Amenability of Finitely Generated Semigroups

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- Definition: (Von Neumann) S is left amenable iff there exists a finite additive probability measure $\mu : 2^S \to [0,1]$ such that $\forall s \in S$, $X \subseteq S$, $\mu(X) = \mu(s^{-1}X)$

- Examples: i) Finite groups (uniform measure)
  ii) Compact groups (Haar measure)
  iii) Semigroups w/ zero: $\mu(X) = \begin{cases} 1 & \text{if } 0 \in X \\ 0 & \text{if } 0 \notin X. \end{cases}$
  iv) Commutative Semigroups (Day)
  v) Bicyclic monoids.

- Non-examples: i) Free groups
  ii) 2-element left zero semigroup.

Note: If I is a right ideal, $i \in I$, then $\mu(I) = \mu(i^{-1}I) = \mu(S) = 1$, so cant have disjoint R-I

- Definition: S is left reversible if every pair of right ideals intersects.

Fact: left amenable $\Rightarrow$ left reversible ($(\Leftarrow)$ holds for finite semigroups)

- Følner Conditions: (FC) For all $H \subseteq_f S$ (finite subset), $\varepsilon > 0$, $\exists F \subseteq_f S$ such that $\forall s \in S$, $|sF \setminus F| \leq \varepsilon |F|$.

Strong FC (SFC): $\forall H \subseteq_f S$, $\varepsilon > 0$, $\exists F \subseteq_f S$ such that $\forall s \in H$, $|F \setminus sF| \leq \varepsilon |F|$.

Theorem: (Følner, Day, A+W): (SFC) $\Rightarrow$ left amenable $\Rightarrow$ (FC)

· Definition: $E \subseteq S$ is "left thick" if $\forall F \subseteq_f S$ $\exists t \in S$ such that $Ft \subseteq E$.

· Examples: i) left ideals
ii) right ideals $\iff$ S is left reversible.

· Definition: S is "near left cancellative" (NLC) if $\forall s \in S$ $\exists$ left thick $E \subseteq S$ such that $\forall x, y \in E$,
$$Sx = Sy \Rightarrow x = y$$

· Examples: i) left cancellative $(E = S)$
ii) Semigroups w/ zero $(E = \{0\})$
iii) Inverse semigps:

Proposition: Let S be left reversible such that every ideal contains an idempotent. Then S is NLC.

Proof: Let $s \in S$. Choose $e \in SsS$ $(e \in E(S))$, say, $e = xsy$. Then $e \mathcal{R} xsyxs$, so choose $f = f^2 \mathcal{L} xsyxs$. Then $f \leq_{\mathcal{L}} s$, write $f = ts$. Consider $E = fS$ ▭

Theorem: If S is NLC then (SFC) $\iff$ left amenable.

· Growth: Let S be f.g. by $X \subseteq S$. We say S has "subexponential growth" if the function $\mathbb{N} \to \mathbb{N}$ given by $n \mapsto |X^n|$ is not bounded below by an increasing exponential

Lemma: S NLC, $a, b \in S$. Then either ① $aS \cap bS \neq \emptyset$ or ② $\langle a, b \rangle$ is free.

Theorem: If S is NLC of subexp growth then S is left amenable.