Amenability of Finitely Generated Semigroups

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\textbf{Definition:} (Von Neumann) \( S \) is left amenable iff there exists a finite additive probability measure \( \mu : 2^S \to [0, 1] \) such that \( \forall x \in S, \ X \subseteq S, \ \mu(X) = \mu(X \setminus x) \)

\textbf{Examples:}

1) Finite groups (uniform measure)
2) Compact groups (Haar measure)
3) Semigroups w/ zero:
   \( \mu(x) = \begin{cases} 1 & \text{if } 0 \in x \\ 0 & \text{if } 0 \notin x \end{cases} \)
4) Commutative Semigroups (Day)
5) Bicyclic monoids.

\textbf{Non-examples:}

1) Free groups
2) 2-element left zero semigroup.

\textbf{Note:} If \( I \) is a right ideal, \( i \in I \), then \( \mu(I) = \mu(i \setminus I) = \mu(S) = 1 \), so can't have disjoint \( R \)

\textbf{Definition:} \( S \) is left reversible if every pair of right ideals intersects.

\textbf{Fact:} Left amenable \( \implies \) left reversible
(\( \implies \) holds for finite semigroups)

\textbf{Følner Conditions:} (Fc) For all \( H \subseteq_s S \), \( \varepsilon > 0 \), \( \exists F \subseteq S \)
such that \( \forall x \in S \), \( 1_F \setminus F \leq \varepsilon 1_F \).

\textbf{Strong Fc (SFC)}: For all \( H \subseteq_s S \), \( \varepsilon > 0 \), \( \exists F \subseteq S \)
such that \( \forall x \in H \), \( 1_F \setminus F \leq \varepsilon 1_F \).

\textbf{Theorem:} (Følner, Day, Atkinson) (SFC) \( \implies \) left amenable \( \implies \) (Fc)
**Definition:** E S is "left thick" if \( \forall x \in S \exists y \in E \) such that \( F \leq E \).

**Examples:**
- i) left ideals
- ii) right ideals \( \Rightarrow S \) is left reversible.

**Definition:** S is "near left cancellative" (NLc) if \( \forall s \in S \exists \) left thick \( E \subseteq S \) such that \( \forall x, y \in E, Sx = Sy \Rightarrow x = y \)

**Examples:**
- i) left cancellative (E=S)
- ii) semigroups w/ zero (E=S03)
- iii) inverse semigroups.

**Proposition:** Let S be left reversible such that every ideal contains an idempotent. Then S is NLc.

**Proof:** Let \( s \in S \). Choose \( e \in S \) \( e \in E(s) \), say, \( e = xsy \).
Then \( e \not\in xsyxs \), so choose \( f = f^2 \not\in xsyxs \).
Then \( f \leq s \), write \( f = ts \). Consider \( E = tS \).

**Theorem:** If S is NLc then \( (SFC) \Rightarrow \) left amenable.

**Growth:** Let S be f.g. by \( X \subseteq S \). We say S has "subexponential growth" if the function \( \text{Incl}(N \rightarrow N) \) given by \( n \rightarrow |X|^n \) is not bounded below by an increasing exponential.

**Lemma:** S NLc, \( a, b \in S \). Then either \( \circ aS \not\subseteq \not\subseteq \) or \( \circ (a, b) \) is free.

**Theorem:** If S is NLc of subexp growth then S is left amenable.