Engel elements in groups of automorphisms of rooted trees

Marialaura Noce

joint work with G. A. Fernández Alcober, A. Garreta, A. Tortora, and G. Tracey

October 2019, University of York

Università di Salerno - Euskal Herriko Unibertsitatea

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Relation between these sets: Heineken's results

- $\bar{R}(G)^{-1} \subseteq \bar{L}(G)$
- $R(G)^{-1} \subseteq L(G)$

- If $L(G) = G$ or $R(G) = G$, then G is an Engel group.

Engel groups and Burnside-like problems

- If $L(G) = G$ or $R(G) = G$, then G is an Engel group.
- Locally nilpotent groups are obviously Engel groups.

Is every Engel group locally nilpotent?

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- Compare this to the General Burnside Problem.

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n-Engel groups that are locally nilpotent:

- All n-Engel groups for $n \leq 4$ (Hopkins, 1929, $n = 2$; Heineken, 1961, $n = 3$; Havas, Vaughan-Lee, 2003, $n = 4$)
- Residually finite groups (Wilson, 1991)

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- Remark: for $n \geq 5$ is still not known if n-Engel groups are locally nilpotent (see Rip's talk on YouTube).

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From Robinsons book A course in the theory of groups:

The major goal of Engel theory is to find conditions which will guarantee that $L(G)$ and $\bar{L}(G)$ are subgroups [...].

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For $\bar{L}(G)$, $R(G)$, and $\bar{L}(G)$:

- Open.

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In both cases the set of let Engel elements is not a subgroup.

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Automorphisms of regular rooted trees

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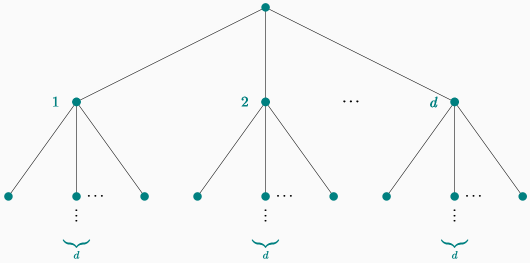

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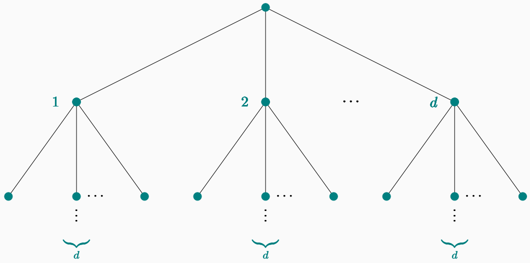

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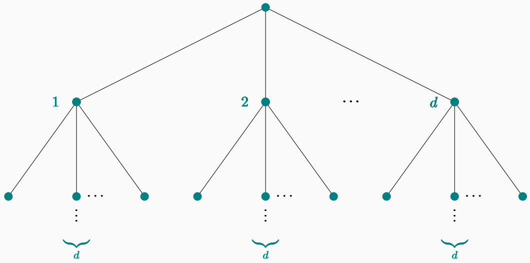
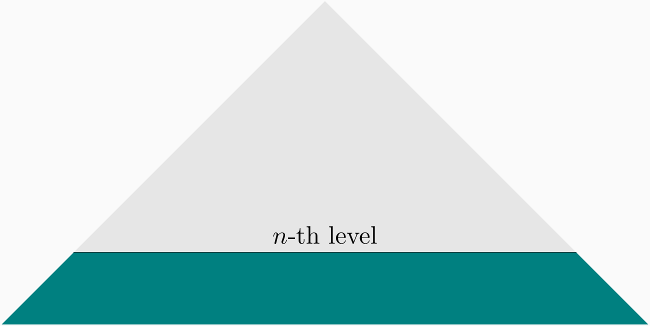


Automorphisms of $\mathcal{T}_d$

Bijections of the vertices that preserve incidence.

- The set $\mathrm{Aut}\,\mathcal{T}_d$ of all automorphisms of $\mathcal{T}_d$ is a group with respect to composition between functions.

The stabilizer of $\operatorname{Aut} \mathcal{T}_d$



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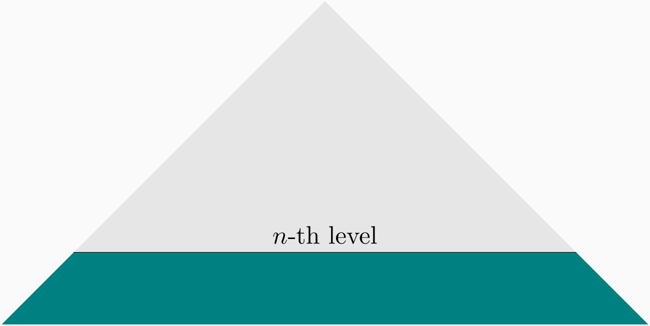
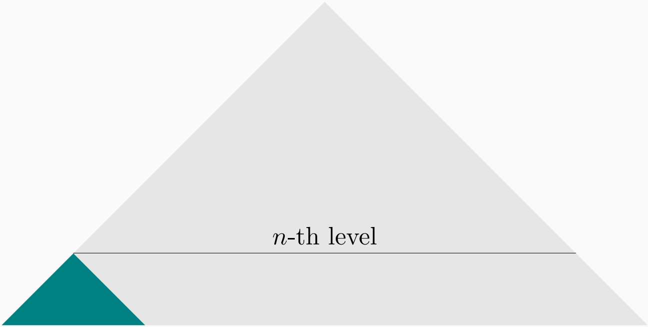


- The nth level stabilizer $\mathrm{st}(n)$ fixes all vertices up to level n.
- If $H \leq \operatorname{Aut}\mathcal{T}$, we define $\mathrm{st}_H(n) = H \cap \mathrm{st}(n)$.

Rigid stabilizers

The *rigid stabilizer* of the vertex u is

$$\mathrm{rst}_G(u) = \{g \in G : g \text{ fixes all vertices outside } \mathcal{T}_u\}$$



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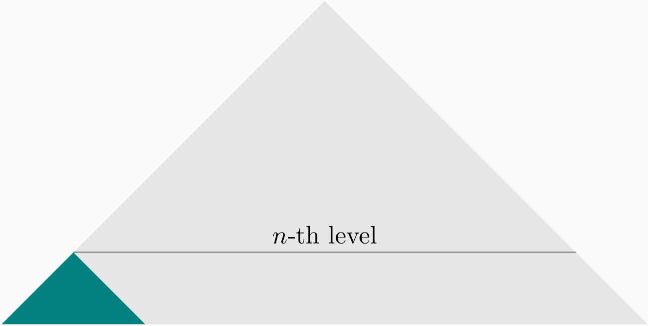

The *rigid stabilizer* of the nth level is $\mathrm{rst}_G(n) = \prod_{u \in X^n} \mathrm{rst}_G(u)$.

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Also:

1. We have $\operatorname{Aut}\mathcal{T}_d \cong \mathrm{st}(1) \rtimes S_d$
2. If $n \in \mathbb{N}$, we define the isomorphism

$$\psi_n : \mathrm{st}(n) \longrightarrow \operatorname{Aut}\mathcal{T}_d \times \overset{d^n}{\cdots} \times \operatorname{Aut}\mathcal{T}_d.$$

Describing elements of $\operatorname{Aut}\mathcal{T}$, II

- Any $g \in \operatorname{Aut}\mathcal{T}_d$ can be seen as

$$g = h\sigma, \quad \sigma \in S_d, \quad h \in \operatorname{Aut}\mathcal{T}_d \times \overset{d}{\dots} \times \operatorname{Aut}\mathcal{T}_d$$

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- This can be used to define automorphisms, and the definition can be recursive.
- If $\mathcal{T}$ is the binary tree and a is rooted corresponding to (12), let

$$b = (1, b)a.$$

  How does b act on $\mathcal{T}$?

The Grigorchuk group

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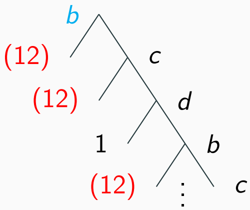

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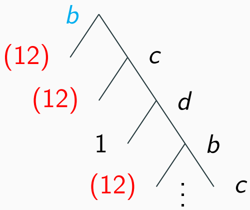

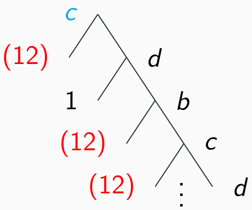

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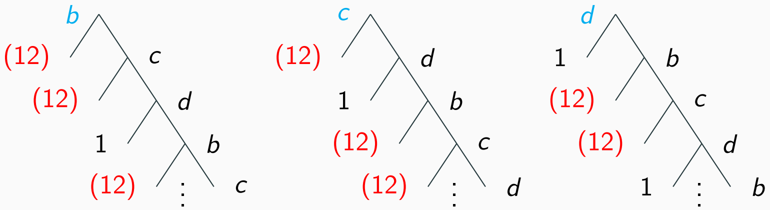

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- The case of the vector $\mathbf{e} = (1, -1, 0, \dots, 0)$ is the famous Gupta-Sidki p-group.

Two classes of subgroups of $\mathrm{Aut}\,\mathcal{T}_d$

- Fractal groups
- (Weakly) Branch groups

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It is natural to search inside $\operatorname{Aut}\mathcal{T}$ for groups where the Engel sets are not subgroups.

First negative results

Theorem (Barholdi, 2016)

Let G be the Gupta-Sidki 3-group. We have

$$L(G) = \bar{L}(G) = R(G) = \bar{R}(G) = 1.$$

Theorem (N, Tortora, 2018)

Let Γ be the first Grigorchuk group. We have

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Engel elements in fractal groups

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Let G be a fractal group such that $|G' : \mathrm{st}_G(1)'| = \infty$. Then $L(G) = 1$.

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Application to specific fractal groups

In the following groups one has $L(G) = \bar{L}(G) = R(G) = \bar{R}(G) = 1$.

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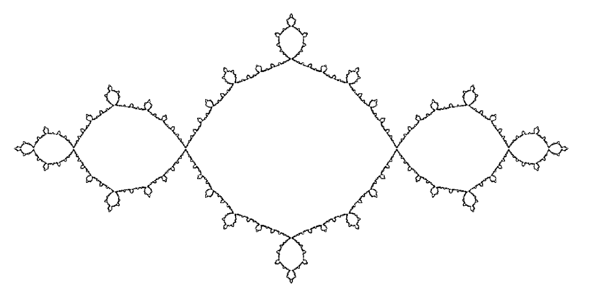

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Engel elements in (weakly) branch groups

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Example: again the Grigorchuk group Γ.

- $L(\Gamma)$ consists of all elements of order 2.
- Γ is a 2-group.

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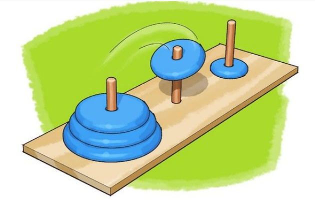
- The Hanoi tower group $\mathcal{H}$ satisfies $L(\mathcal{H}) = 1$.

The Hanoi tower game

The tower of Hanoi was invented by a French mathematician Édouard Lucas in the 19th century.



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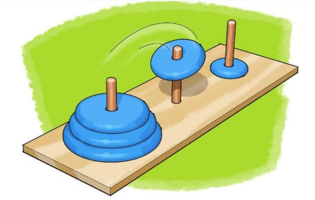

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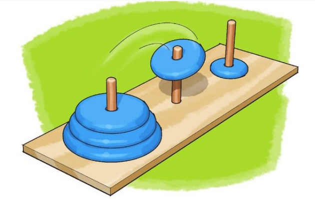

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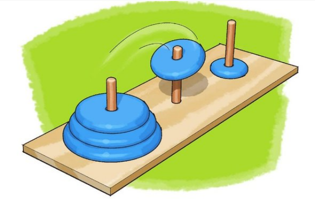

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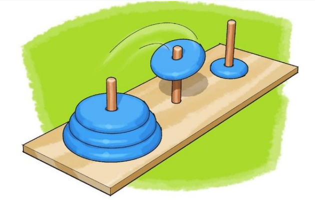

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- The rules:
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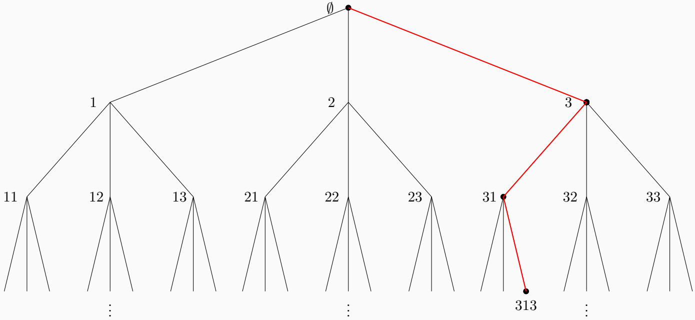
  - No disk may be placed on top of a smaller disk.

The Hanoi towers game

- Let 3 be the number of pegs, then consider $X = \{1, 2, 3\}$. A word in X is a configuration of the disks and the length of the word is the number of disks.
- Example: 231123 (blackboard)
- Goal: to send $11 \ldots 1$ to $33 \ldots 3$.

The Hanoi towers game

- Configurations (sequences of length n of $1, 2, 3$) can be seen as vertices on the n-th level in a rooted ternary tree.



- Any move takes one vertex on the n-th level on the tree to another vertex on the n-th level. Then each move can be thought of as an automorphism of the rooted ternary tree.

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where $a = (a, 1, 1)(23), b = (1, b, 1)(13), c = (1, 1, c)(12)$.

Conlusions

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- Is $R(G) = 1$ in every weakly branch group?

Grazie.
Eskerrik asko.