma = 0$ and $\alpha\beta \neq 0$. Then Im $\alpha\beta \subseteq$ Im $\beta \subseteq M_i$ for some i and (Im $\alpha\beta)\gamma = 0$. Now

$$I = \{m \in M_i : m\gamma = 0\}$$

is an ideal of M_i which is non-zero since Im $\alpha\beta \subseteq I$. But M_i is 0-simple so that $I = M_i$ and hence $\beta\gamma = 0$. Thus T is a C-semigroup in this case.

We now return to a class of examples introduced in [FG]. Let S be a monoid, I be a non-empty set and for each $i \in I$ let A_i be an S-system. Let 0 be a symbol not in A_i and put $B_i = \{0\} \cup A_i$. We make B_i into a centred S-system by defining $s0 = 0$ for all $s \in S$. In the category of centred S-systems we form the coproduct $B = \bigcup^0 \{B_i : i \in I\}$ and denote the monoid of 0-preserving S-endomorphisms of B by $\mathrm{End}^0_S B$. We now define subsemigroups T and V of $\mathrm{End}^0_S B$ as follows:

$$T = \{\alpha \in \ \mathrm{End}^0_S B : B\alpha \subseteq B_i \text{ for some } i\},$$

$$V = \{\alpha \in T : \text{ for each } j \in I, \text{ either } A_j\alpha = 0 \text{ or } A_j\alpha \cap 0 = \emptyset\}.$$

In [FG] it was assumed that each A_i has at least two elements but this played no role in the proof that T and V are always LIB semigroups and that V is always a C-semigroup. We gave examples where T and V are not RIB and where T is not a C-semigroup. We now provide examples where $T = V$ is an IB C-semigroup.

Let $S = \{1, z\}$ where $z^2 = z$; let $I = \{1, 2\}$ with $A_1 = \{e_0, e_1\}$ where $ze_i = e_1$ and $A_2 = \{f\}$. Forming B as above, it is easy to see that $T = V$. Any member of T is determined by its action on e_0 and f so that we can specify all the elements of T by the following table:

$\cdot$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	α	ζ	η_1	β	η_2
e_0	e_0	e_0	e_1	e_1	0	0	f	f	0
f	e_1	0	e_1	0	e_1	0	f	0	f

It is then easily verified that T is the 0-direct union

$$T = \epsilon_1 T \cup \epsilon_2 T \cup \eta_2 T$$

so that T is RIB as required.

Now let G be a group and H a non-trivial subgroup. Let $I = \{1, 2\}$ and take $A_1 = G, A_2 = G/H$ with G acting by multiplication on the left. Form B as above. Again, it is clear that $T = V$. Any member of T is specified by its effect on $1 \in A_1$ and $H \in A_2$. The idempotents of T are ϵ, η, α_x for $x \in G$ and the zero map where $1\epsilon = 1, H\epsilon = 0; 1\eta = 0, H\eta = H; 1\alpha_x = xH, H\alpha_x = H$. Of course $\alpha_x = \alpha_y$ if and only if $x^{-1}y \in H$. It is now straightforward to verify that

$$T = \epsilon T \cup \eta T \cup \bigcup_{x \in C} \alpha_x T,$$

where C is a set of coset representatives for H, and that this is a 0-direct union. Thus T is RIB as required.

4 Double blocked Rees matrix semigroups

In this section we show how to construct double blocked Rees matrix semigroups. These form a class of C-semigroups built up from specific ingredients and are a natural generalisation of blocked Rees matrix semigroups, which were introduced in [F]. These latter semigroups themselves are a generalisation of Rees matrix semigroups. We then give sufficient conditions for a double blocked Rees matrix semigroup to be idempotent bounded. One of these conditions is not strictly necessary, but enables us to locate easily the $\mathcal{R}'$-generators and the $\mathcal{L}'$-generators.

Let I, Λ and Γ denote non-empty sets where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j, 1 \leq i, j \leq 3$. Suppose that $\Gamma_1 \cup \Gamma_2$ indexes a partition of Λ into subsets Λ_α and $\Gamma_1 \cup \Gamma_3$ indexes a partition of I into subsets I_β. Let $M = \bigcup\{M_{\alpha\beta} : \alpha, \beta \in \Gamma\}$ be a partial semigroup such that for $x, y \in M$, xy is defined if and only if $x \in M_{\alpha\beta}, y \in M_{\beta\gamma}$ for some $\alpha, \beta, \gamma \in \Gamma$ and then in this case, $xy \in M_{\alpha\gamma}$. Suppose also that for all $\alpha \in \Gamma, M_{\alpha\alpha} = T_\alpha$ is a monoid with identity g_α and for all $\alpha, \beta \in \Gamma, M_{\alpha\beta} = \emptyset$ or is a (T_α, T_β)-bisystem, that is, T_α acts on $M_{\alpha\beta}$ on the left, T_β acts on $M_{\alpha\beta}$ on the right and $(tm)t' = t(mt')$ for all $t \in T_\alpha, m \in M_{\alpha\beta}, t' \in T_\beta$. Let 0 be a new symbol and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix over $M \cup \{0\}$ where for $\lambda \in \Lambda_\alpha, i \in I_\beta, p_{\lambda i} \in M_{\alpha\beta}$.

Put $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P) =$

$$\{(i, m, \lambda) : i \in I_\alpha, \lambda \in \Lambda_\beta, \alpha \in \Gamma_1 \cup \Gamma_3, \beta \in \Gamma_1 \cup \Gamma_2, m \in M_{\alpha\beta}\} \cup \{0\}.$$

A multiplication on $\mathcal{DM}^0$ is defined by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

$$0x = x0 = 0 \text{ for all } x \in \mathcal{DM}^0.$$

The proof of the following straightforward lemma is left to the reader.

Lemma 4.1 *Constructed as above, $\mathcal{DM}^0$ is a semigroup which is categorical at zero.*

We say that $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$ is a *double blocked Rees matrix semigroup*, or a *DB Rees matrix semigroup.*

Lemma 4.2 *Let $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$ be a DB Rees matrix semigroup. Then $\mathcal{DM}^0$ is idempotent bounded and such that I indexes the non-zero $\mathcal{R}'$-classes and Λ indexes the non-zero $\mathcal{L}'$-classes if and only if $\mathcal{DM}^0$ satisfies conditions (RIB) and (LIB).*

(RIB) For all $\alpha \in \Gamma_1 \cup \Gamma_3$ and $i \in I_\alpha$, there exists $\lambda \in \Lambda$, say $\lambda \in \Lambda_{t(\alpha)}$, and $u \in M_{\alpha t(\alpha)}$ with $up_{\lambda i}a = a$ for all $a \in M_{\alpha\gamma}, \gamma \in \Gamma_1 \cup \Gamma_2$.

(LIB) For all $\alpha \in \Gamma_1 \cup \Gamma_2$ and $\lambda \in \Lambda_\alpha$, there exists $i \in I$, say $i \in I_{s(\alpha)}$, and $v \in M_{s(\alpha)\alpha}$ with $ap_{\lambda i}v = a$ for all $a \in M_{\gamma\alpha}, \gamma \in \Gamma_1 \cup \Gamma_3$.

Proof Suppose that $\mathcal{DM}^0$ is right idempotent bounded and I indexes the non-zero $\mathcal{R}'$-classes . Let $i \in I_\alpha$ where $\alpha \in \Gamma_1 \cup \Gamma_3$. Then there is an $\mathcal{R}'$-generator (i, u, λ) where $\lambda \in \Lambda_{t(\alpha)}$ say and so $u \in M_{\alpha t(\alpha)}$. If $\gamma \in \Gamma_1 \cup \Gamma_2$ and $a \in M_{\alpha\gamma}$, then picking $\mu \in \Lambda_\gamma$ we have $(i, a, \mu) \in \mathcal{DM}^0$ and $(i, u, \lambda)(i, a, \mu) = (i, a, \mu)$. This gives that $up_{\lambda i}a = a$. Thus (RIB) holds and dually, if $\mathcal{DM}^0$ is left idempotent bounded and Λ indexes the non-zero $\mathcal{L}'$-classes, then (LIB) holds.

Conversely, suppose that $\mathcal{DM}^0$ satisfies (RIB) and (LIB). Let $i \in I$, say $i \in I_\alpha$. By (RIB) there exists some $t(\alpha) \in \Gamma_1 \cup \Gamma_2$ and $\lambda \in \Lambda_{t(\alpha)}, u \in M_{\alpha t(\alpha)}$ with $up_{\lambda i}a = a$ for all $a \in M_{\alpha\gamma}, \gamma \in \Gamma_1 \cup \Gamma_2$. Then $(i, u, \lambda) \in \mathcal{DM}^0$ and if $(i, a, \mu) \in \mathcal{DM}^0$, we have that $(i, u, \lambda)(i, a, \mu) = (i, a, \mu)$. It is then clear that

$$((i, u, \lambda)\mathcal{DM}^0)^* = \{(i, a, \mu) : \mu \in \Lambda_\gamma, \gamma \in \Gamma_1 \cup \Gamma_2, a \in M_{\alpha\gamma}\}$$

is an $\mathcal{R}'$-class of $\mathcal{DM}^0$ and (i, u, λ) is an $\mathcal{R}'$-generator. The dual argument completes the proof of the lemma.

If $\mathcal{DM}^0$ satisfies (RIB) then in certain cases we can make a stronger assertion.

Corollary 4.3 *Let $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$ be a DB Rees matrix semigroup satisfying (RIB). Then for all $\alpha \in \Gamma_1$ and $i \in I_\alpha$ there exists $\lambda \in \Lambda$, say $\lambda \in \Lambda_{t(\alpha)}$ and $u \in M_{\alpha t(\alpha)}$ with $up_{\lambda i} = g_\alpha$.*

Proof Let $t(\alpha), \lambda$ and u be as in condition (RIB). Now $g_\alpha \in M_{\alpha\alpha}$ and $\alpha \in \Gamma_1 \cup \Gamma_2$ so that $up_{\lambda i} g_\alpha = g_\alpha$. But $up_{\lambda i} \in M_{\alpha\alpha}$ and hence $g_\alpha = up_{\lambda i} g_\alpha = up_{\lambda i}$.

We now give the promised condition that controls the $\mathcal{D}$-classes of the $\mathcal{R}'$-generators and the $\mathcal{L}'$-generators.

$(\mathrm{D})_{II}$ For all $\alpha, \beta \in \Gamma_1 \cup \Gamma_3$ and $a \in M_{\alpha\beta}, b \in M_{\beta\alpha}$, if $abm = m$ and $ban = n$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$ and $m \in M_{\alpha\gamma}, n \in M_{\beta\gamma}$, then $\alpha = \beta$.

$(\mathrm{D})_{I\Lambda}$ For all $\alpha \in \Gamma_1 \cup \Gamma_3, \gamma \in \Gamma_1 \cup \Gamma_2$ amd $c \in M_{\alpha\gamma}, d \in M_{\gamma\alpha}$, if $cdm = m$ and $ndc = n$ for all $\xi \in \Gamma_1 \cup \Gamma_2, \zeta \in \Gamma_1 \cup \Gamma_3$ and $m \in M_{\alpha\xi}, n \in M_{\zeta\gamma}$, then $\alpha = \gamma$.

$(\mathrm{D})_{\Lambda\Lambda}$ For all $\gamma, \delta \in \Gamma_1 \cup \Gamma_2$ and $c \in M_{\gamma\delta}, d \in M_{\delta\gamma}$, if $mcd = m$ and $ndc = n$ for all $\zeta \in \Gamma_1 \cup \Gamma_3$ and $m \in M_{\zeta\gamma}, n \in M_{\zeta\delta}$, then $\gamma = \delta$.

Condition (D) is the conjunction of these three conditions.

We can now prove the main result of this section.

Theorem 4.4 *Let $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$ be a DB Rees matrix semigroup satisfying conditions (RIB), (LIB) and (D). Then*
(i) if $i \in I_\alpha, \lambda \in \Lambda_\beta$ and $v \in M_{\alpha\beta}$ then (i, v, λ) is an $\mathcal{R}'$-generator if and only if $vp_{\lambda i} a = a$ for all $a \in M_{\alpha\gamma}, \gamma \in \Gamma_1 \cup \Gamma_2$;
(ii) if $i \in I_\alpha, \lambda \in \Lambda_\beta$ and $v \in M_{\alpha\beta}$ then (i, v, λ) is an $\mathcal{L}'$-generator if and only if $ap_{\lambda i} v = a$ for all $a \in M_{\gamma\alpha}, \gamma \in \Gamma_1 \cup \Gamma_3$;
(iii) if e_i, e_j are $\mathcal{R}'$-generators and f_λ, f_μ are $\mathcal{L}'$-generators where $i \in I_\alpha, j \in I_\beta, \lambda \in \Lambda_\gamma$ and $\mu \in \Lambda_\delta$, then

$$e_i \mathcal{D} f_\lambda \text{ if and only if } \alpha = \gamma,$$

$$e_i \mathcal{D} e_j \text{ if and only if } \alpha = \beta,$$

$$f_\lambda \mathcal{D} f_\mu \text{ if and only if } \gamma = \delta.$$

Proof It is clear that (i) and (ii) hold.

Let e_i, e_j be $\mathcal{R}'$-generators where $i \in I_\alpha$ and $j \in I_\beta$. Suppose first that $e_i \mathcal{D} e_j$. By Proposition II 3.6 of [H] there is an element $c = (i, a, \nu)$ with inverse $c' = (j, b, \rho)$ such that $e_i = cc' = (i, ap_{\nu j}b, \rho)$ and $e_j = c'c = (j, bp_{\rho i}a, \nu)$. Let $\overline{a} = ap_{\nu j}$ and $\overline{b} = bp_{\rho i}$ so that $\overline{a} \in M_{\alpha\beta}$ and $\overline{b} \in M_{\beta\alpha}$. Note that $\alpha, \beta \in \Gamma_1 \cup \Gamma_3$. Let $\xi \in \Gamma_1 \cup \Gamma_2$ and $m \in M_{\alpha\xi}$. Then $\overline{a}\overline{b}m = (ap_{\nu j}bp_{\rho i})m = m$ since e_i is an $\mathcal{R}'$-generator. Similarly, $\overline{b}\overline{a}n = n$ for all $n \in M_{\beta\xi}$ so that $\alpha = \beta$ by $(\mathrm{D})_{II}$.

Conversely, suppose that $\alpha = \beta$. Writing $e_i = (i, x, \rho)$ and $e_j = (j, y, \nu)$, we know that $k = (j, x, \rho)$ and $\ell = (i, y, \nu)$ are in $\mathcal{DM}^0$. Now

$$k\ell = (j, x, \rho)(i, y, \nu) = (j, xp_{\rho i}y, \nu) = (j, y, \nu) = e_j$$

$$\ell k = (i, y, \nu)(j, x, \rho) = (i, yp_{\nu j}x, \rho) = (i, x, \rho) = e_i$$

and

$$ke_i = (j, x, \rho)(i, x, \rho) = (j, x, \rho) = k$$

so that $e_j \mathcal{R} k \mathcal{L} e_i$ as required. The dual argument using $(\mathrm{D})_{\Lambda\Lambda}$ shows that if f_λ, f_μ are $\mathcal{L}'$-generators where $\lambda \in \Lambda_\gamma, \mu \in \Lambda_\delta$, then $f_\lambda \mathcal{D} f_\mu$ if and only if $\gamma = \delta$.

Consider now an $\mathcal{R}'$-generator e_i and an $\mathcal{L}'$-generator f_λ where $i \in I_\alpha$ and $\lambda \in \Lambda_\gamma$. If $e_i \mathcal{D} f_\lambda$ then there is an element $d = (i, e, \lambda)$ with inverse $d' = (k, f, \iota)$ such that

$$e_i = dd' = (i, ep_{\lambda k}f, \iota)$$

and

$$f_\lambda = d'd = (k, fp_{\iota i}e, \lambda).$$

Note that $e \in M_{\alpha\gamma}$ and $p_{\lambda k}fp_{\iota i} \in M_{\gamma\alpha}$. Put $\overline{c} = e$ and $\overline{d} = p_{\lambda k}fp_{\iota i}$,so that $\overline{c}\overline{d} \in T_\alpha$ and $\overline{d}\overline{c} \in T_\gamma$.

Certainly $\alpha \in \Gamma_1 \cup \Gamma_3$ and if $\xi \in \Gamma_1 \cup \Gamma_2$ and $m \in M_{\alpha\xi}$ then $\overline{c}\overline{d}m = ep_{\lambda k}fp_{\iota i}m = m$ since e_i is an $\mathcal{R}'$-generator. Further, $\gamma \in \Gamma_1 \cup \Gamma_2$ and if $\zeta \in \Gamma_1 \cup \Gamma_3$ and $n \in M_{\zeta\gamma}$, then $n\overline{d}\overline{c} = np_{\lambda k}fp_{\iota i}e = n$, since f_λ is an $\mathcal{L}'$-generator. Thus $(\mathrm{D})_{I\Lambda}$ gives that $\alpha = \gamma$.

Conversely, if $\alpha = \gamma$, then writing $e_i = (i, z, \iota)$ and $f_\lambda = (k, t, \lambda)$ we know that $(i, g_\alpha, \lambda) \in \mathcal{DM}^0$ so that $zp_{\iota i} = g_\alpha$ and $p_{\lambda k}t = g_\alpha$. Also, $(k, tz, \iota) \in \mathcal{DM}^0$,

$$(k, tz, \iota)(i, g_\alpha, \lambda) = (k, tzp_{\iota i}g_\alpha, \lambda) = (k, t, \lambda) = f_\lambda$$

and

$$(i, g_\alpha, \lambda)(k, tz, \iota) = (i, g_\alpha p_{\lambda k}tz, \iota) = (i, z, \iota) = e_i$$

so that $f_\lambda \mathcal{L}(i, g_\alpha, \lambda) \mathcal{R} e_i$ as required.

A DB Rees matrix semigroup satisfying (RIB),(LIB) and (D) will be referred to as an *IB-DB Rees matrix semigroup*. We show in the next section that any IB C-semigroup is isomorphic to an IB-DB Rees matrix semigroup. First we determine the conditions under which two IB-DB Rees matrix semigroups are isomorphic.

Proposition 4.5 *Let $\mathcal{DM}_i^0 = \mathcal{DM}_i^0(M^i; I^i, \Lambda^i, \Gamma^i; P^i), i = 1, 2$ be IB-DB Rees matrix semigroups. Then $\mathcal{DM}_1^0 \cong \mathcal{DM}_2^0$ if and only if there are bijections $\phi : I^1 \to I^2, \psi : \Lambda^1 \to \Lambda^2, \rho : \Gamma^1 \to \Gamma^2$ such that*
(i) $\rho|_{\Gamma_i^1} : \Gamma_i^1 \to \Gamma_i^2$ is a bijection, $i = 1, 2, 3$;
(ii) if $\alpha \in \Gamma_1^1 \cup \Gamma_3^1$ and $i \in I_\alpha^1$, then $\phi(i) \in I^2_{\rho(\alpha)}$;
(iii) if $\alpha \in \Gamma_1^1 \cup \Gamma_2^1$ and $\lambda \in \Lambda_\alpha^1$, then $\psi(\lambda) \in \Lambda^2_{\rho(\alpha)}$;
(iv) for any $\alpha \in \Gamma_1^1 \cup \Gamma_3^1, \beta \in \Gamma_1^1 \cup \Gamma_2^1, M^1_{\alpha\beta} \neq \emptyset$ if and only if $M^2_{\rho(\alpha)\rho(\beta)} \neq \emptyset$, and if $M_{\alpha\beta} \neq \emptyset$ then for all $i \in I_\alpha, \lambda \in \Lambda_\beta$ there is a bijection $\nu_{i\lambda} : M^1_{\alpha\beta} \to M^2_{\rho(\alpha)\rho(\beta)}$;
(v) for all $i \in I^1, \lambda \in \Lambda^1, p^1_{\lambda i} = 0$ if and only if $p^2_{\psi(\lambda)\phi(i)} = 0$;
(vi) for all $i, j \in I^1$ with $i \in I^1_\alpha, j \in I^1_\beta$ and $\lambda, \mu \in \Lambda^1$ with $\lambda \in \Lambda^1_\gamma, \mu \in \Lambda^1_\delta$ and $p^1_{\lambda j} \neq 0$,

$$\nu_{i\mu}(a p^1_{\lambda j} b) = \nu_{i\lambda}(a) p^2_{\psi(\lambda)\phi(j)} \nu_{j\mu}(b)$$

for all $a \in M_{\alpha\gamma}, b \in M_{\beta\delta}$.

Proof Suppose first that $\theta : \mathcal{DM}_1^0 \to \mathcal{DM}_2^0$ is an isomorphism. Clearly θ induces a bijection between the $\mathcal{L}'$-classes of $\mathcal{DM}_1^0$ and $\mathcal{DM}_2^0$ so that by Lemma 4.2, there is a bijection $\psi : \Lambda^1 \to \Lambda^2$. Dually, there is a bijection $\phi : I^1 \to I^2$.

Define $\rho : \Gamma^1 \to \Gamma^2$ by $\rho(\alpha) = \overline{\alpha}$ where $\alpha \in \Gamma_1^1 \cup \Gamma_2^1$ and $\psi(\lambda) \in \Lambda^2_{\overline{\alpha}}$ for some $\lambda \in \Lambda^1_\alpha$ and $\rho(\beta) = \overline{\beta}$ where $\beta \in \Gamma_3^1$ and $\phi(i) \in I^2_{\overline{\beta}}$ for some $i \in I^1_\beta$.

To see that ρ is well-defined, suppose that $\alpha \in \Gamma_1^1 \cup \Gamma_2^1$ and $\lambda, \mu \in \Lambda^1_\alpha$. By Theorem 4.4 the $\mathcal{L}'$-generators f_λ and f_μ are $\mathcal{D}$-related in $\mathcal{DM}_1^0$, so that their images, $f_{\psi(\lambda)}$ and $f_{\psi(\mu)}$ are $\mathcal{D}$-related $\mathcal{L}'$-generators in $\mathcal{DM}_2^0$. This gives that $\psi(\lambda), \psi(\mu) \in \Lambda^2_{\overline{\alpha}}$ for some $\overline{\alpha} \in \Gamma_1^2 \cup \Gamma_2^2$. Similarly, if $\beta \in \Gamma_3^1$ then $\rho(\beta)$ can have only one value.

Let $\alpha, \beta \in \Gamma_1^1 \cup \Gamma_2^1$ and suppose that $\rho(\alpha) = \rho(\beta)$. By definition there are elements $\lambda \in \Lambda^1_\alpha$ and $\mu \in \Lambda^1_\beta$ with $\psi(\lambda), \psi(\mu) \in \Lambda^2_{\rho(\alpha)}$. Thus the $\mathcal{L}'$-generators $f_{\psi(\lambda)}, f_{\psi(\mu)}$ are $\mathcal{D}$-related and as $f_{\psi(\lambda)} = \theta(f_\lambda), f_{\psi(\mu)} = \theta(f_\mu)$, we have that f_λ

and f_μ are $\mathcal{D}$-related $\mathcal{L}'$-generators so that $\alpha = \beta$. Similarly, if $\alpha, \beta \in \Gamma_3^1$ and $\rho(\alpha) = \rho(\beta)$, then $\alpha = \beta$. Now if $\alpha \in \Gamma_1^1 \cup \Gamma_2^1$ and $\beta \in \Gamma_3^1$ and $\rho(\alpha) = \rho(\beta)$, then there are elements $\lambda \in \Lambda_\alpha^1, i \in I_\beta^1$ with $\psi(\lambda) \in \Lambda_{\rho(\alpha)}^2$ and $\phi(i) \in I_{\rho(\alpha)}^2$. Choosing an $\mathcal{L}'$-generator $f_{\psi(\lambda)}$ and an $\mathcal{R}'$-generator $e_{\phi(i)}$, Theorem 4.4 gives that $f_{\psi(\lambda)} \mathcal{D} e_{\phi(i)}$ and it follows there is an $\mathcal{L}'$-generator f_λ and an $\mathcal{R}'$-generator e_i with $f_\lambda \mathcal{D} e_i$ in $\mathcal{DM}_1^0$. Thus $\alpha = \beta$ and we have shown that ρ is one-one. It is easy to see that ρ is onto and so ρ is a bijection. Further, an argument similar to the above shows that (i),(ii) and (iii) hold.

To see that (iv) holds, let $\alpha \in \Gamma_1^1 \cup \Gamma_3^1, \beta \in \Gamma_1^1 \cup \Gamma_2^1$ and suppose that $a \in M_{\alpha\beta}^1$. Then for any $i \in I_\alpha^1, \lambda \in \Lambda_\beta^1, (i, a, \lambda) \in \mathcal{DM}_1^0$ so that $\theta((i, a, \lambda)) = (\phi(i), \nu_{i\lambda}(a), \psi(\lambda)) \in \mathcal{DM}_2^0$ where $\phi(i) \in I_{\rho(\alpha)}^2, \psi(\lambda) \in \Lambda_{\rho(\beta)}^2$ so that $\nu_{i\lambda}(a) \in M_{\rho(\alpha)\rho(\beta)}^2$. Thus $M_{\rho(\alpha)\rho(\beta)}^2 \neq \emptyset$; moreover, it is clear that $\nu_{i\lambda} : M_{\alpha\beta}^1 \to M_{\rho(\alpha)\rho(\beta)}^2$ is a bijection. Similarly, if $M_{\rho(\alpha)\rho(\beta)}^2 \neq \emptyset$, then $M_{\alpha\beta}^1 \neq \emptyset$.

It is easy to see that (v) holds. Finally, the truth of (vi) is established by using the fact that θ is a homomorphism.

Conversely, assume that bijections $\phi : I^1 \to I^2, \psi : \Lambda^1 \to \Lambda^2, \rho : \Gamma^1 \to \Gamma^2$ exist satisfying conditions (i) to (vi). Define $\theta : \mathcal{DM}_1^0 \to \mathcal{DM}_2^0$ by $\theta((i, a, \lambda)) = (\phi(i), \nu_{i\lambda}(a), \psi(\lambda))$. Then it is easy to see that θ is properly defined and is an isomorphism.

5 Idempotent bounded C-semigroups

The aim of this section is to show that any idempotent bounded C-semigroup is isomorphic to an IB-DB Rees matrix semigroup. Our major tool is Theorem 2.1, which gives a structure theorem for LIB C-semigroups in terms of LIB-LB Rees matrix semigroups.

Throughout this section S denotes a fixed IB C-semigroup.

Theorem 5.1 *Every IB-DB Rees matrix semigroup is an IB C-semigroup. Conversely, any IB C-semigroup is isomorphic to an IB-DB Rees matrix semigroup.*

Proof It remains only to show the converse; the direct part holding by Lemma 4.2.

Let S be a fixed IB C-semigroup. Certainly S is LIB and so by Theorem 2.1, S is isomorphic to a LIB-LB Rees matrix semigroup $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$. By Proposition 2.2 , $I(\Lambda)$ index the non-zero $\mathcal{R}'(\mathcal{L}')$-classes. So given $i \in I$

there is an $\mathcal{R}'$-generator $e_i = (i, v_i, \mu_i)$ and given $\lambda \in \Lambda$ there is an $\mathcal{L}'$-generator $f_\lambda = (j_\lambda, u_\lambda, \lambda)$.

Let $\overline{\Gamma}$ index the $\mathcal{D}$-classes of $\mathcal{LM}^0$ containing at least one $\mathcal{L}'$-generator or $\mathcal{R}'$-generator. In view of Proposition 2.2, $\overline{\Gamma} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where $\Gamma = \Gamma_1 \cup \Gamma_2$ indexes the $\mathcal{D}$-classes containing $\mathcal{L}'$-generators and $\Gamma_1 \cup \Gamma_3$ indexes the $\mathcal{D}$-classes containing $\mathcal{R}'$-generators and $\Gamma_i \cap \Gamma_j = \emptyset, i \neq j, 1 \leq i, j \leq 3$. Note that $\Gamma = \Gamma_1 \cup \Gamma_2$ induces a partition of Λ and $\Gamma_1 \cup \Gamma_3$ induces a partition $P(\Gamma_1 \cup \Gamma_3) = \{I_\alpha : \alpha \in \Gamma_1 \cup \Gamma_3\}$ of I.

For $\alpha \in \Gamma_1 \cup \Gamma_2$ pick an $\mathcal{L}'$-generator f_{λ_α} with $\lambda_\alpha \in \Lambda_\alpha$ and $f_{\lambda_\alpha} = (j_{\lambda_\alpha}, u_{\lambda_\alpha}, \lambda_\alpha)$. Put $h_\alpha = f_{\lambda_\alpha}$ which we write more simply as $h_\alpha = f_\alpha = (i_\alpha, u_\alpha, \lambda_\alpha)$. In view of LIB(ii) we may assume that $p_{\lambda_\alpha i_\alpha}(u_\alpha) = g_\alpha \in M_{s(\alpha)\alpha}$. For $\alpha \in \Gamma_3$ pick an $\mathcal{R}'$-generator e_{i_α} with $i_\alpha \in I_\alpha$ and put $h_\alpha = e_{i_\alpha} = (i_\alpha, v_{i_\alpha}, \mu_{i_\alpha})$, which again we write more simply as $h_\alpha = e_\alpha = (i_\alpha, u_\alpha, \lambda_\alpha)$. Here $u_\alpha = u_\alpha p_{\lambda_\alpha i_\alpha}(u_\alpha)$ so that $g_\alpha = p_{\lambda_\alpha i_\alpha}(u_\alpha) \in M_{s(\overline{\alpha})\overline{\alpha}}$ is idempotent, where $\lambda_\alpha \in \Lambda_{\overline{\alpha}}$.

Now suppose that $\alpha \in \Gamma_1 \cup \Gamma_3$ and $k \in I_\alpha$. Then $e_k \mathcal{D} h_\alpha$ so that there is an element (k, w_k, λ_α) with inverse $(i_\alpha, \overline{w}_k, \mu_k)$ such that

$$e_k = (k, w_k, \lambda_\alpha)(i_\alpha, \overline{w}_k, \mu_k)$$

and

$$h_\alpha = (i_\alpha, \overline{w}_k, \mu_k)(k, w_k, \lambda_\alpha).$$

If $x \in U_k$, then as e_k is an $\mathcal{R}'$-generator,

$$x = v_k p_{\mu_k k}(x) = w_k p_{\lambda_\alpha i_\alpha}(\overline{w}_k) p_{\mu_k k}(x) = w_k p_{\lambda_\alpha i_\alpha}(\overline{x})$$

where $\overline{x} = \overline{w}_k p_{\mu_k k}(x) \in M_{i_\alpha \beta}$ for $x \in M_{k\beta}$. In addition, if $x = w_k p_{\lambda_\alpha i_\alpha}(y)$ where $y \in M_{i_\alpha \beta}$, then

$$\overline{w}_k p_{\mu_k k}(w_k) p_{\lambda_\alpha i_\alpha}(\overline{x}) = \overline{w}_k p_{\mu_k k}(w_k) p_{\lambda_\alpha i_\alpha}(y)$$

so that

$$u_\alpha p_{\lambda_\alpha i_\alpha}(\overline{x}) = u_\alpha p_{\lambda_\alpha i_\alpha}(y)$$

and hence

$$p_{\lambda_\alpha i_\alpha}(u_\alpha) p_{\lambda_\alpha i_\alpha}(\overline{x}) = p_{\lambda_\alpha i_\alpha}(u_\alpha) p_{\lambda_\alpha i_\alpha}(y),$$

that is,

$$g_\alpha p_{\lambda_\alpha i_\alpha}(\overline{x}) = g_\alpha p_{\lambda_\alpha i_\alpha}(y).$$

Our goal is to construct an IB-DB Rees matrix semigroup $\mathcal{DM}^0$ isomorphic to S. We already have the ingredients $I, \Lambda, \overline{\Gamma} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. For $\alpha, \beta \in \overline{\Gamma}$ put $\overline{M}_{\alpha\beta} = g_\alpha M g_\beta$ and put $\overline{M} = \bigcup\{\overline{M}_{\alpha\beta} : \alpha, \beta \in \overline{\Gamma}\}$. Then $\overline{M}$ becomes a partial semigroup where we define the product xy only if $x \in \overline{M}_{\alpha\beta}, y \in \overline{M}_{\beta\gamma}$ for some $\alpha, \beta, \gamma \in \Gamma$ and where defined, we take the product as the product in M. It is easy to see that in this case, $xy \in \overline{M}_{\alpha\gamma}$. Further, it is clear that for each $\alpha \in \overline{\Gamma}, \overline{M}_{\alpha\alpha} = \overline{T}_\alpha$ is a monoid with identity g_α and for $\alpha, \beta \in \overline{\Gamma}$, if $\overline{M}_{\alpha\beta} \neq \emptyset$ then $\overline{M}_{\alpha\beta}$ is a (T_α, T_β)-bisystem.

The final ingredient necessary is a sandwich matrix $\overline{P}$. For $\lambda \in \Lambda$ and $i \in I$, say $\lambda \in \Lambda_\alpha$ and $i \in I_\beta$, put

$$\overline{p}_{\lambda i} = \begin{cases} g_\alpha p_{\lambda i}(w_i) g_\beta & \text{if } p_{\lambda i} \neq 0 \\ 0 & \text{if } p_{\lambda i} \neq 0. \end{cases}$$

Thus we can construct a DB Rees matrix semigroup $\mathcal{DM}^0 = \mathcal{DM}^0(\overline{M}; I, \Lambda, \overline{\Gamma}; \overline{P})$. Our next task is to show that S is isomorphic to $\mathcal{DM}^0$. Define $\theta : \mathcal{LM}^0 \to \mathcal{DM}^0$ by $\theta(0) = 0$ and

$$\theta((k, r, \nu)) = (k, g_\alpha p_{\lambda_\alpha i_\alpha}(t_r), \nu)$$

where $k \in I_\alpha, \nu \in \Lambda_\beta$ and $r = w_k p_{\lambda_\alpha i_\alpha}(t_r)$ and $t_r \in M_{i_\alpha \beta}$. Note that $g_\alpha p_{\lambda_\alpha i_\alpha}(t_r) = g_\alpha p_{\lambda_\alpha i_\alpha}(t_r) g_\beta$ so that $\theta((k, r, \nu)) \in \mathcal{DM}^0$ and by a previous comment, θ is well-defined. A tedious but straightforward verification gives that θ is an isomorphism. Since $I(\Lambda)$ indexes the non-zero $\mathcal{R}'(\mathcal{L}')$-classes of $\mathcal{LM}^0$ and hence of $\mathcal{DM}^0$, and certainly $\mathcal{DM}^0$ is idempotent bounded as S is, we have by Lemma 4.2 that $\mathcal{DM}^0$ satisfies (LIB) and (RIB).

It remains to show that $\mathcal{DM}^0$ satisfies (D). We show first that $(\mathrm{D})_{II}$ holds.

Let $\alpha, \beta \in \Gamma_1 \cup \Gamma_3$ and $a \in \overline{M}_{\alpha\beta}, b \in \overline{M}_{\beta\alpha}$ where $ab = k_\alpha$, $ba = k_\beta$ and k_α, k_β are such that $k_\alpha m = m$ and $k_\beta n = n$ for all $\gamma \in \Gamma_1 \cup \Gamma_2, m \in \overline{M}_{\alpha\gamma}$ and $n \in \overline{M}_{\beta\gamma}$. Pick $i \in I_\alpha$ and $j \in I_\beta$; then there are $\mathcal{R}'$-generators (i, u, λ) and (j, v, μ) where $\lambda \in \Lambda_\gamma$ and $\mu \in \Lambda_\delta$. Then $(i, av, \mu), (j, bu, \lambda) \in \mathcal{DM}^0$ and

$$(i, av, \mu)(j, bu, \lambda) = (i, avp_{\mu j}bu, \lambda)$$

$$= (i, abu, \lambda) = (i, k_\alpha u, \lambda) = (i, u, \lambda),$$

further,

$$(j, bu, \lambda)(i, u, \lambda) = (j, bup_{\lambda i}u, \lambda) = (j, bu, \lambda)$$

so that $(i, u, \lambda)\mathcal{L}(j, bu, \lambda)$. Similarly, $(j, bu, \lambda)(i, av, \mu) = (j, v, \mu)$ so that $(j, bu, \lambda)\mathcal{R}(j, v, \mu)$ and $(i, u, \lambda)\mathcal{D}(j, v, \mu)$. By construction of $\overline{\Gamma}$ it is immediate that $\alpha = \beta$, so that $(\mathrm{D})_{II}$ holds. The dual argument gives that $(\mathrm{D})_{\Lambda\Lambda}$ holds.

Finally, suppose that $\alpha \in \Gamma_1 \cup \Gamma_3, \beta \in \Gamma_1 \cup \Gamma_2, a \in \overline{M}_{\alpha\beta}, b \in \overline{M}_{\beta\alpha}$ where $ab = k_\alpha, ba = k_\beta$ and k_α, k_β are such that $k_\alpha m = m$ for all $m \in \overline{M}_{\alpha\gamma}, \gamma \in \Gamma_1 \cup \Gamma_2$ and $nk_\beta = n$ for all $n \in \overline{M}_{\delta\beta}, \delta \in \Gamma_1 \cup \Gamma_3$. Pick $i \in I_\alpha$ and $\lambda \in \Lambda_\beta$. Then there is an $\mathcal{R}'$-generator (i, u, μ) where $\mu \in \Lambda_\gamma$ and an $\mathcal{L}'$-generator (j, v, λ) where $j \in I_\delta$. Now $(i, a, \lambda), (j, vbu, \mu) \in \mathcal{DM}^0$ and

$$(i, a, \lambda)(j, vbu, \mu) = (i, ap_{\lambda j}vbu, \mu)$$

$$= (i, abu, \mu) = (i, k_\alpha u, \mu) = (i, u, \mu)$$

and

$$(j, vbu, \mu)(i, a, \lambda) = (j, vbup_{\mu i}a, \lambda)$$

$$(j, vba, \lambda) = (j, vk_\beta, \lambda) = (j, v, \lambda)$$

so that $(i, u, \mu)\mathcal{R}(i, a, \lambda)\mathcal{L}(j, v, \lambda)$. Thus $\alpha = \beta$ and $(\mathrm{D})_{I\Lambda}$ holds. This completes the proof that $\mathcal{DM}^0$ satisfies (D) and so $\mathcal{DM}^0$ is an IB-DB Rees matrix semigroup isomorphic to S.

6 Some special cases

This final section uses Theorem 5.1 to investigate some special cases of IB C-semigroups.

An IB C-semigroup is *balanced* if every $\mathcal{R}'$-generator is $\mathcal{D}$-related to an $\mathcal{L}'$-generator and vice versa.

Proposition 6.1 *Let S be an IB C-semigroup. Then S is balanced if and only if S is isomorphic to an IB-DB Rees matrix semigroup in which $\Gamma = \Gamma_1$.*

Proof By Theorem 5.1, S is isomorphic to an IB-DB Rees matrix semigroup $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$. If $\mathcal{DM}^0$ is balanced, $\alpha \in \Gamma_1 \cup \Gamma_3$ and $i \in I_\alpha$, then the $\mathcal{R}'$-generator e_i is $\mathcal{D}$-related to some $\mathcal{L}'$-generator f_λ. Now Theorem 4.4 says that $\lambda \in \Lambda_\alpha$ so that $\alpha \in \Gamma_1$. Similarly, $\Gamma_2 = \emptyset$ so that $\Gamma = \Gamma_1$ as required.

Conversely, if $\Gamma = \Gamma_1$ and e_i is an $\mathcal{R}'$-generator of $\mathcal{DM}^0$, then $i \in I_\alpha$ for some $\alpha \in \Gamma_1$; but Γ partitions Λ and so choosing $\lambda \in \Lambda_\alpha$ we again use Theorem 4.4 to yield $e_i \mathcal{D} f_\lambda$, where f_λ is an $\mathcal{L}'$-generator. Together with the dual argument this completes the proof of the proposition.

DB Rees matrix semigroups $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$ in which $\Gamma = \Gamma_1$ were introduced in [F] , where they are referred to as *blocked Rees matrix semigroups* and written as $\mathcal{M}^0(M; I, \Lambda, \Gamma; P)$. We retain our notation $\mathcal{DM}^0$.

Corollary 6.2 *A semigroup S is a balanced IB C-semigroup if and only if S is isomorphic to a blocked Rees matrix semigroup satisfying (RIB)′,(LIB)′ and (D)′.*

(RIB)′ For all $\alpha \in \Gamma$ and $i \in I_\alpha$ there exists $\beta \in \Gamma, \lambda \in \Lambda_\beta$ and $u \in M_{\alpha\beta}$ with $up_{\lambda i} = g_\alpha$.

(LIB)′ For all $\alpha \in \Gamma$ and $\lambda \in \Lambda_\alpha$ there exists $\beta \in I, i \in I_\beta$ and $u \in M_{\beta\alpha}$ with $p_{\lambda i}u = g_\alpha$.

(D)′ For all $\alpha, \beta \in \Gamma$ and $a \in M_{\alpha\beta}, b \in M_{\beta\alpha}$, if $ab = g_\alpha$ and $ba = g_\beta$ then $\alpha = \beta$.

Proof If S is a balanced IB C-semigroup then by Proposition 6.1 and the remark following, S is isomorphic to a blocked Rees matrix semigroup satisfying (RIB),(LIB) and (D). Corollary 4.3 and its dual give that (RIB)′ and (LIB)′ hold. To see that (D)′ holds, let $\alpha, \beta \in \Gamma, a \in M_{\alpha\beta}, b \in M_{\beta\alpha}$ and suppose that $ab = g_\alpha$ and $ba = g_\beta$. Then it is immediate from $(\mathrm{D})_{II}$ that $\alpha = \beta$.

Conversely, suppose that S is isomorphic to a blocked Rees matrix semigroup, which we contrive to write as $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$, satisfying (RIB)′, (LIB)′ and (D)′. Clearly $\mathcal{DM}^0$ satisfies (RIB) and (LIB).

Suppose now that $\alpha, \beta \in \Gamma, a \in M_{\alpha\beta}, b \in M_{\beta\alpha}$ and $abm = m, ban = n$ for all $\gamma \in \Gamma, m \in m_{\alpha\gamma}, n \in M_{\beta\gamma}$. Since $g_\alpha \in M_{\alpha\alpha}, g_\alpha = abg_\alpha = ab$ and similarly, $g_\beta = ba$. Thus by (D)′, $\alpha = \beta$. This shows that $\mathcal{DM}^0$ satisfies $(\mathrm{D})_{II}$ and corresponding arguments give that $\mathcal{DM}^0$ satisfies $(\mathrm{D})_{I\Lambda}$ and $(\mathrm{D})_{\Lambda\Lambda}$, that is, $\mathcal{DM}^0$ satisfies (D).

We have shown that $\mathcal{DM}^0$ is an IB-DB Rees matrix semigroup and certainly $\Gamma = \Gamma_1$, so that by Proposition 6.1, S is a balanced IB C-semigroup.

Our next result is very simple. Recall that a semigroup is *prime* if $aSb \neq 0$ for all $a, b \in S^*$.

Proposition 6.3 *Let S be an IB C-semigroup. Then S is prime if and only if S is isomorphic to an IB-DB Rees matrix semigroup in which $M_{\alpha\beta} \neq \emptyset$ for all $\alpha \in \Gamma_1 \cup \Gamma_3, \beta \in \Gamma_1 \cup \Gamma_2$.*

Proof Certainly S is isomorphic to an IB-DB Rees matrix semigroup $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$. Suppose first that S is prime. Let $\alpha \in \Gamma_1 \cup \Gamma_3$ and $\beta \in \Gamma_1 \cup \Gamma_2$ and choose $i \in I_\alpha, \lambda \in \Lambda_\beta$. Then if e_i is an $\mathcal{R}'$-generator and f_λ is an $\mathcal{L}'$-generator we have $e_i\mathcal{DM}^0 f_\lambda \neq 0$ and so there is an element (i, a, λ) in $\mathcal{DM}^0$. But then $a \in M_{\alpha\beta}$.

Conversely we assume that for all $\alpha \in \Gamma_1 \cup \Gamma_3, \beta \in \Gamma_1 \cup \Gamma_2, M_{\alpha\beta} \neq \emptyset$. Let $(i, a, \lambda), (j, b, \mu) \in \mathcal{DM}^0$. Using (RIB) and (LIB) we can certainly choose $k \in I, \nu \in \Lambda$ with $p_{\lambda k} \neq 0$ and $p_{\nu j} \neq 0$. Suppose that $k \in I_\alpha$ and $\nu \in \Lambda_\beta$. Then $\alpha \in \Gamma_1 \cup \Gamma_3, \beta \in \Gamma_1 \cup \Gamma_2$ so that there is some $c \in M_{\alpha\beta}$. Thus $(k, c, \nu) \in \mathcal{DM}^0$ and $(i, a, \lambda)(k, c, \nu)(j, b, \mu) \neq 0$, giving that $\mathcal{DM}^0$ and so S are prime.

We now turn our attention to primitive abundant semigroups. On a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $(a, b) \in \mathcal{L}^*$ if and only if the elements a, b of S are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually. A semigroup is *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class contains an idempotent.

An abundant semigroup is *primitive* if the non-zero idempotents are primitive under the natural partial order on the idempotents. As a special case of the main theorem of this paper, one can derive the following result of [F].

Theorem 6.4 (F, Theorem 5.1) *A semigroup S is primitive and abundant if and only if S is isomorphic to a blocked Rees matrix semigroup $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$ satisfying (C),(U) and (R).*
(C) If $a, a_1, a_2 \in M_{\alpha\beta}, b, b_1, b_2 \in M_{\beta\gamma}$, then $ab_1 = ab_2$ implies $b_1 = b_2$, and $a_1 b = a_2 b$ implies $a_1 = a_2$.
(U) For each $\alpha \in \Gamma$ and each $\lambda \in \Lambda_\alpha (i \in I_\alpha)$ there is a member i of I_α (λ of Λ_α) such that $p_{\lambda i}$ is a unit in T_α.
(R) If $M_{\alpha\beta}, M_{\beta\alpha}$ are both non-empty where $\alpha \neq \beta$, then $aba \neq a$ for all $a \in M_{\alpha\beta}, b \in M_{\beta\alpha}$.

Finally we turn our attention to IB-DB Rees matrix semigroups for which $|\Gamma| = 1$. Adapting the terminology of Steinfeld [S1], we say that left ideals L_1, L_2 of a semigroup S are *left similar* if there is a bijection $\phi : L_1 \to L_2$ such that $\phi(sx) = s\phi(x)$ for all $x \in L_1$ and for all $s \in S$; that is, L_1 and L_2 are isomorphic as left S^1-systems. Then an IB C-semigroup S is *left similarly decomposable* if Sf_λ and Sf_μ are left similar for all $\mathcal{L}'$-generators f_λ, f_μ. The notion of *right similarly decomposable* is defined dually and S is *similarly decomposable* if S is both left and right similarly decomposable.

The following lemma is Proposition 4.10 of [S2].

Lemma 6.5 *[S2] Let S be a semigroup and $e, f \in E(S)$. Then $e\mathcal{D}f$ if and only if Se and Sf are left similar.*

Proposition 6.6 *Let S be an IB C-semigroup, where S is isomorphic to $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$. Then S is left (right) similarly decomposable if and only if $|\Gamma_1 \cup \Gamma_2| = 1(|\Gamma_1 \cup \Gamma_3| = 1)$. Further, S is similarly decomposable and balanced if and only if $|\Gamma| = 1$.*

Proof We know that S is left similarly decomposable if and only if for all $\mathcal{L}'$-generators $f_\lambda, f_\mu, Sf_\lambda$ and Sf_μ are left similar. By Lemma 6.5, this is equivalent to $f_\lambda \mathcal{D} f_\mu$ for all $\mathcal{L}'$-generators f_λ, f_μ. From Theorem 4.4, this is true if and only if $|\Gamma_1 \cup \Gamma_2| = 1$.

From the above, S is similarly decomposable if and only if $|\Gamma_1 \cup \Gamma_2| = 1$ and $|\Gamma_1 \cup \Gamma_3| = 1$. If in addition S is balanced, than $\Gamma = \Gamma_1$ so that $|\Gamma| = 1$. Conversely, if $|\Gamma| = 1$ then clearly as $\Gamma_1 \cup \Gamma_2 \neq \emptyset$ and $\Gamma_1 \cup \Gamma_3 \neq \emptyset$, we must have that $\Gamma = \Gamma_1$ and so $|\Gamma_1 \cup \Gamma_2| = 1 = |\Gamma_1 \cup \Gamma_3|$. Thus S is balanced and similarly decomposable.

Abundant semigroups that are primitive and similarly decomposable are said to be *primitively decomposable*. It is now straightforward to deduce the following result of [F] from Theorem 6.4 and Proposition 6.6.

Corollary 6.7 (F, Theorem 4.9) *A semigroup S is abundant and primitively decomposable if and only if S is isomorphic to a blocked Rees matrix semigroup $\mathcal{DM}^0 = \mathcal{DM}^0(M; I, \Lambda, \Gamma; P)$ which satisfies (U) and where $|\Gamma| = 1$ and M is a cancellative monoid.*

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