INVERSE SEMIGROUPS OF LEFT I-QUOTIENTS

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Abstract. We examine, in a general setting, a notion of inverse semigroup of left quotients, which we call *left I-quotients*. This concept has appeared, and has been used, as far back as Clifford's seminal work describing bisimple inverse monoids in terms of their right unit subsemigroups. As a consequence of our approach, we find a straightforward way of extending Clifford's work to bisimple inverse semigroups (a step that has previously proved to be awkward). We also put some earlier work on Gantos into a wider and clearer context, and pave the way for further progress.

Introduction

The notion of quotient plays an important role in algebra. As far as semigroup theory is concerned, it occurs in its simplest form as the concept of a *group* G of left quotients of a subsemigroup S, which requires that every $g \in G$ can be written as $g = a^{-1}b$ where $a, b \in S$. A well known result of Ore and Dubreil [4] says that a semigroup S has a group of left quotients if and only if it is right reversible and cancellative, where *right reversible* means that for any $a, b \in S$, $Sa \cap Sb \neq \emptyset$. The notion of group of left quotients was extended to that of semigroup of left quotients by Fountain and Petrich in [8]; this idea has been extensively developed by a number of authors. If Q is a semigroup of quotients of a subsemigroup S, then every $q \in G$ can be written as $q = a^{\sharp}b$ where $a, b \in S$ and $a^{\sharp}$ is the inverse of *a in a subgroup of* Q. It is a hard problem to obtain a general characterisation of those semigroups possessing a semigroup of left quotients. The best results in this direction consider semigroups having a semigroup of left quotients *lying in a particular class*; the result of Ore and Dubreil being such an example.

The object of this paper is to begin a systematic review of an alternative, but very natural, notion of quotient. The focus here will be on inverse semigroups, and we aim to develop a concept of quotient that will utilise the natural involution that an inverse semigroup possesses. For an element a of an inverse semigroup Q, a^{-1} will always denote the inverse of a in the sense of inverse semigroup theory. If a lies in a subgroup, then $a^{\sharp} = a^{-1}$, but a^{-1} may exist without a lying in a subgroup.

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Definition 0.1. Let S be a subsemigroup of an inverse semigroup Q. Then S is a *left I-order* in Q and Q is a *semigroup of left I-quotients* of S if every $q \in Q$ can be written as $q = a^{-1}b$ where $a, b \in S$.

We stress that this notion is not new - it has been effectively defined by a number of authors, without being made fully explicit. Perhaps the first time this idea appeared was in [3], an article that (in our terminology) considered right cancellative monoids as left I-orders in bisimple inverse monoids. The results of [3] are arrived at via explicit construction of quotients from equivalence classes of ordered pairs of elements of S. An alternative approach uses inverse hulls of right cancellative monoids, and was pursued (and taken further) in, for example, [17, 16] and [2]. For inverse semigroups that do not have an identity, one cannot take the right unit subsemigroup as a natural right cancellative submonoid. To overcome this, Reilly [19], introduced the notion of quotient of an RP-system, where an RP-system corresponds (according to his result) to an $\mathcal{R}$-class of a bisimple inverse semigroup Q. Reilly's approach is subsumed in Lawson's use of category actions to construct inverse semigroups [15]. It is easy to see (and we show this enroute in Section 1) that if Q is a bisimple inverse semigroup and R is an $\mathcal{R}$-class of Q, then every element of Q can be written as $a^{-1}b$ where a and b lie in R; from the above, R is an RP-system but will not, in general, be a subsemigroup. Nevertheless, Reilly successfully used RP-systems to characterise congruences on bisimple inverse semigroups [20]. In a different direction, Gantos [9] extended the work of Clifford to semilattices of right cancellative monoids.

The above mentioned articles, and others, all consider left I-orders in particular classes of semigroups. Here, after Section 1 of preliminaries, we begin in Section 2 in a rather more abstract way, by asking the natural questions that arise when one introduces a notion of quotient. For example, if S is a left I-order in Q, under what conditions is $a^{-1}b = c^{-1}d$ where $a, b, c, d \in S$? If S possesses a semigroup of left I-quotients, when is this unique? We then apply our findings in a number of different ways. We first show that Brandt semigroups of left I-quotients of a given semigroup S are unique up to isomorphism, thus extending the result of [7] that states that Brandt semigroups of left quotients of a given S are unique.

In Section 3 we focus on left ample semigroups. A semigroup S is left ample if and only if it embeds into an inverse semigroup Q in such a way that if $a \in S$, then $aa^{-1} \in S$. Right cancellative monoids are precisely left ample semigroups possessing a single idempotent. A left ample semigroup S has a natural representation as partial one-one maps of S, from which we can construct its inverse hull $\Sigma(S)$. We find necessary and sufficient conditions for a left ample semigroup to be a left I-order in its inverse hull, namely that for any $a, b \in S$, $Sa \cap Sb = Sc$ for some $c \in S$; we call this Condition (LC). Thus, (LC) is a rather stronger condition than being right reversible. Our result corresponds exactly to that of Clifford for right cancellative monoids.

We divert a little in Section 4 to look at strong semilattices of left ample semigroups with (LC). Let S be a strong semilattice Y of left ample semigroups $S_\alpha, \alpha \in Y$, such that each S_α has (LC); using the general results of Section 2, we show that S is a left I-order in Q, where Q a strong semilattice Y of the inverse hulls $\Sigma(S_\alpha)$ of the semigroups $S_\alpha, \alpha \in Y$, if and only if the connecting morphisms are *LC-preserving*, and this is equivalent to S having (LC). In this case, Q is the inverse hull of S. Part of our result extends that of [9] in a rather simple way.

In the final section, we consider left I-orders in bisimple inverse semigroups. Building on the work of Section 3, we define a category **LAC** of left ample semigroups with (LC), and the category **BIS** of bisimple inverse semigroups, and show that **LAC** and **BIS** are equivalent. This result may be be specialised to show that the corresponding category of right cancellative monoids is equivalent to the category of bisimple inverse monoids.

It will be useful in future work to make adjustments to Definition 0.1 in the case where Q has a zero. To avoid complications in the current paper we make no further mention of this.

1. Preliminaries and inverse hulls

For any semigroup Q we denote the quasi-orders associated with Green's relations $\mathcal{R}$ and $\mathcal{L}$ by $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$, respectively. To avoid ambiguity we may use the superscript Q to indicate that a relation applies to Q, so that, for example, $a \leq_{\mathcal{R}}^{Q} b$ if and only if $aQ^1 \subseteq bQ^1$.

The relation $\mathcal{R}^*$ is defined on a semigroup S by the rule that for any $a, b \in S$, $a \,\mathcal{R}^*\, b$ in S if and only if $a \,\mathcal{R}\, b$ in some oversemigroup of S. The following alternative characterisation of $\mathcal{R}^*$ is well known.

Lemma 1.1. *The following are equivalent for elements a, b of a semigroup S:*
(i) $a \,\mathcal{R}^*\, b$;
(ii) for all $x, y \in S^1$,

$$xa = ya \text{ if and only if } xb = yb.$$

It is easy to see that $\mathcal{R}^*$ is a left congruence, $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{R} = \mathcal{R}^*$ if S is regular. In general, however, the inclusion can be strict.

In a semigroup with commuting idempotents, it is clear that any $\mathcal{R}^*$-class contains at most one idempotent. Where it exists we denote the (unique) idempotent in the $\mathcal{R}^*$-class of a by a^+. If *every* $\mathcal{R}^*$-class contains an idempotent, $^+$ is then a unary operation on S and we may regard S as an algebra of type $(2, 1)$; as such, morphisms must preserve the unary operation of $^+$ (and hence the relation $\mathcal{R}^*$). We may refer to such morphisms as '$(2, 1)$-morphisms' if there is danger of ambiguity. Of course, any semigroup isomorphism must preserve $^+$. We remark here that if S is inverse, then $a^+ = aa^{-1}$ for all $a \in S$.

Definition 1.2. A semigroup S is *left ample* if $E(S)$ is a semilattice, every $\mathcal{R}^*$-class contains a (necessarily unique) idempotent a^+ and the *left ample* identity

(AL) holds:

$$xy^+ = (xy^+)^+x \quad \text{(AL)}.$$

It is easy to see that a semigroup is left ample and unipotent (that is, it contains exactly one idempotent) if and only if it a right cancellative monoid.

Remark 1.3. The class of left ample semigroups forms a quasi-variety of algebras of type $(2, 1)$.

Since any inverse semigroup is left ample, any subsemigroup of such that is closed under $^+$ must therefore be left ample. The converse is also true: left ample semigroups are (up to isomorphism), precisely the submonoids of (symmetric) inverse semigroups closed under $^+$. As the representation we use is needed in some later sections, we give a brief outline. Further details can be found in, for example, [12]. It is useful to recall that in a symmetric inverse semigroup $\mathcal{I}_X$ we have $\alpha \,\mathcal{R}\, \beta$ if and only if $\operatorname{dom} \alpha = \operatorname{dom} \beta$, and $\alpha \,\mathcal{L}\, \beta$ if and only if $\operatorname{im} \alpha = \operatorname{im} \beta$.

Let S be left ample. We construct an embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ as follows. For each $a \in S$ we let $\rho_a \in \mathcal{I}_S$ be given by

$$\operatorname{dom} \rho_a = Sa^+ \text{ and } \operatorname{im} \rho_a = Sa$$

and for any $x \in \operatorname{dom} \rho_a$.

$$x\rho_a = xa.$$

Then the map $\theta_S : S \to \mathcal{I}_S$ is a $(2, 1)$-embedding.

Definition 1.4. Let S be a left ample semigroup. Then the *inverse hull* $\Sigma(S)$ of S is the inverse subsemigroup of $\mathcal{I}_S$ generated by $\operatorname{im} \theta_S$.

We pause to consider a special case. Let M be a right cancellative monoid. Then for any $a \in M$, we have that $\rho_a : M \to Ma$, so that $\operatorname{dom} \rho_a = M = \operatorname{dom} I_M$, giving that $\operatorname{im} \theta_S \subseteq R_1$, where R_1 is the $\mathcal{R}$-class of I_M in $\mathcal{I}_M$.

2. Left I-orders

The classical notion of quotients in [8], developed in a number of further articles, tells us that if we want close relationship between a left order and its semigroup of left quotients, then we may need to insist that the left order be *straight* [11]. Extrapolating this idea gives us the following:

Definition 2.1. Let S be a left I-order in Q. Then S is *straight* in Q if every $q \in Q$ can be written as $q = a^{-1}b$ where $a, b \in S$ and $a \,\mathcal{R}\, b$ in Q.

If S is a left I-order in Q and S has a right identity, then this must be an identity of Q (and hence of S). For any $q \in Q$ we have that $q = a^{-1}b$ where $a, b \in S$, so that

$$qe = a^{-1}be = a^{-1}b = q$$

and

$$eq = ea^{-1}b = (ae)^{-1}b = a^{-1}b = q.$$

We remark that if a semigroup S is a left order in Q in the sense of [8], then it is certainly a left I-order. For, if S is a left order in Q, then we insist that any $q \in Q$ can be written as $q = a^{\sharp}b$ where $a, b \in S$ and $a^{\sharp}$ is the inverse of *a in a subgroup of S*. Our notion of left I-order is more general, as we now demonstrate with an easy example.

Example 2.2. Let B be the Bicyclic Semigroup and let $S = R_{(0,0)}$, the $\mathcal{R}$-class of the identity. It is clear that S is a subsemigroup of B. For any $(a, b) \in B$ we have that

$$(a, b) = (a, 0)(0, b) = (0, a)^{-1}(0, b)$$

so that S is a left I-order in B. On the other hand, the only element of S lying in a subgroup is $(0, 0)$ and $(0, 0)^{\sharp}(0, n) = (0, n)$, for any $(0, n) \in S$. Thus S is not a left order in B.

The fact that $R_{(0,0)}$ is a left I-order in B is a very special case of the result of [3] mentioned in the Introduction, which we shall revisit. The semigroup B is bisimple and we shall see that bisimple inverse semigroups play an important role in this theory. Suppose that Q is bisimple and we pick an $\mathcal{R}$-class $R = R_e$ of Q, where $e \in E(S)$. Let $q \in Q$. As Q is bisimple we can find mutually inverse elements $x, x^{-1} \in Q$ such that $xx^{-1} = e$ and $x^{-1}x = qq^{-1}$. Then $q = x^{-1}xq$ and $xq \,\mathcal{R}\, xqq^{-1} = x \,\mathcal{R}\, e$. Thus any element of Q can be written as a quotient of elements *chosen from any $\mathcal{R}$-class.*

For an example of a different flavour, we present the following. For later purposes, it is useful to recall that if B is a Brandt semigroup and $a, e \in B$ with $e = e^2$, then $ea \neq 0\,(ae \neq 0)$ implies that $ea = a\,(ae = a)$.

Example 2.3. Let H be a left order in a group G, and let $\mathcal{B}^0 = \mathcal{B}^0(G, I)$ be a Brandt semigroup over G where $|I| \geq 2$. Fix $i \in I$ and let

$$S_i = \{(i, h, j) : h \in H, j \in I\} \cup \{0\}.$$

Then S_i is a straight left I-order in $\mathcal{B}^0$.

To see this, notice that S_i is a subsemigroup, $0 = 0^{-1}0$, and for any $(j, g, k) \in \mathcal{B}^0$, we may write $g = a^{-1}b$ where $a, b \in H$ and then

$$(j, g, k) = (i, a, j)^{-1}(i, b, k)$$

where $(i, a, j), (i, b, k) \in S_i$.

Again, it is easy to see that S_i is not a left order in $\mathcal{B}^0$.

Notice that if S is a left I-order in Q and $a, b \in S$ with $a \,\mathcal{R}\, b$ in Q, then $a^{-1} \,\mathcal{R}\, a^{-1}b \,\mathcal{L}\, b$ in Q, so that if S is straight in Q, then S *intersects every $\mathcal{L}$-class of Q.*

In this initial article we will be primarily interested in left ample semigroups that are left I-orders. In such cases *the relation $\mathcal{R}^*$ will always refer to the left I-order.*

Lemma 2.4. *Let S be a left ample semigroup, embedded (as a $(2,1)$-algebra) in an inverse semigroup Q. If S is a left I-order in Q, then S is straight.*

Proof. Let $q = a^{-1}b \in Q$ where $a, b \in S$. Then

$$q = (a^+a)^{-1}(b^+b) = a^{-1}a^+b^+b = a^{-1}b^+a^+b = (b^+a)^{-1}(a^+b).$$

We have

$$a^+b\,\mathcal{R}^*\,a^+b^+ = b^+a^+\,\mathcal{R}^*\,b^+a$$

and so $a^+b\,\mathcal{R}\,b^+a$ in Q and S is straight. □

We know from the classical case that a semigroup may be a left I-order in non-isomorphic semigroups of left I-quotients (see, for example, [5]). For the remainder of this section we concentrate on determining when two semigroups of straight left I-quotients of a given semigroup are isomorphic. More generally we introduce the following notion.

Definition 2.5. Let S be a subsemigroup of Q and let $\phi : S \to P$ be a morphism from S to a semigroup P. If there is a morphism $\overline{\phi} : Q \to P$ such that $\overline{\phi}|_S = \phi$, then we say that ϕ *lifts* to domain Q and ϕ *lifts* to $\overline{\phi}$. If ϕ lifts to an isomorphism, then we say that Q and P are *isomorphic over* S.

To achieve our goal, we must first examine when two quotients $a^{-1}b$ and $c^{-1}d$ are equal, where $a, b, c, d \in S$ and S is a left I-order in Q. Notice that if $a\,\mathcal{R}^Q\,b$ and $c\,\mathcal{R}^Q\,d$ and $a^{-1}b = c^{-1}d$, then an earlier remark gives that $a\,\mathcal{L}^Q\,c$ and $b\,\mathcal{L}^Q\,d$. In Lemma 2.7 below, we give conditions on S such that $a^{-1}b = c^{-1}d$; the use of Green's relations in Q in our conditions will be 'internalised' to S at a later point. First, a preliminary remark that, given its usefulness, more than merits the name of lemma.

Lemma 2.6. *Let b, c, x, y be elements of an inverse semigroup Q such that $x\,\mathcal{R}\,y$. If $bc^{-1} = x^{-1}y$, then $xb = yc$.*

Proof. We have that

$$bc^{-1}cb^{-1} = (bc^{-1})(bc^{-1})^{-1} = (x^{-1}y)(x^{-1}y)^{-1} = x^{-1}yy^{-1}x = x^{-1}x$$

as $x\,\mathcal{R}\,y$. Hence

$$bc^{-1}c = bb^{-1}bc^{-1}c = bc^{-1}cb^{-1}b = x^{-1}xb$$

and so $xbc^{-1}c = xb$. From $y = xbc^{-1}$ we have

$$xb = xbc^{-1}c = yc.$$

□

Lemma 2.7. *Let S be a straight left I-order in Q. Let $a, b, c, d \in S$ with $a\,\mathcal{R}^Q\,b$ and $c\,\mathcal{R}^Q\,d$ in Q. Then $a^{-1}b = c^{-1}d$ if and only if there exist $x, y \in S$ with $xa = yc$ and $xb = yd$ and such that $a\,\mathcal{R}^Q\,x^{-1}$, $x\,\mathcal{R}^Q\,y$ and $y\,\mathcal{L}^Q\,c^{-1}$ in Q.*

Proof. Suppose first that $a^{-1}b = c^{-1}d$. Then as above, $b\,\mathcal{L}^Q\,d$ and $a\,\mathcal{L}^Q\,c$. Let $x, y \in S$ be such that $ac^{-1} = x^{-1}y$ and $x\,\mathcal{R}^Q\,y$. Then

$$b = ac^{-1}d = x^{-1}yd$$

so that $xb = yd$. From Lemma 2.6, $xa = yc$. Also,

$$a\,\mathcal{R}^Q\,aa^{-1}\,\mathcal{R}^Q\,ac^{-1} = x^{-1}y\,\mathcal{R}^Q\,x^{-1},$$

and

$$y = xac^{-1}\,\mathcal{L}^Q\,a^{-1}ac^{-1} = c^{-1}cc^{-1} = c^{-1}$$

as required.

Conversely, if $xa = yc$, $xb = yd$ for some $x, y \in S$ with $x\,\mathcal{R}^Q\,y$, $a\,\mathcal{R}^Q\,x^{-1}$ and $y\,\mathcal{L}^Q\,c^{-1}$, then $a = x^{-1}yc, b = x^{-1}yd$ and

$$\begin{array}{rcll}
a^{-1}b & = & (x^{-1}yc)^{-1}(x^{-1}yd) & \\
& = & c^{-1}y^{-1}xx^{-1}yd & \\
& = & c^{-1}y^{-1}yd & \text{as } x\,\mathcal{R}^Q\,y \\
& = & c^{-1}cc^{-1}d & \text{as } y\,\mathcal{L}^Q\,c^{-1} \\
& = & c^{-1}d. &
\end{array}$$

□

Let S be a subsemigroup of an inverse semigroup Q. We use Green's relations on Q to define binary relations $\leq^Q_{\mathcal{R},S}, \mathcal{R}^Q_S, \leq^Q_{\mathcal{L},S}$ and $\mathcal{L}^Q_S$ and a ternary relation $\mathcal{T}^Q_S$ on S by the rules that:

$$\leq^Q_{\mathcal{R},S} = \leq^Q_{\mathcal{R}} \cap (S \times S) \text{ and } \leq^Q_{\mathcal{L},S} = \leq^Q_{\mathcal{L}} \cap (S \times S),$$

so that $\leq^Q_{\mathcal{R},S}$ and $\leq^Q_{\mathcal{L},S}$ are, respectively, left and right compatible quasi-orders. We then define $\mathcal{R}^Q_S$ and $\mathcal{L}^Q_S$ to be the associated equivalence relations, so that

$$\mathcal{R}^Q_S = \mathcal{R}^Q \cap (S \times S) \text{ and } \mathcal{L}^Q_S = \mathcal{L}^Q \cap (S \times S).$$

Consequently, $\mathcal{R}^Q_S$ and $\mathcal{L}^Q_S$ are left and right compatible. We define $\mathcal{T}^Q_S$ by the rule that for any $a, b, c \in S$,

$$(a, b, c) \in \mathcal{T}^Q_S \text{ if and only if } ab^{-1}Q \subseteq c^{-1}Q.$$

Lemma 2.8. *Let S and T be subsemigroup of inverse semigroups Q and P respectively, and let $\phi : S \to T$ be a morphism. If for all $a, b, c \in S$,*

$$(a, b, c) \in \mathcal{T}^Q_S \Rightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}^P_T,$$

then for all $u, v \in S$,

$$u \leq^Q_{\mathcal{R}} v^{-1} \Rightarrow u\phi \leq^P_{\mathcal{R}} (v\phi)^{-1}.$$

Proof. Suppose that $u, v \in S$ and $uQ \subseteq v^{-1}Q$. Then $uu^{-1}Q \subseteq v^{-1}Q$, so that $(u, u, v) \in \mathcal{T}^Q_S$. By assumption, $(u\phi, u\phi, v\phi) \in \mathcal{T}^P_T$, so that

$$u\phi P = u\phi(u\phi)^{-1}P \subseteq (v\phi)^{-1}P$$

and $u\phi \leq^P_{\mathcal{R}} (v\phi)^{-1}$ as required. □

We use the relation $\mathcal{T}_S^Q$ to prove our rather general result below. *As in the classical case, $\mathcal{T}_S^Q$ can be avoided in some special cases of interest.*

Theorem 2.9. *Let S be a straight left I-order in Q and let T be a subsemigroup of an inverse semigroup P. Suppose that $\phi : S \to T$ is a morphism. Then ϕ lifts to a (unique) morphism $\overline{\phi} : Q \to P$ if and only if for all $(a,b,c) \in S$:*
(i) $(a,b) \in \mathcal{R}_S^Q \Rightarrow (a\phi, b\phi) \in \mathcal{R}_T^P$;
(ii) $(a,b,c) \in \mathcal{T}_S^Q \Rightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}_T^P$.
If (i) and (ii) hold and $S\phi$ is a left I-order in P, then $\overline{\phi} : Q \to P$ is onto.

Proof. If ϕ lifts to a morphism $\overline{\phi}$, then as morphisms between inverse semigroups preserve inverses and Green's relations, it is easy to see that (i) and (ii) hold.

Conversely, suppose that (i) and (ii) hold. We define $\overline{\phi} : Q \to P$ by the rule that

$$(a^{-1}b)\overline{\phi} = (a\phi)^{-1}b\phi$$

where $a, b \in S$ and $a \,\mathcal{R}^Q\, b$.

To show that $\overline{\phi}$ is well defined, suppose that

$$a^{-1}b = c^{-1}d$$

where $a, b, c, d \in S$, $a \,\mathcal{R}^Q\, b$ and $c \,\mathcal{R}^Q\, d$. Then by Lemma 2.7, there exist $x, y \in S$ with $xa = yc$ and $xb = yd$ and such that $a \,\mathcal{R}^Q\, x^{-1}$, $x \,\mathcal{R}^Q\, y$ and $y \,\mathcal{L}^Q\, c^{-1}$. Applying ϕ, we have that $x\phi\, a\phi = y\phi\, c\phi$ and $x\phi\, b\phi = y\phi\, d\phi$. By (i) we also have that $x\phi \,\mathcal{R}^P\, y\phi, a\phi \,\mathcal{R}^P\, b\phi$ and $c\phi \,\mathcal{R}^P\, d\phi$, and by (ii) and Lemma 2.8, it follows that $b\phi \leq_{\mathcal{R}}^P (x\phi)^{-1}$ and $d\phi \leq_{\mathcal{R}}^P (y\phi)^{-1}$.

From $x\phi\, b\phi = y\phi\, d\phi$ we can now deduce that $b\phi = (x\phi)^{-1}y\phi\, d\phi$ so that

$$\begin{aligned}
(a\phi)^{-1}b\phi &= (a\phi)^{-1}(x\phi)^{-1}y\phi\, d\phi \\
&= (x\phi\, a\phi)^{-1}y\phi\, d\phi \\
&= (y\phi\, c\phi)^{-1}y\phi\, d\phi \\
&= (c\phi)^{-1}(y\phi)^{-1}y\phi\, d\phi \\
&= (c\phi)^{-1}d\phi,
\end{aligned}$$

so that $\overline{\phi}$ is well defined.

To see that $\overline{\phi}$ lifts ϕ, let $h \in S$; then $h = k^{-1}\ell$ for some $k, \ell \in S$ with $k \,\mathcal{R}^Q\, \ell$. We have that $kh = \ell$ and $h \leq_{\mathcal{R}}^Q k^{-1}$, so that $k\phi h\phi = \ell\phi$ and by Lemma 2.8, $h\phi \leq_{\mathcal{R}}^P (k\phi)^{-1}$. It follows that $h\phi = (k\phi)^{-1}\ell\phi = h\overline{\phi}$.

We need to show that $\overline{\phi}$ is a morphism. To this end, let $a^{-1}b, c^{-1}d \in Q$ with $a \,\mathcal{R}^Q\, b$ and $c \,\mathcal{R}^Q\, d$. By (i) we have that $c\phi \,\mathcal{R}^P\, d\phi$. Now $bc^{-1} = u^{-1}v$ for some $u, v \in S$ with $u \,\mathcal{R}^Q\, v$. By Lemma 2.6, $ub = vc$, so that $u\phi\, b\phi = v\phi\, c\phi$. Further, $(b, c, u) \in \mathcal{T}_S^Q$, so by assumption (ii), we have that $(b\phi, c\phi, u\phi) \in \mathcal{T}_T^P$. Then from $u\phi\, b\phi(c\phi)^{-1} = v\phi\, c\phi(c\phi)^{-1}$ we obtain $b\phi(c\phi)^{-1} = (u\phi)^{-1}v\phi\, c\phi(c\phi)^{-1}$.

Multiplying, we have

$$(a^{-1}b)(c^{-1}d) = a^{-1}(bc^{-1})d = a^{-1}(u^{-1}v)d = (a^{-1}u^{-1})(vd) = (ua)^{-1}vd,$$

and

$$ua \, \mathcal{R}^Q \, ub = vc \, \mathcal{R}^Q \, vd.$$

Hence

$$\begin{aligned} ((a^{-1}b)(c^{-1}d))\overline{\phi} &= ((ua)^{-1}vd)\overline{\phi} \\ &= ((ua)\phi)^{-1}(vd)\phi \\ &= (a\phi)^{-1}(u\phi)^{-1}v\phi \, d\phi \\ &= (a\phi)^{-1}(u\phi)^{-1}v\phi \, c\phi \, (c\phi)^{-1}d\phi \\ &= (a\phi)^{-1}b\phi(c\phi)^{-1}d\phi \\ &= (a^{-1}b)\overline{\phi}(c^{-1}d)\overline{\phi}, \end{aligned}$$

so that $\overline{\phi}$ is a morphism as required.

If (i) and (ii) hold and $S\phi$ is a left I-order in P, then for any $p \in P$ we have $p = (a\phi)^{-1}b\phi$ for some $a, b \in S$, so that $p = (a^{-1}b)\overline{\phi}$. $\square$

Corollary 2.10. *Let S be a straight left I-order in Q and let $\phi : S \to P$ be an embedding of S into an inverse semigroup P such that $S\phi$ is a straight left I-order in P. Then Q is isomorphic to P over S if and only if for any $a, b, c \in S$:*

(i) $(a, b) \in \mathcal{R}^Q_S \Leftrightarrow (a\phi, b\phi) \in \mathcal{R}^P_{S\phi}$; and

(ii) $(a, b, c) \in \mathcal{T}^Q_S \Leftrightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}^P_{S\phi}$.

Proof. If Q is isomorphic to P over S then (i) and (ii) hold from Theorem 2.9.

Suppose now that (i) and (ii) hold. From Theorem 2.9, ϕ lifts to a morphism $\overline{\phi} : Q \to P$, where $(a^{-1}b)\overline{\phi} = (a\phi)^{-1}b\phi$. Dually, $\phi^{-1} : S\phi \to Q$ lifts to a morphism $\overline{\phi^{-1}} : P \to Q$, where $((a\phi)^{-1}b\phi)\overline{\phi^{-1}} = a^{-1}b$. Clearly $\overline{\phi}$ and $\overline{\phi^{-1}}$ are mutually inverse. $\square$

Where S is left ample, and ϕ preserves $^+$, then we note that (i) in Theorem 2.9 and Corollary 2.10 is redundant. Further redundancies become apparent in the next section.

For an alternative use of Theorem 2.9, we consider the case of left I-orders in Brandt semigroups.

Theorem 2.11. *Let S be a left I-order in a Brandt semigroup $\mathcal{B}^0 = \mathcal{B}^0(G, I)$. Then S contains a zero and is straight in $\mathcal{B}^0$.*

If $\phi : S \to T$ is an isomorphism where T is a left I-order in $\mathcal{B}^0_1 = \mathcal{B}^0(H, J)$, then ϕ lifts to an isomorphism $\overline{\phi} : \mathcal{B}^0 \to \mathcal{B}^0_1$.

Proof. Let S be a left I-order in $\mathcal{B}^0$. Clearly $S \neq \{0\}$. Suppose that S is contained within a non-zero group $\mathcal{H}$-class of $\mathcal{B}^0$, say $S \subseteq H_{(i,1,i)}$ (where 1 is the identity of G). Then $S^{-1}S = \{a^{-1}b : a, b \in S\} \subseteq H_{(i,1,i)}$, a contradiction as we must have $0 \in S^{-1}S$. It follows that either $0 \in S$, or there exists $(i, g, j) \in S$ for some $i, j \in I$ with $i \neq j$ and $g \in G$. But in the latter case, we again have $0 = (i, g, j)(i, g, j) \in S$.

Clearly $0 = 0^{-1}0$. For any $(i, g, j) \in \mathcal{B}^0$, we have $(i, g, j) = a^{-1}b$ for some $a, b \in S$. We must have that

$$a^{-1} = (i, u, \ell) \text{ and } b = (\ell, v, j)$$

so that $a = (\ell, u^{-1}, i) \mathcal{R} (\ell, v, j) = b$ in $\mathcal{B}^0$. Thus S is straight in $\mathcal{B}^0$.

Suppose now that $b \in S$ and $b \neq 0$. Then $bb^{-1} \neq 0$ and $bb^{-1} = a^{-1}c$ for some $a, c \in S \setminus \{0\}$. Since $aa^{-1} \neq 0$ and $\mathcal{B}^0$ is categorical at zero, we have $ab \neq 0$. We deduce that $Sb \neq \{0\}$.

Let $\phi : S \to T$ be as given. Let $a, b \in S \setminus \{0\}$ with $a \mathcal{R} b$ in $\mathcal{B}^0$. Then there exists $c \in S$ with $ca \neq 0$ and hence $cb \neq 0$. It follows that $(c\phi)(a\phi), (c\phi)(b\phi)$ are non-zero in $\mathcal{B}^0_1$, so that $a\phi \mathcal{R} b\phi$ in $\mathcal{B}^0_1$.

We also show that ϕ preserves $\mathcal{L}$; for if $a, b \in S \setminus \{0\}$ and $a \mathcal{L} b$ in $\mathcal{B}^0$, then $ba^{-1} \neq 0$, whence $ba^{-1} = u^{-1}v$ for some $u, v \in S \setminus \{0\}$. It follows that $ub = va \neq 0$ and so $(u\phi)(b\phi) = (v\phi)(a\phi) \neq 0$ in T. Consequently, $a\phi \mathcal{L} b\phi$ in $\mathcal{B}^0_1$.

It remains to show that ϕ preserves $\mathcal{T}_S^{\mathcal{B}^0}$. Suppose therefore that $a, b, c \in S$ and $ab^{-1}\mathcal{B}^0 \subseteq c^{-1}\mathcal{B}^0$. Then either $ab^{-1} = 0$, or $ab^{-1} \mathcal{R} c^{-1}$ in $\mathcal{B}^0$. In the former case, a and b are not $\mathcal{L}$-related in $\mathcal{B}^0$ and so by the previous paragraph (applying the argument to ϕ^{-1}), $a\phi$ and $b\phi$ are not $\mathcal{L}$-related in $\mathcal{B}^0_1$, giving $(a\phi)(b\phi)^{-1} = 0$ and so $(a\phi)(b\phi)^{-1}\mathcal{B}^0_1 \subseteq (c\phi)^{-1}\mathcal{B}^0$. On the other hand, if $ab^{-1} \neq 0$, then we have $a \mathcal{L} b$ and $a \mathcal{R} c^{-1}$ in $\mathcal{B}^0$. It follows that $ca \neq 0$ and so $a\phi \mathcal{L} b\phi$ and $(c\phi)(a\phi) \neq 0$ in $\mathcal{B}^0_1$. Consequently,

$$(a\phi)(b\phi)^{-1}\mathcal{B}^0 = (a\phi)\mathcal{B}^0 = (c\phi)^{-1}\mathcal{B}^0_1.$$

Since ϕ (and, dually, ϕ^{-1}) preserve both $\mathcal{R}$ and $\mathcal{T}$, it follows from Corollary 2.10 that ϕ lifts to an isomorphism $\overline{\phi} : \mathcal{B}^0 \to \mathcal{B}^0_1$. $\square$

A characterisation of left I-orders in Brandt semigroups appears as a consequence of the study of left I-orders in primitive inverse semigroups in [10]. The statement of the result in the case of Brandt semigroups was also communicated to the second author by A. Cegarra [1].

3. Inverse hulls of left ample semigroups

Let S be a left ample semigroup. Where convenient we identify S with its image under θ in $\Sigma(S)$. We begin with four simple but useful observations.

Remark 3.1. First observe that for any $a, b \in S$,

$$\begin{aligned} \operatorname{dom} \rho_a^{-1}\rho_b &= (\operatorname{im} \rho_a^{-1} \cap \operatorname{dom} \rho_b)(\rho_a^{-1})^{-1} \\ &= (\operatorname{dom} \rho_a \cap \operatorname{dom} \rho_b)\rho_a \\ &= (Sa^+ \cap Sb^+)\rho_a \\ &= (Sa^+b^+)\rho_a \\ &= Sb^+a \end{aligned}$$

and $\operatorname{im} \rho_a^{-1}\rho_b = (Sb^+a)\rho_a^{-1}\rho_b = Sb^+a^+\rho_b = Sa^+b$, and for any $yb^+a \in Sb^+a$,

$$(yb^+a)\rho_a^{-1}\rho_b = (yb^+a^+)\rho_b = ya^+b.$$

It follows that if $a \mathcal{R}^* b$, then

$$\operatorname{dom} \rho_a^{-1}\rho_b = Sa,\ \operatorname{im} \rho_a^{-1}\rho_b = Sb \text{ and } (ya)\rho_a^{-1}\rho_b = yb.$$

Remark 3.2. We also observe that for any $a, b \in S$,

$$\begin{aligned} \rho_a \,\mathcal{L}\, \rho_b \text{ in } \Sigma(S) &\Leftrightarrow \operatorname{im}\rho_a = \operatorname{im}\rho_b \\ &\Leftrightarrow Sa = Sb \\ &\Leftrightarrow a \,\mathcal{L}\, b \text{ in } S. \end{aligned}$$

Remark 3.3. If $b, c \in S$ and $Sb \cap Sc = Sw$, where $ub = vc = w$ and $ub^+ = u, vc^+ = v$, then

$$\operatorname{dom}\rho_b\rho_c^{-1} = (\operatorname{im}\rho_b \cap \operatorname{dom}\rho_c^{-1})\rho_b^{-1} = (Sb \cap Sc)\rho_b^{-1} = Sw\rho_b^{-1} = (Sub)\rho_b^{-1} = Sub^+ = Su,$$

and for any $su \in Su$,

$$(su)\rho_b\rho_c^{-1} = (sub)\rho_c^{-1} = (svc)\rho_c^{-1} = svc^+ = sv,$$

so that in particular, $\operatorname{im}\rho_b\rho_c^{-1} = Sv$. Notice that

$$u = ub^+ \,\mathcal{R}^*\, ub = vc \,\mathcal{R}^*\, vc^+ = v.$$

It follows from Remark 3.1 that $\rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v$.

It is known [3], although our terminology is new, that a right cancellative monoid is a left I-order in its inverse hull if and only if it satisfies Condition (LC), which we now define for arbitrary semigroups.

Definition 3.4. We say that a semigroup S satisfies Condition (LC) if for any $a, b \in S$, there exists $c \in S$ with $Sa \cap Sb = Sc$.

Before proving the analogue of Clifford's result, we give two preliminary lemmas, of which we will make much use.

Lemma 3.5. *For any semigroup S, $\mathcal{R}^* \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}^*$.*

Proof. Let $a, b \in S$ with $a \,\mathcal{R}^* \circ \mathcal{L}\, b$. Then there exists an element $c \in S$ with $a \,\mathcal{R}^*\, c \,\mathcal{L}\, b$. Either $c = b$ in which case $a \,\mathcal{L}\, a \,\mathcal{R}^*\, b$ or there exist $u, v \in S$ with $c = ub$ and $b = vc$. Hence $c = ub = uvc$ so that as $a \,\mathcal{R}^*\, c$ we deduce $a = uva$ and thus $a \,\mathcal{L}\, va$. But $va \,\mathcal{R}^*\, vc = b$ so that $a \,\mathcal{L} \circ \mathcal{R}^* b$ and $\mathcal{R}^* \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}^*$. The proof of the dual inclusion is very similar. □

Lemma 3.6. *Let S be a left ample semigroup that is a left I-order in an inverse semigroup Q, such that S is a union of $\mathcal{R}$-classes of Q. Then*

(i) S is a $(2,1)$-subalgebra of Q;
(ii) for $a, b \in S$ with $a \,\mathcal{R}^*\, b$, $a^{-1}b$ is idempotent if and only if $a = b$;
(iii) for any $a, b \in S$, $Sa \subseteq Sb$ if and only if $Qa \subseteq Qb$;
(iv) for any $a, b, c \in S$, $Sa \cap Sb = Sc$ if and only if $Qa \cap Qb = Qc$;
(v) S satisfies Condition (LC);
(vi) Q is bisimple if and only if

$$\mathcal{L}^S \circ \mathcal{R}^* = S \times S;$$

and

(vii) Q is simple if and only if for all $a, b \in S$ there exists $c \in S$ with

$$a\,\mathcal{R}^*\,c \leq^S_{\mathcal{L}} b.$$

Proof. (i) We need only show that if $a \in S$, then $aa^{-1} = a^+$. We have that $a\,\mathcal{R}^Q\,aa^{-1}$ and S is a union of $\mathcal{R}^Q$-classes, giving $aa^{-1} \in S$. As $a\,\mathcal{R}^*\,aa^{-1}$ we must have that $aa^{-1} = a^+$.

(ii) If $a^{-1}b$ is idempotent, then as $a^{-1}b\,\mathcal{R}^Q\,a^{-1}a$, we must have that $a^{-1}b = a^{-1}a$. Multiplying with a on the left gives $b = bb^{-1}b = aa^{-1}b = aa^{-1}a = a$. The converse is clear.

(iii) If $a, b \in S$ and $Sa \subseteq Sb$, then clearly $Qa \subseteq Qb$. On the other hand, if $Qa \subseteq Qb$, then we have that $a = h^{-1}kb$ for some $h, k \in S$. It follows that $a = ((kb)^+h)^{-1}h^+kb$ and

$$((kb)^+h)^{-1}\,\mathcal{R}^Q\,((kb)^+h)^{-1}((kb)^+h)\,\mathcal{R}^Q\,((kb)^+h)^{-1}h^+kb = a,$$

so that as S is a union of $\mathcal{R}^Q$-classes, $((kb)^+h)^{-1} \in S$. It follows that $Sa \subseteq Sb$.

(iv) Suppose that $a, b \in S$ and $Sa \cap Sb = Sc$. Then $c \in Sa \cap Sb \subseteq Qa \cap Qb$, so that $Qc \subseteq Qa \cap Qb$. Conversely, if $h^{-1}ka = u^{-1}vb \in Qa \cap Qb$, where $h, k, u, v \in S$, $h\,\mathcal{R}\,k$ and $u\,\mathcal{R}\,v$ in Q, then $ka = hu^{-1}vb$ and $hu^{-1} = s^{-1}t$ say, where $s, t \in S$ and $s\,\mathcal{R}\,t$ in Q. This gives that

$$ska = tvb \in Sa \cap Sb = Sc,$$

and so $ska = tvb = xc$, where $x \in S$. Now

$$ka = hu^{-1}vb = s^{-1}tvb = s^{-1}xc$$

and then $h^{-1}ka = h^{-1}s^{-1}xc \in Qc$. Hence $Qa \cap Qb \subseteq Qc$ so that $Qa \cap Qb = Qc$.

Conversely, suppose that $a, b \in S$ and $Qa \cap Qb = Qc$. From $Qc \subseteq Qa$ and $Qc \subseteq Qb$, (iii) gives that $Sc \subseteq Sa \cap Sb$. On the other hand, if $u = xa = yb \in Sa \cap Sb$ for some $x, y \in S$, then $u = qc$ for some $q \in Q$, whence $Qu \subseteq Qc$. Again from (iii), $Su \subseteq Sc$ so that $Sa \cap Sb \subseteq Sc$ and we have $Sa \cap Sb = Sc$ as required.

(v) Let $a, b \in S$. Then

$$Qa \cap Qb = Qa^{-1}a \cap Qb^{-1}b = Qa^{-1}ab^{-1}b = Qab^{-1}b,$$

but $ab^{-1} = s^{-1}t$ for some $s, t \in S$ with $s\,\mathcal{R}\,t$ in Q, and so

$$Qa \cap Qb = Qs^{-1}tb = Qtb.$$

From (iv) we now have that $Sa \cap Sb = Stb$ and S has Condition (LC).

(vi) We have observed in Lemma 2.4 that S is straight in Q. Let $a^{-1}b, c^{-1}d \in Q$, where $a\,\mathcal{R}^*\,b$ and $c\,\mathcal{R}^*\,d$. Then

$$\begin{aligned}
a^{-1}b\,\mathcal{D}^Q\,c^{-1}d \;&\Leftrightarrow\; a^{-1}b\,\mathcal{R}^Q\,x^{-1}y\,\mathcal{L}^Q\,c^{-1}d \text{ for some } x, y \in S \text{ with } x\,\mathcal{R}^*\,y\\
&\Leftrightarrow\; a^{-1}\,\mathcal{R}^Q\,x^{-1} \text{ and } y\,\mathcal{L}^Q\,d \text{ for some } x, y \in S \text{ with } x\,\mathcal{R}^*\,y\\
&\Leftrightarrow\; a\,\mathcal{L}^Q\,x \text{ and } y\,\mathcal{L}^Q\,d \text{ for some } x, y \in S \text{ with } x\,\mathcal{R}^*\,y\\
&\Leftrightarrow\; a\,\mathcal{L}^S\,x\,\mathcal{R}^*\,y\,\mathcal{L}^S\,d \text{ for some } x, y \in S.
\end{aligned}$$

It follows that Q is bisimple if and only if $\mathcal{L}^S \circ \mathcal{R}^* \circ \mathcal{L}^S$ is universal. But from Lemma 3.5, $\mathcal{L}$ and $\mathcal{R}^*$ commute on S, so that Q is bisimple if and only if $\mathcal{L}^S \circ \mathcal{R}^* = S \times S$.

(vii) Since Q is inverse, it follows from [4, Theorem 8.33] that Q is simple if and only if for any $e, f \in E(Q)$, there is an element $q \in Q$ with $e = qq^{-1}$ and $q^{-1}q \leq f$. Let $e, f \in Q$ so that by (ii), $e = a^{-1}a, f = b^{-1}b$ for some $a, b \in S$. Then Q is simple if and only if there exists $q = c^{-1}d$ (where $u, v \in S$ and $c\,\mathcal{R}^*\,d$) such that

$$e = qq^{-1} = c^{-1}c \text{ and } d^{-1}d = q^{-1}q \leq f.$$

It follows that Q is simple if and only if for any $a, b \in S$ there exist $c, d \in S$ with $c\,\mathcal{R}^*\,d$ such that $Sa = Sc$ and $Sd \subseteq Sb$. Again using Lemma 3.5, we obtain the given condition. $\square$

We can now extend from right cancellative monoids to left ample semigroups the classic result for inverse hulls.

Theorem 3.7. *Let S be a left ample semigroup. Then $S\theta_S$ is a left I-order in its inverse hull if and only if S has Condition (LC).*

If Condition (LC) holds, then $S\theta_S$ is a union of $R^{\Sigma(S)}$-classes.

Proof. Suppose that S is a left I-order in $\Sigma(S)$. The for any $b, c \in S$, $\rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v$ where $u\,\mathcal{R}^*\,v$. By Remark 3.1 $\mathrm{dom}(\rho_b\rho_c^{-1}) = Su$, so that

$$Su = (\mathrm{im}\,\rho_b \cap \mathrm{dom}\,\rho_c^{-1})\rho_b^{-1} = (Sb \cap Sc)\rho_b^{-1}.$$

But $\rho_b^{-1}\rho_b$ is the identity on $Sb = \mathrm{im}\,\rho_b$, and so

$$Sub = (Su)\rho_b = (Sb \cap Sc)\rho_b^{-1}\rho_b = Sb \cap Sc,$$

and S has Condition (LC).

Conversely, suppose that S has Condition (LC). Let

$$Q = \{\rho_a^{-1}\rho_b : a, b \in S\} \subseteq \Sigma(S).$$

Observe that for any $a \in S$, $\rho_a = \rho_{a^+a} = \rho_{a^+}^{-1}\rho_a$, so that $S\theta_S \subseteq Q$.

Consider $b, c \in S$. By Condition (LC), there exist $u, v \in S$ with $Sb \cap Sc = Sub$ and $ub = vc$ with $ub^+ = u$ and $vc^+ = v$. By Remark 3.3, $\rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v$.

It follows that if $\rho_a^{-1}\rho_b, \rho_c^{-1}\rho_d \in Q$, then

$$(\rho_a^{-1}\rho_b)(\rho_c^{-1}\rho_d) = \rho_a^{-1}(\rho_b\rho_c^{-1})\rho_d = \rho_a^{-1}(\rho_u^{-1}\rho_v)\rho_d = (\rho_u\rho_a)^{-1}(\rho_v\rho_d) = \rho_{ua}^{-1}\rho_{vd},$$

so that Q is closed under multiplication. Clearly Q is closed under taking inverses, so that $\Sigma(S) \subseteq Q$ from definition of inverse hull, and so $Q = \Sigma(S)$ as required.

Finally, if $e \in E(S)$ and $\rho_e\,\mathcal{R}^{\Sigma(S)}\,\rho_a^{-1}\rho_b$, where $a, b \in S$ and $a\,\mathcal{R}^*\,b$, then $\mathrm{dom}\,\rho_e = \mathrm{dom}\,\rho_a^{-1}\rho_b$, so that $Se = Sa$ and a is regular in S. Any inverse c of a in S must be such that ρ_c is the unique inverse of ρ_a in Q, so that $\rho_a^{-1} \in S\theta_S$ and hence $\rho_a^{-1}\rho_b \in S\theta_S$. $\square$

Corollary 3.8. *The following conditions are equivalent for a left ample semigroup S:*
(i) $\Sigma(S)$ is bisimple;
(ii) S has Condition (LC) and $\mathcal{R}^* \circ \mathcal{L} = S \times S$;
(iii) S is a left I-order in $\Sigma(S)$ and $\mathcal{R}^* \circ \mathcal{L} = S \times S$.

Proof. We recall that the embedding of S into $\Sigma(S)$ is via what, in the terminology of [16], are called one-one *partial right translations*. It follows that $\Sigma(S)$ is an inverse subsemigroup of the inverse semigroup $\hat{S}$ of one-one partial right translations. Thus for any $\alpha \in \Sigma(S)$, $\operatorname{dom}\alpha$ is a left ideal and for any $a \in \operatorname{dom}\alpha$ and $x \in S$, $(xa)\alpha = x(a\alpha)$.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are immediate from Lemma 3.6 and Theorem 3.7.

$(i) \Rightarrow (ii)$. Suppose that $\Sigma(S)$ is bisimple, and let $e \in E(S)$. For any $\alpha \in \Sigma(S)$, we know that $\alpha \,\mathcal{D}\, \rho_e$, so that $\alpha \,\mathcal{R}\, \beta \,\mathcal{L}\, \rho_e$ in $\Sigma(S)$. Then $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $\operatorname{im}\beta = Se$. It follows that $\operatorname{dom}\beta = S(e\beta^{-1})$ so that $\operatorname{dom}\alpha$ is principal. Now let $a, b \in S$; then
$$\operatorname{dom}(\rho_a\rho_b^{-1}) = (Sa \cap Sb)\rho_a^{-1} = Sw$$
for some $w \in S$, and so
$$Swa = Sa \cap Sb$$
and S has (LC). From Theorem 3.7, $\mathcal{R}^* \circ \mathcal{L}$ is universal on S. $\square$

We recall that a left ample semigroup S is *proper* if $\mathcal{R}^* \cap \sigma = \iota$, where σ is the least right cancellative congruence on S, and where σ is given by the formula that for any $a, b \in S$,
$$a \,\sigma\, b \Leftrightarrow ea = eb \text{ for some } e \in E(S).$$
Clearly, if S is a subsemigroup in an inverse semigroup Q, then if $a \,\sigma\, b$ in S, we have that $a \,\sigma\, b$ in Q, but the converse may not be true. In other words, there is a natural morphism from S/σ to Q/σ, but this may not be an embedding.

Theorem 3.9. *Let S be a left ample semigroup such that S is a left I-order in Q where S is a union of $\mathcal{R}$-classes of Q. Then the following conditions are equivalent:*
(i) Q is E-unitary;
(ii) S is proper and S/σ embeds naturally in Q/σ;
(iii) S is proper and S/σ is cancellative.

Proof. $(i) \Rightarrow (ii)$ Suppose that Q is E-unitary, and $a, b \in S$ are such that $a \,\sigma\, b$ *in* Q. Then $ea = eb$ for some $e \in Q$, so that $eb^+a = ea^+b$. But $b^+a \,\mathcal{R}^Q\, a^+b$ and so $b^+a = a^+b$. This gives that $a \,\sigma\, b$ *in* S.

Clearly, if $a, b \in S$ and $a\,(\mathcal{R}^* \cap \sigma)\,b$ in S, then $a\,(\mathcal{R} \cap \sigma)\,b$ in Q, whence $a = b$ and S is proper.

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (i)$ Let $a^{-1}b, c^{-1}d \in Q$, where $a,b,c,d \in S$, $a\,\mathcal{R}^*\,b$ and $c\,\mathcal{R}^*\,d$. Suppose that $a^{-1}b\,(\mathcal{R}\cap\sigma)\,c^{-1}d$ in Q. Then there exists $x \in S$ such that

$$x^{-1}xa^{-1}b = x^{-1}xc^{-1}d$$

and $a^{-1}b\,\mathcal{R}^Q\,c^{-1}d$. From the former, $xa^{-1}b = xc^{-1}d$ and from the latter, $a\,\mathcal{L}\,c$ in Q. Hence $a\,\mathcal{L}\,c$ in S and so there exist $u, v \in S$ with $a = uc$ and $c = va$. We may choose u, v such that $a^+u = u$ and $c^+v = v$. Now $a = uc = uva$ so that $a^+ = uva^+$, whence $u = a^+u = uva^+u = uvu$. Similarly, $v = vuv$, so that u and v are mutually inverse in both S and Q.

From $xa^{-1}b = xc^{-1}d$ we have that

$$xa^{-1}b = x(va)^{-1}d = xa^{-1}v^{-1}d = xa^{-1}ud.$$

But $xa^{-1}\,\mathcal{L}^Q\,y$ for some $y \in S$, so that $yb = yud$ and as S/σ is cancellative, $b\,\sigma\,ud$ in S. Also, $b\,\mathcal{R}^*\,a = uc\,\mathcal{R}^*\,ud$ so that as S is proper, $b = ud$. Now

$$a^{-1}b = a^{-1}ud = a^{-1}v^{-1}d = (va)^{-1}d = c^{-1}d$$

and Q is E-unitary as required.

□

We remark that if the conditions of Theorem 3.9 hold, then for any $q = [a^{-1}b] \in Q/\sigma$, we have that $q = [a]^{-1}[b]$ and so the cancellative monoid S/σ is a left order in the group Q/σ.

The following result is classic; most of it follows from Theorem 3.7 and Corollary 3.8.

Corollary 3.10. [3, 17, 16]. *The following conditions are equivalent for a right cancellative monoid S:*

(i) $\Sigma(S)$ is bisimple;

(ii) S has Condition (LC);

(iii) S is a left I-order in $\Sigma(S)$.

If the above conditions hold, then S is the $\mathcal{R}$-class of the identity of Q. Further, $\Sigma(S)$ is E-unitary if and only if S is cancellative.

Conversely, the $\mathcal{R}$-class of the identity of any bisimple inverse monoid is right cancellative with Condition (LC).

Proof. The equivalence of $(i), (ii)$ and (iii) follows from Corollary 3.8 and the fact that $\mathcal{R}^*$ is universal on S.

Suppose that $(i), (ii)$ and (iii) hold. Let e be the identity of S. As remarked in Section 1, $\Sigma(S)$ is a monoid with identity e. Since S is a single $\mathcal{R}^*$-class, and the embedding of S into $\Sigma(S)$ is a $(2,1)$-embedding, we have $S \subseteq R_e^{\Sigma(S)}$. Again by Theorem 3.7, we have that $R_e^{\Sigma(S)} \subseteq S$, so that $S = R_e^{\Sigma(S)}$.

Since $\sigma = \iota$ on S, it is clear that S is proper and $S/\sigma \cong S$. From Theorem 3.9 $\Sigma(S)$ is E-unitary if and only if S is cancellative.

Conversely, let R be the $\mathcal{R}$-class of the identity of a bisimple inverse monoid Q. It is easy to see that R is a right cancellative monoid, and from a comment in Section 1, we have that R is a left I-order in Q. Lemma 3.6 tells us that R has (LC). $\square$

We now give a promised simplification of Theorem 2.9. First, we say that a $(2,1)$-morphism $\phi : S \to T$, where S and T are left ample semigroups with Condition (LC) is *(LC)-preserving* if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that

$$T(b\phi) \cap T(c\phi) = T(w\phi).$$

This condition is not new: it appeared originally in [21] for right cancellative monoids with (LC), where it was called an *sl homomorphism* and subsequently (or variations thereof, and under different names) in, for example, [9] and [16]. Using the fact that for idempotents e, f of an inverse semigroup Q, we have that $Qe \cap Qf = Qef$, it is easy to verify that any morphism between inverse semigroups is (LC)-preserving.

The following result was first proved in the special case of S and T being right cancellative in [21].

Theorem 3.11. *Let S and T be left ample semigroups with Condition (LC) and let Q and P be their inverse hulls. Suppose that $\phi : S \to T$ is a $(2,1)$-morphism. Then ϕ lifts to a morphism $\overline{\phi} : Q \to P$ if and only if ϕ is (LC)-preserving.*

Proof. For ease in this proof we identify S and T with $S\theta_S$ and $T\theta_T$, respectively. We have remarked that any such ϕ preserves $\mathcal{R}^*$, and since $(\mathcal{R}^*)^S = \mathcal{R}^Q \cap (S \times S)$ and $(\mathcal{R}^*)^T = \mathcal{R}^P \cap (T \times T)$, (i) of Theorem 2.9 holds. It remains to show that (ii) of that theorem holds if and only if ϕ is (LC)-preserving.

Suppose first that ϕ is (LC)-preserving. If $(a, b, c) \in \mathcal{T}_S^Q$, then $ab^{-1}Q \subseteq c^{-1}Q$. Now S has (LC) so that $Sa \cap Sb = Sw$ for some $w \in S$ and $ua = vb = w$ for some $u, v \in S$ with $ua^+ = u$ and $vb^+ = v$. From Remark 3.3, $ab^{-1} = u^{-1}v$ and $u\,\mathcal{R}^*\,v$. Hence $u^{-1}vQ \subseteq c^{-1}Q$ so that $Su \subseteq Sc$ from Lemma 3.6. Clearly, $T(u\phi) \subseteq T(v\phi), u\phi a\phi = v\phi b\phi = w\phi, u\phi(a\phi)^+ = u\phi$ and $v\phi(b\phi)^+ = v\phi$. As ϕ is (LC)-preserving, $T(a\phi) \cap T(b\phi) = T(w\phi)$ whence $a\phi(b\phi)^{-1} = (u\phi)^{-1}v\phi$ and it follows that $(a\phi, b\phi, c\phi) \in \mathcal{T}_T^P$.

Conversely, suppose that (ii) of Theorem 2.9 holds, so that ϕ lifts to a morphism $\overline{\phi} : Q \to P$. Supppose that $b, c \in S$ and $Sb \cap Sc = Sw$. We have $ub = vc = w$ for some $u, v \in S$ with $ub^+ = u, vc^+ = v$ and $u\,\mathcal{R}^*\,v$. This gives that $bc^{-1} = u^{-1}v$ and so, applying $\overline{\phi}$, $b\phi(c\phi)^{-1} = (u\phi)^{-1}v\phi$. As T has (LC), we certainly have that $b\phi(c\phi)^{-1} = h^{-1}k$ for some $h, k \in T$ with $h(b\phi) = k(c\phi) = z, T(b\phi) \cap T(c\phi) = Tz$ and $h\,\mathcal{R}^*\,k$ in T. From Lemma 3.6, $u\phi\,\mathcal{L}\,h$ in T, so that $w\phi = (ub)\phi = u\phi\, b\phi\,\mathcal{L}\,h(b\phi) = z$ in T. We now have that

$$T(b\phi) \cap T(c\phi) = Tz = T(w\phi)$$

and ϕ is (LC)-preserving. $\square$

The above result could (via a series of intermediate steps) be deduced from Theorem 2.6 of [16]. For, the ample condition ensures that a left ample semigroup is embedded in the semigroup $\hat{S}$ of one-to-one partial right translations of S via the right regular representation described in Section 1. Further, the image of S is contained in $J(S)$, the set of join irreducible elements of $\hat{S}$. By [16, Proposition 1.14], if S has (LC), then $J(S)$ is an inverse semigroup, which is isomorphic to our $\Sigma(S)$. The restriction of θ in [16, Theorem 2.6] to $J(S)$, with a slight adaptation of the notion of *permissible homomorphism*, will now give our Theorem 3.11.

4. Semilattices of inverse semigroups

We begin by setting up our notation. Let Y be a semilattice and let S be a semigroup such that S is the disjoint union of subsemigroups $S_\alpha, \alpha \in Y$, and is such that for any $\alpha, \beta \in Y$, $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Then we say that S is a *semilattice* Y *of subsemigroups* $S_\alpha, \alpha \in Y$ and write $S = \mathcal{S}\big(Y; S_\alpha\big)$. *We make the convention that if we write* $x_\alpha \in S$ *(for any symbol* x *and any* $\alpha \in Y$*), then we mean that* $x_\alpha \in S_\alpha$.

If there exists a set of morphisms $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ for $\alpha \geq \beta$ such that

(i) $\phi_{\alpha,\alpha} = I_{S_\alpha}$ for all $\alpha \in Y$;

and

(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, such that the binary operation in S is given by the rule that

$$a_\alpha b_\beta = (a_\alpha \phi_{\alpha,\alpha\beta})(b_\beta \phi_{\beta,\alpha\beta}),$$

where the last product is taken in $S_{\alpha\beta}$, then we say that S is a *strong semilattice* Y *of semigroups* $S_\alpha, \alpha \in Y$*, with connecting morphisms* $\phi_{\alpha,\beta}, \alpha \geq \beta$ and write $S = \mathcal{S}\big(Y; S_\alpha; \phi_{\alpha,\beta}\big)$.

Let $Q = \big(Y; Q_\alpha; \psi_{\alpha,\beta}\big)$ be a strong semilattice of *bisimple inverse monoids* $Q_\alpha, \alpha \in Y$, such that the connecting morphisms are monoid morphisms. It follows that the set E of identities $E = \{e_\alpha : \alpha \in Y\}$ forms a subsemigroup, indeed a semilattice isomorphic to Y. Let R_α denote the $\mathcal{R}$-class of e_α in Q_α. Then it is easy to see that $S = \mathcal{S}\big(Y; R_\alpha; \phi_{\alpha,\beta}\big)$ is a strong semilattice of right cancellative monoids, where $\phi_{\alpha,\beta} = \psi_{\alpha,\beta}|_{R_\alpha}$. In [9], Gantos showed how to recover the structure of Q from that of S; in our terminology, Q is a semigroup of left I-quotients of S and the morphisms $\phi_{\alpha,\beta}$ satisfy Condition (LC).

In this section we revisit and generalise Gantos's result. We believe that its correct context is that of inverse hulls of left ample semigroups, and we show that his result can be naturally extended to strong semilattices of left ample semigroups. Gantos uses an explicit construction of quotients involving ordered pairs subject to an equivalence relation - we avoid all such technicalities by using our results concerning lifting of morphisms.

We first observe that the 'strong' in Gantos's result is automatic. The proof of the following is entirely routine, but we provide it for completeness.

Lemma 4.1. *Let $P = \mathcal{S}\big(Y; M_\alpha\big)$ where each M_α is a monoid with identity e_α, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of P. Then E is a semilattice isomorphic to Y and E is central in P.*

If we define $\phi_{\alpha,\beta} : M_\alpha \to M_\beta$ by $a_\alpha\phi_{\alpha,\beta} = a_\alpha e_\beta$, where $\alpha \geq \beta$, then each $\phi_{\alpha,\beta}$ is a monoid morphism, and $P = \mathcal{S}\big(Y; M_\alpha; \phi_{\alpha,\beta}\big)$.

Proof. Let $a_\alpha \in M_\alpha$ and suppose first that $\alpha \geq \beta$. Then

$$a_\alpha e_\beta = e_\beta(a_\alpha e_\beta) = (e_\beta a_\alpha)e_\beta = e_\beta a_\alpha.$$

Now, for arbitrary e_γ,

$$a_\alpha e_\gamma = (a_\alpha e_\gamma)e_{\alpha\gamma} = a_\alpha(e_\gamma e_{\alpha\gamma}) = a_\alpha e_{\alpha\gamma} = e_{\alpha\gamma}a_\alpha = (e_{\alpha\gamma}e_\gamma)a_\alpha = e_{\alpha\gamma}(e_\gamma a_\alpha) = e_\gamma a_\alpha,$$

so that E is central in P.

It is easy to see that for $\alpha \geq \beta$, $\phi_{\alpha,\beta} : M_\alpha \to M_\beta$ is a monoid morphism, $\phi_{\alpha,\alpha} = I_{M_\alpha}$ and for $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta}\phi_{\beta,\gamma}$. Let $Q = \mathcal{S}\big(Y; M_\alpha; \phi_{\alpha,\beta}\big)$ and denote the binary operation in Q by $*$.

For $a_\alpha, b_\beta \in M$ we have

$$a_\alpha * b_\beta = (a_\alpha\phi_{\alpha,\alpha\beta})(b_\beta\phi_{\beta,\alpha\beta}) = (a_\alpha e_{\alpha\beta})(b_\beta e_{\alpha\beta}) = (a_\alpha b_\beta)e_{\alpha\beta} = a_\alpha b_\beta,$$

as required. $\square$

Proposition 4.2. *Let $S = \mathcal{S}\big(Y; S_\alpha; \phi_{\alpha,\beta}\big)$, where each S_α is left ample and the connecting morphisms are $(2,1)$-morphisms.*

(i) The semigroup S is left ample, and for any $a, b \in S$, $a\,\mathcal{R}^*\,b$ in S if and only if $a, b \in S_\alpha$ for some $\alpha \in Y$ and $a\,\mathcal{R}^*\,b$ in S_α.

(ii) If each S_α has (LC), then S has (LC) if and only if every $\phi_{\alpha,\beta}, \alpha \geq \beta$, is (LC)-preserving.

Proof. (i) Let $f_\alpha, g_\beta \in E(S)$; then

$$f_\alpha g_\beta = (f_\alpha\phi_{\alpha,\alpha\beta})(g_\beta\phi_{\beta,\alpha\beta}) = (g_\beta\phi_{\beta,\alpha\beta})(f_\alpha\phi_{\alpha,\alpha\beta}) = g_\beta f_\alpha,$$

using the fact that $E(S_{\alpha\beta})$ is a semilattice. Thus $E(S)$ is a semilattice.

Suppose now that $a_\alpha\,(\mathcal{R}^*)^S\,b_\beta$. Let f_α be the idempotent in the $(\mathcal{R}^*)^{S_\alpha}$-class of a_α. Then as $f_\alpha a_\alpha = a_\alpha$ we must also have that $f_\alpha b_\beta = b_\beta$ so that $\beta \leq \alpha$. With the dual we obtain that $\alpha = \beta$; clearly, then $a_\alpha\,(\mathcal{R}^*)^{S_\alpha}\,b_\alpha$.

Conversely, suppose that $a_\alpha\,(\mathcal{R}^*)^{S_\alpha}\,b_\alpha$ and $x_\gamma a_\alpha = y_\delta a_\alpha$. Then $\gamma\alpha = \delta\alpha = \mu$, say, and $(x_\gamma\phi_{\gamma,\mu})(a_\alpha\phi_{\alpha,\mu}) = (y_\delta\phi_{\delta,\mu})(a_\alpha\phi_{\alpha,\mu})$. But $\phi_{\alpha,\mu}$ is a $(2,1)$-morphism, and $a_\alpha\,(\mathcal{R}^*)^{S_\alpha}\,b_\alpha$, so that $a_\alpha\phi_{\alpha,\mu}\,(\mathcal{R}^*)^{S_\mu}\,b_\alpha\phi_{\alpha,\mu}$. We thus obtain that $(x_\gamma\phi_{\gamma,\mu})(b_\alpha\phi_{\alpha,\mu}) = (y_\delta\phi_{\delta,\mu})(b_\alpha\phi_{\alpha,\mu})$ and hence $x_\gamma b_\alpha = y_\delta b_\alpha$. Making an easy adjustment for $x_\gamma = 1$ yields that $a_\alpha\,(\mathcal{R}^*)^S\,b_\alpha$.

Notice that from the above, there is no ambiguity in the use of the superscript $^+$. To see that S is left ample, let $a_\alpha \in S$ and $f_\beta \in E(S)$. Then

$$(a_\alpha f_\beta)^+a_\alpha = ((a_\alpha\phi_{\alpha,\alpha\beta})(f_\beta\phi_{\beta,\alpha\beta}))^+(a_\alpha\phi_{\alpha,\alpha\beta}) = (a_\alpha\phi_{\alpha,\alpha\beta})(f_\beta\phi_{\beta,\alpha\beta}),$$

using the fact that $S_{\alpha\beta}$ is left ample, so that $(a_\alpha f_\beta)^+a_\alpha = a_\alpha f_\beta$ as required.

(*ii*) Suppose that each S_α has (LC).

Assume first that each $\phi_{\alpha,\beta}$ is (LC)-preserving. Let $a_\alpha, b_\beta \in S$ and let $\gamma = \alpha\beta$. As S_γ has (LC) we know that

$$S_\gamma a_\alpha \cap S_\gamma b_\beta = S_\gamma(a\phi_{\alpha,\gamma}) \cap S_\gamma(b\phi_{\beta,\gamma}) = S_\gamma c_\gamma,$$

for some c_γ. We claim that $Sa_\alpha \cap Sb_\beta = Sc_\gamma$.

Certainly $c_\gamma = x_\gamma a_\alpha = y_\gamma b_\beta$ for some $x_\gamma, y_\gamma \in S_\gamma$, so that $c_\gamma \in Sa_\alpha \cap Sb_\beta$ and so

$$Sc_\gamma \subseteq Sa_\alpha \cap Sb_\beta.$$

On the other hand, let $d \in Sa_\alpha \cap Sb_\beta$; then there are elements $u_\mu, v_\nu \in S$ with $d = u_\mu a_\alpha = v_\nu b_\beta$. Let $\tau = \mu\alpha = \nu\beta$, so that $\tau \leq \gamma$. Then

$$d = d_\tau = d_\tau^+ d_\tau = (d_\tau^+ u_\mu)a_\alpha = (d_\tau^+ v_\nu)b_\beta \in S_\tau a_\alpha \cap S_\tau b_\beta.$$

Now $\phi_{\gamma,\tau}$ is (LC)-preserving, so that $S_\tau a_\alpha \cap S_\tau b_\beta = S_\tau c_\gamma$. This gives that $d = z_\tau c_\gamma \in Sc_\gamma$. Hence $Sa_\alpha \cap Sb_\beta = Sc_\gamma$ as required.

Conversely, assume that S has (LC) and suppose that $\alpha \geq \beta$; we must show that $\phi_{\alpha,\beta}$ is (LC)-preserving.

We first show that for any $a_\alpha, b_\alpha, c_\alpha \in S$,

$$S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha \Leftrightarrow Sa_\alpha \cap Sb_\alpha = Sc_\alpha.$$

($\Leftarrow$) If $Sa_\alpha \cap Sb_\alpha = Sc_\alpha$, we have that

$$c_\alpha = ua_\alpha = vb_\alpha = (ua_\alpha^+)a_\alpha = (vb_\alpha^+)b_\alpha \in S_\alpha a_\alpha \cap S_\alpha b_\alpha,$$

so that $S_\alpha c_\alpha \subseteq S_\alpha a_\alpha \cap S_\alpha b_\alpha$. On the other hand, if $x_\alpha, y_\alpha \in S_\alpha$ and

$$x_\alpha a_\alpha = y_\alpha b_\alpha \in S_\alpha a_\alpha \cap S_\alpha b_\alpha \subseteq Sa_\alpha \cap Sb_\alpha,$$

then

$$x_\alpha a_\alpha = y_\alpha b_\alpha = zc_\alpha = (zc_\alpha^+)c_\alpha \in S_\alpha c_\alpha.$$

Thus $S_\alpha a_\alpha \cap S_\alpha b_\alpha \subseteq S_\alpha c_\alpha$ and we have $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ as desired.
($\Rightarrow$) Conversely, suppose that $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$. We also know that $Sa_\alpha \cap Sb_\alpha = Sd_\beta$ for some $d_\beta \in S$. As $d_\beta \in Sa_\alpha$ we have that $\beta \leq \alpha$, but $c_\alpha \in Sd_\beta$, so that $\alpha = \beta$. From ($\Leftarrow$) we have that $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha d_\alpha$, so that $c_\alpha \,\mathcal{L}\, d_\alpha$ in S_α and hence in S. Consequently, $Sa_\alpha \cap Sb_\alpha = Sc_\alpha$.

We now return to the argument that $\phi_{\alpha,\beta}$ is (LC)-preserving (for $\alpha \geq \beta$). Suppose that $a_\alpha, b_\alpha, c_\alpha \in S$ and $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$. We know that $c_\alpha = x_\alpha a_\alpha = y_\alpha b_\alpha$ for some x_α, y_α, so that $c_\alpha\phi_{\alpha,\beta} = (x_\alpha\phi_{\alpha,\beta})(a_\alpha\phi_{\alpha,\beta}) = (y_\alpha\phi_{\alpha,\beta})(b_\alpha\phi_{\alpha,\beta})$, giving that

$$c_\alpha\phi_{\alpha,\beta} \in S_\beta(a_\alpha\phi_{\alpha,\beta}) \cap S_\beta(b_\alpha\phi_{\alpha,\beta}) = S_\beta d_\beta$$

for some d_β. From the above, $Sa_\alpha \cap Sb_\alpha = Sc_\alpha$. We have that for some u_β, v_β,

$$d_\beta = u_\beta(a_\alpha\phi_{\alpha,\beta}) = v_\beta(b_\alpha\phi_{\alpha,\beta}) = u_\beta a_\alpha = v_\beta b_\alpha \in Sa_\alpha \cap Sb_\alpha = Sc_\alpha$$

so that $d_\beta = z_\gamma c_\alpha = ((z_\gamma c_\alpha)^+ z_\gamma)(c_\alpha\phi_{\alpha,\beta}) \in S_\beta(c_\alpha\phi_{\alpha,\beta})$. It follows that

$$S_\beta(a_\alpha\phi_{\alpha,\beta}) \cap S_\beta(b_\alpha\phi_{\alpha,\beta}) = S_\beta d_\beta = S_\beta(c_\alpha\phi_{\alpha,\beta})$$

and $\phi_{\alpha,\beta}$ has (LC). $\square$

We can now give the main result of this section.

Theorem 4.3. *Let $S = \mathcal{S}\big(Y; S_\alpha; \phi_{\alpha,\beta}\big)$ be a strong semilattice of left ample semigroups S_α, such that the connecting morphisms are $(2,1)$-morphisms. Suppose that each $S_\alpha, \alpha \in Y$ has (LC) and that S has (LC).*

For each $\alpha \in Y$, let Q_α be the inverse hull of S_α. Then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, we have that $\phi_{\alpha,\beta}$ lifts to a morphism $\overline{\phi_{\alpha,\beta}} : Q_\alpha \to Q_\beta$. Further, $Q = \mathcal{S}\big(Y; Q_\alpha; \overline{\phi_{\alpha,\beta}}\big)$ is a strong semilattice of inverse semigroups, such that S is a straight left I-order in Q.

Moreover, Q is isomorphic to the inverse hull of S.

Proof. By Theorem 3.7, each $S_\alpha\theta_{S_\alpha}$ is a left I-order in its inverse hull - we identify S_α with $S_\alpha\theta_{S_\alpha}$ and write the inverse hull of S_α as Q_α. By Lemma 2.4, S_α is straight in Q_α.

From Proposition 4.2, S is left ample and as S has (LC), the connecting morphisms are (LC)-preserving. By Theorem 3.11, each $\phi_{\alpha,\beta}(\alpha \geq \beta)$ lifts to a morphism $\overline{\phi_{\alpha,\beta}} : Q_\alpha \to Q_\beta$. Clearly $\overline{\phi_{\alpha,\alpha}}$ is the identity map and for any $\alpha \geq \beta \geq \gamma$, $\overline{\phi_{\alpha,\beta}}\,\overline{\phi_{\beta,\gamma}} = \overline{\phi_{\alpha,\gamma}}$. Thus $Q = \mathcal{S}\big(Y; Q_\alpha; \overline{\phi_{\alpha,\beta}}\big)$ is a strong semilattice of inverse semigroups and S is a straight left I-order in Q.

It remains to show that Q is isomorphic to the inverse hull $P = \Sigma(S)$ of S. First, it is easy to check that S is a union of $\mathcal{R}$-classes of Q.

For any $a, b \in S$,

$$\begin{array}{rcl} a\,\mathcal{R}^Q_S\,b & \Leftrightarrow & a, b \in S_\alpha \text{ for some } \alpha \text{ and } a\,\mathcal{R}^{Q_\alpha}_{S_\alpha}\,b \\ & \Leftrightarrow & a, b \in S_\alpha \text{ for some } \alpha \text{ and } a\,(\mathcal{R}^*)^{S_\alpha}\,b \\ & \Leftrightarrow & a\,(\mathcal{R}^*)^S\,b \\ & \Leftrightarrow & a\theta_S\,\mathcal{R}^P\,b\theta_S. \end{array}$$

Let $a_\alpha, b_\beta \in S$; we show that

$$Sa_\alpha \cap Sb_\beta = S(a_\alpha\phi_{\alpha,\alpha\beta}) \cap S(b_\beta\phi_{\beta,\alpha\beta}).$$

Let

$$x = u_\gamma a_\alpha = v_\delta b_\beta \in Sa_\alpha \cap Sb_\beta;$$

then $\gamma\alpha = \delta\beta = \tau$ say, so that $\tau \leq \alpha\beta$ and

$$\begin{aligned} x = x^+x &= (x^+u_\gamma)a_\alpha = (x^+v_\delta)b_\beta = (x^+u_\gamma)(a_\alpha\phi_{\alpha,\tau}) = (x^+v_\delta)(b_\beta\phi_{\beta,\tau}) \\ &= (x^+u_\gamma)(a_\alpha\phi_{\alpha,\alpha\beta}) = (x^+v_\delta)(b_\beta\phi_{\beta,\alpha\beta}) \in S(a_\alpha\phi_{\alpha,\alpha\beta}) \cap S(b_\beta\phi_{\beta,\alpha\beta}). \end{aligned}$$

Conversely, if

$$y = h_\gamma(a_\alpha\phi_{\alpha,\alpha\beta}) = k_\delta(b_\beta\phi_{\beta,\alpha\beta}) \in S(a_\alpha\phi_{\alpha,\alpha\beta}) \cap S(b_\beta\phi_{\beta,\alpha\beta})$$

then $\gamma\alpha\beta = \delta\alpha\beta = \kappa$ say and

$$y = (y^+h_\gamma)(a_\alpha\phi_{\alpha,\alpha\beta}) = (y^+k_\delta)(b_\beta\phi_{\beta,\alpha\beta}) = (y^+h_\gamma)a_\alpha = (y^+k_\delta)b_\beta \in Sa_\alpha \cap Sb_\beta.$$

Now let $a, b, c \in S$. Consider $ba^{-1} \in Q$; say $b = b_\beta$ and $a = a_\alpha$. Then

$$ba^{-1} = (b\phi_{\beta,\alpha\beta})(a\phi_{\alpha,\alpha\beta})^{-1} = x^{-1}y$$

where $x, y \in S_{\alpha\beta}$, $x = x(b\phi_{\beta,\alpha\beta})^+, y = y(a\phi_{\alpha,\alpha\beta})^+$, $S_{\alpha\beta}(b\phi_{\beta,\alpha\beta}) \cap S_{\alpha\beta}(a\phi_{\alpha,\alpha\beta}) = S_{\alpha\beta}(x(b\phi_{\beta,\alpha\beta}))$ and

$$x(b\phi_{\beta,\alpha\beta}) = y(a\phi_{\alpha,\alpha\beta}) = xb = ya.$$

Also, $x = x(b^+\phi_{\beta,\alpha\beta}) = xb^+$ and similarly, $y = ya^+$. From the proof of Proposition 4.2 and the argument above, we have that $Sb \cap Sa = S(xb)$. It follows from Remark 3.3, that in P, $b\theta_S(a\theta_S)^{-1} = (x\theta_S)^{-1}y\theta_S$.

Now

$$\begin{aligned}
(a, b, c) \in \mathcal{T}_S^Q &\Leftrightarrow ab^{-1}Q \subseteq c^{-1}Q \\
&\Leftrightarrow Qba^{-1} \subseteq Qc \\
&\Leftrightarrow Qx^{-1}y \subseteq Qc \\
&\Leftrightarrow Qy \subseteq Qc \\
&\Leftrightarrow Sy \subseteq Sc \\
&\Leftrightarrow S\theta_S(y\theta_S) \subseteq S\theta_S(c\theta_S) \\
&\Leftrightarrow P(y\theta_S) \subseteq P(c\theta_S) \\
&\Leftrightarrow Pb\theta_S(a\theta_S)^{-1} \subseteq P(c\theta_S) \\
&\Leftrightarrow (a\theta_S)(b\theta_S)^{-1}P \subseteq (c\theta_S)^{-1}P \\
&\Leftrightarrow (a\theta_S, b\theta_S, c\theta_S) \in P.
\end{aligned}$$

From Corollary 2.10, Q is isomorphic to P via an isomorphism lifting θ_S. $\square$

From Lemma 3.6 and Theorem 4.3 we have the following result of Gantos.

Corollary 4.4. *(cf.* [9, Main Theorem]*) Let $S = \mathcal{S}(Y; S_\alpha)$ be a semilattice Y of right cancellative monoids S_α with identity e_α, such that each S_α has (LC). Suppose in addition that for any $\alpha \geq \beta$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, then $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$. For each $\alpha \in Y$, let Q_α be the inverse hull of S_α, so that Q_α is a bisimple inverse monoid, and S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α. Then $Q = \mathcal{S}(Y; Q_\alpha)$ is a semigroup of left I-quotients of S, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup.*

Conversely, let $Q = \mathcal{S}(Y; Q_\alpha)$ be a semilattice Y of bisimple inverse monoids Q_α, with identity e_α, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup. Then $S = \mathcal{S}(Y; R_{e_\alpha})$ is a semilattice of right cancellative monoids R_{e_α}, such that each R_{e_α} has (LC) and for any $\alpha \geq \beta$, if $R_{e_\alpha}a_\alpha \cap R_{e_\alpha}b_\alpha = R_{e_\alpha}c_\alpha$, then $R_{e_\beta}a_\alpha \cap R_{e_\beta}b_\alpha = R_{e_\beta}c_\alpha$.

5. Bisimple inverse semigroups

Let Q be a bisimple inverse semigroup and let S be a subsemigroup of Q that is a union of $\mathcal{R}^Q$-classes. Clearly S is left ample and is embedded as a $(2,1)$-subalgebra of Q and from a remark in Section 1, S is a left I-order in Q. It follows from Lemma 3.6 that S has Condition (LC).

Let **BIS** be the category with objects ordered pairs (Q, S), where Q is a bisimple inverse semigroup and S is a subsemigroup of Q that is a union of $\mathcal{R}^Q$-classes.

A morphism in **BIS** from (Q, S) to (P, T) is a semigroup morphism ψ such that $S\psi \subseteq T$. Now let **LAC** be the category with objects left ample semigroups with Condition (LC) and such that $\mathcal{R}^* \circ \mathcal{L}$ is universal. A morphism in **LAC** from S to T is a $(2,1)$-morphism $\phi : S \to T$ that is (LC)-preserving. It is easy to see that **BIS** and **LAC** are categories.

The next lemma follows from Theorem 3.7.

Lemma 5.1. *Let S be an object in* **LAC**. *Then $(\Sigma(S), S\theta_S)$ is an object in* **BIS**.

Lemma 5.2. *(i) Let $(Q, S), (P, T)$ be objects in* **BIS**, *and suppose that $\phi : S \to T$ is an isomorphism. Then ϕ lifts to an isomorphism from Q to P.*

(ii) Let (Q, S) be an object in **BIS**. *Then $\psi : \Sigma(S) \to Q$ given by $(\rho_a^{-1}\rho_b)\psi = a^{-1}b$ is an isomorphism.*

Proof. We need only prove (i), for then (ii) follows from Lemma 5.1.

For ease we identify S with T and take ϕ to be the identity map on S. Notice that for any $a, b \in S$, $a \,\mathcal{R}^Q\, b$ if and only if $a \,\mathcal{R}^*\, b$ in S, if and only if $a \,\mathcal{R}^P\, b$. If $a \,\mathcal{L}^Q\, b^{-1}$, then $a \,\mathcal{L}^Q\, bb^{-1} = b^+$, so from Lemma 3.6, $a \,\mathcal{L}^S\, b^+$, whence $a \,\mathcal{L}^P\, b^+ \,\mathcal{L}^P\, b^{-1}$. Consequently, $a \,\mathcal{R}^Q\, b^{-1}$ if and only if $a^{-1} \,\mathcal{L}^Q\, b$ if and only if $a^{-1} \,\mathcal{L}^P\, b$ if and only if $a \,\mathcal{R}^P\, b^{-1}$.

We recall from Lemma 2.7 that for $a, b, c, d \in S$ with $a \,\mathcal{R}^Q\, b$ and $c \,\mathcal{R}^Q\, d$, $a^{-1}b = c^{-1}d$ in Q if and only if there exist $x, y \in S$ with $xa = yc$ and $xb = yd$ and such that $a \,\mathcal{R}^Q\, x^{-1}$, $x \,\mathcal{R}^Q\, y$ and $y \,\mathcal{L}^Q\, c^{-1}$. It follows from the above observations that the rule that takes $a^{-1}b \in Q$ (where $a, b \in S$ and $a \,\mathcal{R}^*\, b$) to $a^{-1}b$ in P is a bijection.

Suppose now that $bc^{-1} = x^{-1}y$ in Q, where $b, c, x, y \in S$ and $x \,\mathcal{R}^*\, y$. Notice that $yc^+ = y$. From Lemma 2.6, $xb = yc$. Certainly $Sxb \subseteq Sb \cap Sc$. On the other hand, if $ub = vc \in Sb \cap Sc$, then

$$ub = vc = ubc^{-1}c = ux^{-1}yc$$

so that $Qub \subseteq Qyc$ and so $Sub \subseteq Syc$. It follows that $Sb \cap Sc = Sxb$.

Conversely, if we are given that $Sb \cap Sc = Sxb$, where $xb = yc$, $x \,\mathcal{R}^*\, y$ and $yc^+ = y$, then $b^{-1}bc^{-1}c = (xb)^{-1}xb = b^{-1}x^{-1}xb$ and so

$$bc^{-1} = bb^{-1}x^{-1}xbc^{-1} = x^{-1}xbc^{-1} = x^{-1}ycc^{-1} = x^{-1}y.$$

Suppose now that $a^{-1}b, c^{-1}d \in Q$ with $a, b, c, d \in S$ and $a \,\mathcal{R}^*\, b, c \,\mathcal{R}^*\, d$. Then $bc^{-1} = x^{-1}y$ in Q with $x \,\mathcal{R}^*\, y = yc^+$, $xb = yc$ and $Sxb = Sb \cap Sc$. It follows that $bc^{-1} = x^{-1}y$ in P also. Now

$$(a^{-1}b)(c^{-1}d) = a^{-1}(bc^{-1})d = a^{-1}(x^{-1}y)d = (xa)^{-1}yd$$

in both Q and P. It follows that the map that takes $a^{-1}b \in Q$ (where $a, b \in S$ and $a \,\mathcal{R}^*\, b$) to $a^{-1}b \in P$ is an isomorphism, which clearly restricts to the identity on S. $\square$

Let $F : \textbf{LAC} \to \textbf{BIS}$ be the functor that takes an object S of **LAC** to $(\Sigma(S), S\theta_S)$. If S, T are objects in **LAC** and $\phi : S \to T$ is a morphism in **LAC**, then by Theorem 3.11, ϕ lifts to a morphism $\overline{\phi} : \Sigma(S) \to \Sigma(T)$. More accurately, $\phi' : S\theta_S \to T\theta_T$ given by $\rho_a\phi' = \rho_{a\phi}$ lifts to $\overline{\phi}$, so that $(\rho_a^{-1}\rho_b)\overline{\phi} = \rho_{a\phi}^{-1}\rho_{b\phi}$. Clearly $S\theta_S\overline{\phi} = S\theta_S\phi' \subseteq T\theta_T$, so that $\overline{\phi}$ is a morphism from $(\Sigma(S), S\theta_S)$ to $(\Sigma(T), T\theta_T)$ in **BIS**. It is straightforward to verify that F is indeed a functor.

Let $G : \mathbf{BIS} \to \mathbf{LAC}$ take an object (Q, S) of **BIS** to S and a morphism ψ from (Q, S) to (P, T) in **BIS** to $\phi = \psi|_S$. Clearly ϕ is a $(2, 1)$-morphism.

Lemma 5.3. *The map G defined as above is a functor from* **BIS** *to* **LAC**.

Proof. We need only check that if ψ is a morphism from (Q, S) to (P, T) in **BIS**, then $\phi = \psi_S$ is (LC)-preserving.

Let $a, b, c \in S$ be such that $Sa \cap Sb = Sc$. By Lemma 3.6, $Qa \cap Qb = Qc$, so that as ψ is certainly an (LC)-morphism, $P(a\psi) \cap P(b\psi) = P(c\psi)$ and again by Lemma 3.6, and using the fact $\phi = \psi|_S$, $T(a\phi) \cap T(b\phi) = T(c\phi)$.

□

We now show that FG and GF are naturally isomorphic to $I_{\mathbf{LAC}}$ and $I_{\mathbf{BIS}}$, respectively.

Let S be any object in **LAC**; then

$$SFG = (\Sigma(S), S\theta_S)G = S\theta_S$$

and $\theta_S : SI_{\mathbf{LAC}} \to SFG$ is an isomorphism in **LAC**. If $\phi : S \to T$ is a morphism in **LAC**, then

$$\phi FG = \overline{\phi}G = \overline{\phi}|_{S\theta_S} = \phi',$$

where $\rho_a\phi' = \rho_{a\phi}$. For any $s \in S$ we have that

$$s\theta_S\phi' = \rho_s\phi' = \rho_{s\phi} = s\phi\theta_T$$

so that

$$\begin{tikzcd} S \arrow[r, "\theta_S"] \arrow[d, "\phi"] & SFG \arrow[d, "\phi FG"] \\ T \arrow[r, "\theta_T"] & TFG \end{tikzcd}$$

commutes.

On the other hand, for any object (Q, S) of **BIS**,

$$(Q, S)GF = SF = (\Sigma(S), S\theta_S)$$

and for a morphism $\psi : (Q,S) \to (P,T)$ in **BIS**,

$$\psi GF = \psi|_S F = \overline{\psi|_S},$$

where for $\rho_a^{-1}\rho_b \in \Sigma(S)$, $(\rho_a^{-1}\rho_b)\overline{\psi|_S} = \rho_{a\psi}^{-1}\rho_{b\psi}$. By Lemma 5.2, $\mu_{(Q,S)} : (Q,S) \to (\Sigma(S), S\theta_S)$ given by $(a^{-1}b)\mu_{(Q,S)} = \rho_a^{-1}\rho_b$ is an isomorphims, which clearly lies in **BIS**. We have that for any $a^{-1}b \in Q$,

$$(a^{-1}b)\mu_{(Q,S)}(\psi GF) = (\rho_a^{-1}\rho_b)(\psi GF) = \rho_{a\psi}^{-1}\rho_{b\psi} = \big((a\psi)^{-1}b\psi\big)\mu_{(P,T)} = (a^{-1}b)\psi\mu_{(P,T)},$$

so that

$$\begin{tikzcd}
(Q,S) \arrow[r, "\mu_{(Q,S)}"] \arrow[d, "\psi"'] & (Q,S)GF \arrow[d, "\psi FG"] \\
(P,T) \arrow[r, "\mu_{(P,T)}"] & (P,T)GF
\end{tikzcd}$$

commutes.

We now have the main result of this section.

Theorem 5.4. *The categories* **BIS** *and* **LAC** *are equivalent.*

As a corollary, we have the classical result due to Clifford and Warne, made explicit in [18, Chapter X]. To state this result, we let **BIM** be the full subcategory of **BIS** consisting of all pairs (Q, R_1), where Q is a bisimple inverse monoid and R_1 is the $\mathcal{R}$-class of the identity of Q, and we let **RCC** be the full subcategory of **LAC** consisting of right cancellative monoids with the (LC) condition. Since $F|_{\mathbf{RCC}} : \mathbf{RCC} \to \mathbf{BIM}$ and $G|_{\mathbf{BIM}} : \mathbf{BIM} \to \mathbf{RCC}$, we deduce the following.

Corollary 5.5. [3, 21, 18] *The categories* **BIM** *and* **RCC** *are equivalent.*

Finally, we remark on the connection between this material and Reilly's RP-systems [19]. Reilly defined an RP-system (R,P) to be a 'right partial semigroup' R together with a subsemigroup P of R satisfying certain conditions. He showed that for any RP-system (R,P) there exists a bisimple inverse semigroup $Q(R,P)$, some $\mathcal{R}$-class of which is isomorphic (under the appropriate notion) to R. Conversely, for any $\mathcal{R}$-class R of a bisimple inverse semigroup, there is a subsemigroup P of R such that (R,P) is an RP-system and $Q(R,P)$ is isomorphic to Q. We remark that if S is an object in **LAC**, then any $\mathcal{R}^*$-class of S is an $\mathcal{R}$-class of a bisimple inverse semigroup and so an RP-system. Conversely, if (R,P) is an RP-system, let Q be a bisimple inverse semigroup into which it embeds as an

$\mathcal{R}$-class R_q. Let S be the smallest subsemigroup of Q containing R_q that is a union of $\mathcal{R}$-classes of Q. Then S is left ample, and a left I-order in Q, so that S is an object in **LAC**.

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