Completely 0-simple semigroups of left quotients of a semigroup

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Abstract

In this paper we investigate the class of all completely 0-simple semigroups of left quotients of a given semigroup S. We show that (if the class is non-empty) it has a 'greatest' member $\overline{S}$ which is in a sense the free completely 0-simple semigroup on S, and describe how the other members can be obtained as homomorphic images of $\overline{S}$.
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1 Introduction

One of the oldest constructions in algebra, which arises from extending the notion of number from integers to rationals, is the construction of fractions, or quotients. This procedure is not particular to integers; as is well known, every commutative integral domain has a field of quotients. Together with the more general concept of a *ring* of (left) quotients this has led to many deep results in ring theory. Consequently, the construction is generally considered as ring-theoretic. We claim, however, that this view is somewhat prejudiced. When constructing fractions only the multiplication of integers is used - their addition is needed only to define addition of fractions but not the fractions themselves. Indeed it is true that every commutative cancellative semigroup has a commutative group of quotients.

The definition of quotients relies on defining reciprocals, in the classical construction with respect to a single identity. This identity may not lie in the original ring or semigroup but must exist in the ring or group of quotients and then the reciprocals must lie in the group of units. Since the behaviour of elements in the group of units of a ring has a strong influence on the structure of the whole ring, this approach in ring theory is successful. It can of course be used in the context of semigroups, but has not been as fruitful as in ring theory.

For semigroups the significant factor is the behaviour of *all* the idempotents and those elements which have a reciprocal with respect to *some* idempotent. With this in mind we consider semigroups of quotients in which we take inverses of certain elements (called *square-cancellable*) with respect to *some* idempotent. This construction, which takes into account the existence of *all* subgroups of the semigroup of quotients, may be considered as revealing hidden group structures in a semigroup. Here the term 'group structure' may relate to one group or many. Thus this construction establishes an important connection between groups and semigroups.

The approach we take is that introduced by Fountain and Petrich in [4] which concentrates on completely 0-simple semigroups of quotients, as we do here. However for completeness and for one or two particular points we do define the general notion of a semigroup of quotients.

Completely 0-simple semigroups were the first class to be considered as semigroups of (left) quotients and have been the subject of a number of further papers by Fountain, Petrich and the second author. Much of the effort in these papers is devoted to situations where the semigroup of (left) quotients of a given semigroup is unique up to isomorphism. Here we take a rather different view and investigate the class of *all* completely 0-simple semigroups of left quotients of a given semigroup.

Section 2 consists of preliminaries and some motivating examples. In Section 3 we present the main results: we show that the class of all completely 0-simple semigroups of left quotients of a given semigroup S (if non-empty) has a 'greatest' member $\overline{S}$ which is in a sense the free completely 0-simple semigroup on S. We indicate two constructions of $\overline{S}$ and describe those congruences on $\overline{S}$ that yield semigroups of left quotients of S, obtaining thereby a description of all completely 0-simple semigroups of left quotients of S.

2 Preliminaries and Examples

We assume the reader is familiar with the basics of semigroup theory, as may be found in the earlier chapters of [3] or [8]. As far as possible we follow the notation of [8], with the exception that $a^\sharp$ denotes the *group* inverse of an element a. That is, if a is an element of a semigroup Q then by writing $a^\sharp$ it is implicit that a lies in a subgroup of Q with inverse $a^\sharp$ in this subgroup. In this case $a^\sharp$ is the inverse of a in the group $\mathcal{H}$-class H_a and thus is unique. As in [8] in cases of possible ambiguity we may write the superscript S on a relation $\mathcal{K}$ on a semigroup S to emphasise that S is the semigroup under consideration.

If Q is a semigroup of left quotients of a semigroup S, then we insist that all elements of S satisfying a weak cancellability condition known as *square-cancellability* lie in subgroups of Q. This condition is defined as follows. An element a of S is said to be *square-cancellable* if, for all $x, y \in S^1$,

$$a^2x = a^2y \text{ implies } ax = ay$$

and

$$xa^2 = ya^2 \text{ implies } xa = ya.$$

We denote by $\mathcal{S}(S)$ the set of square-cancellable elements of S.

We are now in a position to define a semigroup of left quotients. Let S be a subsemigroup of a semigroup Q. Then Q is a *semigroup of left quotients* of S if every $q \in Q$ can be written as $q = a^\sharp b$ where $a, b \in S$ and if, in addition, every square-cancellable element of S lies in a subgroup of Q. If this is the case, then, as is easy to check, a and b can always be chosen such that $aa^\sharp b = b$; we use this remark repeatedly. If Q is a semigroup of left quotients of S then we also say that S is a *left order* in Q. *Semigroups of right quotients* and *right orders* are defined dually. If S is both a left order and a right order in Q then we say that Q is a *semigroup of quotients* of S and S is an *order* in Q.

Clearly if S is a commutative cancellative semigroup then its group of quotients is a semigroup of quotients of S. However S may possess other, more complicated, semigroups of quotients. This is shown by the following example.

Example 2.1 .

Consider the additive semigroup of non-negative integers $\mathbf{N}$. By the above comments, the additive group of integers $\mathbf{Z}$ is a semigroup of quotients of $\mathbf{N}$. But if we put $\mathbf{Z}^+ = \mathbf{N} \setminus \{0\}$, then $\mathbf{Z}^+$ has a (semi)group of quotients $\mathbf{Z}'$ isomorphic to $\mathbf{Z}$, with (additive) identity ϕ say. Then adjoining an identity 0 to $\mathbf{Z}'$ we have that $\mathbf{N}$ is an order in $\mathbf{Z}' \cup \{0\}$. Since $\mathbf{Z}' \cup \{0\}$ has two idempotents it is clearly not isomorphic to $\mathbf{Z}$. (We have 'doubled' the number 0).

If S is a subsemigroup of a completely 0-simple semigroup Q and $a \in \mathcal{S}(S)$ then either $a = 0$, in which case a lies in a subgroup of Q, or if $a \neq 0$ then $a^2 \neq 0$. By the nature of completely 0-simple semigroups we thus have that a lies in a subgroup of Q. It follows that S is a left order in Q if every $q \in Q$ can be written as $q = a^\sharp b$ where $a, b \in S$. Similar comments apply to right orders and orders in Q. As mentioned in the Introduction, orders in this sense were first studied in completely 0-simple semigroups, where the notion of square-cancellability is not needed.

As was proved in [4], if a semigroup S is a two-sided order in a completely 0-simple semigroup then the latter is uniquely determined by S. However, looking for semigroups of *one-sided* quotients in the class of completely (0-)simple semigroups does not ensure unicity. An example of this is given in [4]; here we present another, a semigroup S which has continuum many pairwise non-isomorphic semigroups of left quotients, one of which is a group.

Example 2.2 .

Let S be a semigroup which is a left order in a group G and a disjoint union of infinitely many right ideals R_i, $i \in I$. Such semigroups certainly exist: take a cancellative semigroup T which satisfies the left but not the right Ore condition. Then T contains two disjoint right ideals R_1, R_2'. Take any $a_2 \in R_2'$. Since T is cancellative, a_2R_1 and a_2R_2' are disjoint and contained in R_2', so putting $R_2 = a_2R_1$ and $R_3' = a_2R_2'$, $R_1 \cup R_2 \cup R_3'$ is a disjoint union. Now take any $a_3 \in R_3'$ and continue by induction. The union of the right ideals R_i, $i \in \mathbf{Z}^+$ has the desired property. Given this semigroup S, take an arbitrary partitioning $P_J = \{I_j : j \in J\}$ of I and for each $j \in J$ put $S_j = \bigcup\{R_i : i \in I_j\}$. It is easy to see that each principal right ideal of S is a left order in G and it follows that for each $j \in J$, S_j has a group of left quotients G_j isomorphic to G; let $\phi_j : G_j \to G$ be an isomorphism which restricts to the identity on S_j. Then $Q_{P_J} = \bigcup\{G_j : j \in J\}$ is a semigroup of left quotients of $S = \bigcup\{S_j : j \in J\}$ where for $g \in G_j, h \in G_{j'}$ the product gh is defined by

$$gh = (g\phi_j h\phi_{j'})\phi_j^{-1}.$$

Note that Q_{P_J} is a completely simple semigroup and has $|J|$ right ideals.

Example 2.2 also shows that if S is a left order in a completely (0)-simple semigroup Q, then the number of right ideals of Q is not an invariant. However, the number of left ideals of Q *is* invariant, consequent to Proposition 2.6 below.

Necessary and sufficient conditions for a semigroup S to be a left order in a completely 0-simple semigroup Q are given in [1] and [7]. We make some straightforward remarks about such an S. Since Q is regular and $\mathcal{H}$ is a congruence on Q it follows from Proposition 2 of [5] that S is *straight* in Q, that is, if $q \in Q$ then $q = a^\sharp b$ for some $a, b \in S$ with $a\mathcal{R}b$ in Q. We shall make repeated use of the next lemma, taken from [5].

Lemma 2.3 *Let S be a straight left order in Q. Let $q \in Q$ where $q = a^\sharp b$ for some $a, b \in S$ with $a\mathcal{R}b$ in Q. Then $q\mathcal{H}b$ in Q. Thus S has non-empty intersection with every $\mathcal{H}$-class of Q. Further, if H is a group $\mathcal{H}$-class of Q then $S \cap H$ is a left order in H.*

We recall at this point that a semigroup Q with 0 is *prime* if given any non-zero $a, b \in Q$ we have $aQb \neq 0$. Also Q is *categorical at zero* if given any

$a, b, c \in Q$ with $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$. It is well known that if Q is completely 0-simple, then Q is prime and categorical at zero. Thus if S is a left order in Q, then S is also prime and categorical at zero [4]; the former result uses the fact that S intersects every $\mathcal{H}$-class of Q. If $a \in \mathcal{S}(S)$ then putting $H'_a = S \cap H_a$ (where H_a is the $\mathcal{H}$-class of a in Q) we have that H'_a is a left order in H_a. But then $aH'_a a$ is also a left order in H_a, so that as $aH'_a a \subseteq aSa \subseteq H_a \cup \{0\}$ we deduce that aSa is a left order in $H_a \cup \{0\} = eQe$ where $e = a^\sharp a$ [1]. Proposition 2.4 follows immediately from the structure of Q and the fact that S intersects every $\mathcal{H}$-class of Q.

Proposition 2.4 *Let S be a left order in a completely 0-simple semigroup Q and let $a \in \mathcal{S}(S) \setminus \{0\}$. Then $Q = SeQeS = Sa(eQe)aS$ where $e = a^\sharp a$.*

Proposition 2.5 *Let S, Q and a be as in Proposition 2.4 and let $s \in \mathcal{S}(S) \setminus \{0\}$. Since S is prime there exist $u, v \in S$ with $ausva \neq 0$ and then*

$$s = sva(ausva)^\sharp aus.$$

If, in addition, $s \in \mathcal{S}(S)$ then

$$s^\sharp = sva(aus^3va)^\sharp aus.$$

Proof Using the facts that S is categorical at zero and if $s \in S \setminus \{0\}$ then $s \in \mathcal{S}(S)$ if and only if $s^2 \neq 0$, the result follows from a straightforward computation.

We comment that in Proposition 2.4, $eQe \setminus \{0\}$ is a maximal subgroup of Q and Q is 0-bisimple, so that choosing another non-zero square-cancellable element from S would yield a group isomorphic to $eQe \setminus \{0\}$. Furthermore, $eQe \setminus \{0\}$ is, up to isomorphism, independent of the choice of Q, for it is the group of left quotients of $aSa \setminus \{0\}$.

So, once we know that a semigroup S is a left order in some completely 0-simple semigroup Q, we pick an arbitrary $a \in S$ with $a^2 \neq 0$ and get a group G in which $aSa \setminus \{0\}$ is a left order. Now Q (and all completely 0-simple semigroups of left quotients of S) must arise in the form $S * G * S$ under a 'suitable' multiplication $*$. This $*$ can in general be defined in essentially different ways, and this is the reason why S may have non-isomorphic semigroups of left quotients. Note that there is always a 'free' choice for

$*$, described by means of tensor product (see [1]). This 'freeness' will be formulated in exact terms later on.

One of the basic features of the structure of a completely 0-simple semigroup is the fact that it is the 0-disjoint union of 0-minimal left ideals and of 0-minimal right ideals. We exploit this fact in the following way.

Let S be a semigroup with zero. The relations λ and ρ are defined on S by the rule that for any $a, b \in S$,

$$a \;\lambda\; b \text{ if and only if } a = b = 0 \text{ or } Sa \cap Sb \neq 0$$

and

$$a \;\rho\; b \text{ if and only if } a = b = 0 \text{ or } aS \cap bS \neq 0.$$

The symbols λ and ρ will always refer to relations on S. We denote by λ^t and ρ^t the transitive closures of λ and ρ respectively. Clearly, λ^t and ρ^t are equivalence relations.

Proposition 2.6 *Let S be a semigroup with 0 and let $\phi : S \to T$ be a 0-restricted homomorphism from S into a completely 0-simple semigroup T.*
(i) For any $a, b \in S$

$$a\lambda^t b \textit{ implies } a\phi \;\mathcal{L}^T\; b\phi$$

and

$$a\rho^t b \textit{ implies } a\phi \;\mathcal{R}^T\; b\phi.$$

(ii) If S is a left order in Q, where Q is completely 0-simple, then

$$\lambda = \lambda^t = \mathcal{L}^Q \cap (S \times S) \textit{ and } \rho^t \subseteq \mathcal{R}^Q \cap (S \times S).$$

Proof (i) This follows from the fact that ϕ is 0-restricted and T is the 0-disjoint union of 0-minimal left (right) ideals.

(ii) From (i) we have that

$$\lambda^t \subseteq \mathcal{L}^Q \cap (S \times S), \rho^t \subseteq \mathcal{R}^Q \cap (S \times S).$$

Let $a, b \in S$ and suppose that $a\mathcal{L}b$ in Q. Either $a = b = 0$ (so that $a\lambda b$ in S), or there exists $q \in Q$ with $a = qb \neq 0$. Now $q = c^\sharp d$ for some $c, d \in S$ and then $c^2 a = cdb \neq 0$ for Q is categorical at zero, whence $a\lambda b$ in S. The result follows.

We have shown that if S is a left order in a semigroup Q, where Q is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$, then G and Λ are uniquely determined by S. Example 2.2 shows that the same is not true of I.

Proposition 2.7 *Let S be a left order in a completely 0-simple semigroup Q, and take any $a \in \mathcal{S}(S) \setminus \{0\}$. If aSa is a two-sided order in a group G with 0 then S is a (two-sided) order in Q.*

Proof Take any $q \in Q \setminus \{0\}$. We have to write q in the form $uv^\sharp$ with $u \in S$, $v \in \mathcal{S}(S)$. By Proposition 2.4 we can write $q = sagat$ with $s, t \in S$ and $g \in aa^\sharp Qaa^\sharp \setminus \{0\} = H_a$. Since $aSa \setminus \{0\}$ is a two-sided order in a group G, it follows that $aSa \setminus \{0\}$ is a two-sided order in H_a and it is then easy to check that $atSa \setminus \{0\}$ is a two-sided order in H_a. Thus g can be written as $g = h(atva)^\sharp$ where $h \in atSa, v \in S$. Now $q = sagat = sah(atva)^\sharp at = sah(atva)((atva)^\sharp)^2 at = (sahat)(va((atva)^\sharp)^2 at)$. A simple computation gives $va((atva)^\sharp)^2 at = (va^2t)^\sharp$.

Example 2.1 shows that a semigroup S may be a left order in two semigroups Q_1 and Q_2, only one of which is completely 0-simple. As a counterpart to this example we prove:

Proposition 2.8 *Let S be a semigroup with zero. Suppose that S is prime and a left order in Q where for every $a \in \mathcal{S}(S)$, $aSa \setminus \{0\}$ is contained in a subgroup of Q. Then Q is completely 0-simple.*

Proof Let $q \in Q \setminus \{0\}$, so that $q = a^\sharp b$ for some $a, b \in S$. As $ab\mathcal{L}^Q a^\sharp b$ we have that $ab \neq 0$; now as S is prime, $abca \neq 0$ for some $c \in S$. Hence $a = (abca)(abca)^\sharp a$ and $a^3\mathcal{R}a\mathcal{R}ab$ in Q. Thus $aQ^1 = qQ^1$. It follows that Q is regular. Moreover, if $a, c \in \mathcal{S}(S) \setminus \{0\}$ and $aQ \subseteq cQ$, then $aQ = acQ \subseteq cQ$, so that $0 \neq ac = c^\sharp cac$ and $cac \neq 0$. By hypothesis, $c = (cac)(cac)^\sharp c$ so that $cQ \subseteq caQ$ and $c^2\mathcal{R}c\mathcal{R}ca$ in Q. This gives that $c\mathcal{R}^Q c^\sharp ca = a$, that is, $cQ = aQ$. Hence the non-zero idempotents of Q are primitive.

From Lemma 2.3, S has non-empty intersection with every $\mathcal{H}$-class of Q. This gives that Q is also prime. From Theorem III 3.5 of [8], Q is completely 0-simple.

Proposition 2.8 has a natural analogue for semigroups without zero. Example 2.1 is based on the fact that $\mathbf{N} = 0 + \mathbf{N} + 0$ is not contained in a subgroup of $\mathbf{Z}' \cup \{0\}$.

3 The main results

Let S be a left order in a completely 0-simple semigroup. We show that S is also a left order in a completely 0-simple semigroup $\overline{S}$ which is 'greatest' in the sense that if S is a left order in Q, where Q is completely 0-simple, then Q is a homomorphic image of $\overline{S}$. Moreover, $\overline{S}$ is the 'almost' free completely 0-simple semigroup on S. By this we mean that if $\phi : S \to T$ is a 0-*restricted* homomorphism from S to a completely 0-simple semigroup T, then ϕ can be extended to a homomorphism $\overline{\phi} : \overline{S} \to T$.

In Theorem 3.1 below we give a short argument for the existence of $\overline{S}$. This relies on Corollary 5.1 of [7], which gives necessary and sufficient conditions for a semigroup S to be a left order in a completely 0-simple semigroup Q, in terms of the restriction to S of $\mathcal{R}^Q$. The semigroup $\overline{S}$ can then be constructed as in [6]. Alternatively, both Theorem 3.1 and the construction of $\overline{S}$ can also be obtained using the idea of tensor products, as in [1].

Theorem 3.1 *Let S be a left order in a completely 0-simple semigroup Q. Then S is a left order in a completely 0-simple semigroup $\overline{S}$, where $\mathcal{R}^{\overline{S}} \cap (S \times S) = \rho^t$.*

Proof For the undefined terminology we refer the reader to [7]. Since $\mathcal{L}^Q \cap (S \times S) = \lambda$ it is certainly true that λ is a right congruence contained in $\mathcal{L}^*$. From the fact that S is categorical at zero it is easy to deduce that ρ is left compatible with multiplication so that ρ^t is a left congruence and $\rho^t \subseteq \mathcal{R}^Q \cap (S \times S) \subseteq \mathcal{R}^*$. If $a \in S \setminus \{0\}$ then as we have already seen, $a \in \mathcal{S}(S)$ if and only if $a^2 \neq 0$. It follows that $a \in \mathcal{S}(S)$ if and only if $a(\lambda \cap \rho^t)a^2$. Thus (λ, ρ^t) is a 'suitable pair' for S.

If $a, b, c \in S$ where $a\rho^t b$ and $ca = cb \neq 0$, then as $a\mathcal{R}b$ in Q we have $a = b$. As S is prime then if $u, v \in S$ and $uv \neq 0$, we have $uv\rho u$. It is then immediate from Corollary 5.1 of [7] that $\overline{S}$ exists as claimed.

So far we have only shown the existence of $\overline{S}$, we now set out to prove it has the required 'freeness' properties.

Theorem 3.2 *Let S be a left order in a completely 0-simple semigroup Q and let $\phi : S \to T$ be a 0-restricted homomorphism from S to a completely 0-simple semigroup T such that for all $a, b \in S$*

$$a\mathcal{R}^Q b \text{ implies } a\phi \, \mathcal{R}^T \, b\phi.$$

Then ϕ has a unique extension to a homomorphism $\overline{\phi} : Q \to T$. Further, if $S\phi$ is a left order in T, then $\overline{\phi}$ is onto.

Proof Take any $a \in \mathcal{S}(S) \setminus \{0\}$, then by Proposition 2.4, $Q = SaGaS$, where G is the group of left quotients of $aSa \setminus \{0\}$. Since ϕ is 0-restricted, $a\phi$ lies in a subgroup of T so that $(aSa \setminus \{0\})\phi$ is contained in a subgroup of T. Thus $\phi\mid_{aSa\setminus\{0\}}$ extends to a homomorphism $\phi' : G \to T \setminus \{0\}$.

Now take any $x \in Q \setminus \{0\}$. Then $x = sagat$ for some $s, t \in S$ and $g \in G$. Define $\overline{\phi} : Q \to T$ by putting $0\overline{\phi} = 0$ and $x\overline{\phi} = (sa)\phi g\phi'(at)\phi$. To see that $\overline{\phi}$ is well-defined, suppose that $sagat = s'ag'at'$ where $s, s', t, t' \in S$ and $g, g' \in G$. Since S is prime there exist $u, v \in S$ such that $aus \neq 0$ and $tvs \neq 0$. Now Q is categorical at zero so that $ausagatva = aus'ag'at'va \neq 0$, whence

$$(au)\phi(sa)\phi g\phi'(at)\phi(va)\phi = (ausa)\phi g\phi'(atva)\phi =$$
$$(ausagatva)\phi' = (aus'ag'at'va)\phi' =$$
$$(aus'a)\phi g'\phi'(at'va)\phi = (au)\phi(s'a)\phi g'\phi'(at')\phi(va)\phi \neq 0.$$

Certainly $s\mathcal{R}s'$ and $t\mathcal{L}t'$ in Q. By hypothesis and the fact that T is completely 0-simple, from Proposition 2.6 it follows that

$$(sa)\phi g\phi'(at)\phi \; \mathcal{H} \; (s'a)\phi g'\phi'(at')\phi$$

and we deduce that $(sa)\phi g\phi'(at)\phi = (s'a)\phi g'\phi'(at')\phi$ so that $\overline{\phi}$ is well defined. It is straightforward to check that $\overline{\phi} : Q \to T$ is a homomorphism extending ϕ and that $\overline{\phi}$ is unique.

If $S\phi$ is a left order in T, then given any $q \in T \setminus \{0\}$ we have $q = (a\phi)^\sharp b\phi$ for some $a, b \in S$. Now $a^2 \neq 0$ so that $a^\sharp$ exists in Q and $q = (a^\sharp b)\overline{\phi}$. Thus $\overline{\phi}$ is onto.

The proofs of the following corollaries are now immediate.

Corollary 3.3 *Let S be a left order in completely 0-simple semigroups Q_1 and Q_2. Then Q_2 is a homomorphic image of Q_1 under a homomorphism which restricts to the identity on S if and only if $\mathcal{R}^{Q_1} \cap (S \times S) \subseteq \mathcal{R}^{Q_2} \cap (S \times S)$.*

Corollary 3.4 *Let S be a left order in completely 0-simple semigroups Q_1 and Q_2. Then Q_1 is isomorphic to Q_2 under an isomorphism which restricts to the identity on S if and only if $\mathcal{R}^{Q_1} \cap (S \times S) = \mathcal{R}^{Q_2} \cap (S \times S)$.*

We can now establish that the $\overline{S}$ obtained in Theorem 3.1 is the 'almost' free completely 0-simple semigroup on S.

Theorem 3.5 *Let S be a left order in a completely 0-simple semigroup Q. Then*
(i) S is a left order in a completely 0-simple semigroup $\overline{S}$ such that $\mathcal{R}^{\overline{S}} \cap (S \times S) = \rho^t$, and this $\overline{S}$ is unique up to isomorphism;
(ii) if $\phi : S \to T$ is a 0-restricted homomorphism from S to a completely 0-simple semigroup T, then ϕ can be extended to a unique homomorphism $\overline{\phi} : \overline{S} \to T$;
(iii) if S is a left order in P, where P is completely 0-simple, then P is a homomorphic image of $\overline{S}$, under a unique homomorphism which restricts to the identity on S.

Proof The existence of $\overline{S}$ is proved in Theorem 3.1; by Corollary 3.4, $\overline{S}$ is unique up to isomorphism. Proposition 2.6 and Theorem 3.2 together give that (ii) and (iii) hold.

Remark. The completely simple version of Theorem 3.5 has been proved recently by Antonippillai and Pastijn [2]; this corresponds to the special case of our results when Q is a completely simple semigroup with zero adjoined.

In view of Theorem 3.5 it is natural to ask whether a 'smallest' completely 0-simple semigroup of left quotients always exists. The answer to this question is no, as we later illustrate.

In our next result we describe how all completely 0-simple semigroups of left quotients of S can be obtained from $\overline{S}$. To state this result we make use of the relation $\mathcal{R}^*$. Elements a, b of a semigroup S are $\mathcal{R}^*$-related if and only if for all $x, y \in S^1$

$$xa = ya \text{ if and only if } xb = yb.$$

Clearly if $a\mathcal{R}b$ in some oversemigroup Q of S then $a\mathcal{R}^*b$ in S; in fact, this condition is equivalent. Note that if $a\mathcal{R}^*b$ in S then for all $x \in S$, $xa = 0$ if and only if $xb = 0$.

Much of the following theorem is contained in [7] Corollary 5.1, without pointing at $\overline{S}$, however.

Theorem 3.6 *Let S be a left order in a completely 0-simple semigroup and let $\overline{S}$ be the greatest completely 0-simple semigroup of left quotients of S.*

Then the completely 0-simple semigroups of left quotients of S are in one-one correspondence with the left congruences $\mathcal{R}'$ on S such that $\rho^t \subseteq \mathcal{R}' \subseteq \mathcal{R}^*$ and

$$(I) \quad \textit{if } a, b, c \in S \textit{ where } ca = cb \neq 0 \textit{ and } a\mathcal{R}'b, \textit{ then } a = b.$$

For such a left congruence $\mathcal{R}'$, the corresponding completely 0-simple semigroup of left quotients of S is the homomorphic image of $\overline{S}$ under the 0-restricted congruence $\theta_{\mathcal{R}'}$ where for any $a, b \in \overline{S} \setminus 0$, $a\theta_{\mathcal{R}'}b$ if and only if $ca = cb$ for all $c \in \overline{S}$ and there exist elements $a', b' \in S$ with $a\mathcal{R}^{\overline{S}}a'\mathcal{R}'b'\mathcal{R}^{\overline{S}}b$. Moreover, denoting $\overline{S}/\theta_{\mathcal{R}'}$ by Q and identifying S with its image in Q we have that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$.

Proof Let Q be a completely 0-simple semigroup of left quotients of S and put $\mathcal{R}^S_Q = \mathcal{R}^Q \cap (S \times S)$. Then by Proposition 2.6 and by the definition of $\mathcal{R}^*$ we have $\rho^t \subseteq \mathcal{R}^S_Q \subseteq \mathcal{R}^*$ in S. Clearly, $\mathcal{R}^S_Q$ is a left congruence which satisfies the weak cancellativity condition (I).

Conversely, suppose we have a left congruence $\mathcal{R}'$ on S satisfying the required properties. It is straightforward to check that $\theta_{\mathcal{R}'}$ is a congruence on $\overline{S}$; since S is prime and (I) holds, $\theta_{\mathcal{R}'} \cap (S \times S)$ is the identity relation on S. So $Q = \overline{S}/\theta_{\mathcal{R}'}$ is a (completely 0-simple) semigroup of left quotients of S. By the definition of $\theta_{\mathcal{R}'}$ we have $\mathcal{R}^Q \cap (S \times S) \subseteq \mathcal{R}'$. Let now $s\mathcal{R}'t$, $s, t \in S$; we want to show that $s\mathcal{R}t$ in Q. Clearly, $s \neq 0 \neq t$ can be assumed. Take any $a \in \mathcal{S}(S) \setminus 0$, then by the primeness of S there exist $u, v \in S$ such that $sua \neq 0 \neq tva$. Since $sua\mathcal{L}tva$ in $\overline{S}$, we must have $\overline{S}a = \overline{S}sua = \overline{S}tva$.

Consider the mapping

$$\phi : \overline{S}a \to \overline{S}a : xsua \mapsto xtva \quad (x \in \overline{S}).$$

First we show that ϕ is well defined. Assume that $xsua = x'sua \neq 0$, then $xs = x's \neq 0$ for $\overline{S}$ is completely 0-simple, and $x\mathcal{R}^{\overline{S}}x'$. So we can write $x = k^\sharp l$ and $x' = k'^\sharp l'$ with $k, k' \in \mathcal{S}(S)$, $l, l' \in S$ and $k\mathcal{R}^{\overline{S}}k'\mathcal{R}^{\overline{S}}x\mathcal{R}^{\overline{S}}\ell\mathcal{R}^{\overline{S}}\ell'$. Now we have $k^\sharp ls = k'^\sharp l's$, hence $ls = kk'^\sharp l's$ and $k\mathcal{R}^{\overline{S}}kk'^\sharp$. Writing $kk'^\sharp = m^\sharp n$ where $m, n \in S$ we get $k\mathcal{R}^{\overline{S}}m$ and $ls = m^\sharp nl's$, whence $mls = nl's$. Since $s\mathcal{R}'t$ and $\mathcal{R}' \subseteq \mathcal{R}^*$, this implies $mlt = nl't$, and in view of $k'\mathcal{R}^{\overline{S}}k\mathcal{R}^{\overline{S}}l\mathcal{R}^{\overline{S}}m$ we obtain $xt = k^\sharp lt = k^\sharp m^\sharp mlt = k^\sharp m^\sharp nl't = k^\sharp kk'^\sharp l't = k'^\sharp l't = x't$. Next, if $xsua = 0$ then $xs = 0$ for $sua \neq 0$ and $\overline{S}$ is categorical at 0; writing $x = k^\sharp l$

we get, for the same reason, $ls = 0$ whence $lt = 0$ and so $xt = 0$ in view of $s\mathcal{R}^*t$. The dual argument gives that ϕ is one-one.

So ϕ is an automorphism of the left $\overline{S}$-act $\overline{S}a$; in view of $a \in \mathcal{S}(S)$, this means that ϕ^{-1} is induced by multiplication on the right by an appropriate $w \in a\overline{S}a$. So we have $xsua = xtvaw$ for all $x \in \overline{S}$ and $sua\mathcal{R}^{\overline{S}}s\mathcal{R}'t\mathcal{R}^{\overline{S}}tvaw$, hence $sua\theta_{\mathcal{R}'}tvaw$. In Q we have therefore $sua = tvaw$, and thus $s\mathcal{R}sua = tvaw\mathcal{R}t$ in Q as required. All this shows that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$. By Corollary 3.4 this proves that the correspondence between $\mathcal{R}'$ and Q is one-one.

Corollary 3.7 *If S is a left order in a completely 0-simple semigroup then S has a minimal completely 0-simple semigroup of left quotients.*

Proof By Zorn's Lemma, this follows immediately from Theorem 3.6.

Here 'minimal' cannot be replaced by 'smallest' as shown by the following example.

Example 3.8 .

Let T be a cancellative semigroup that is right reversible but not left reversible and let G be its group of left quotients. Let $h, k \in T$ with $hT \cap kT = \emptyset$. Put $I = \{1, 2, 3\}$, $\Lambda = \{\lambda\}$ and $P = (h\ k\ h)$. It is easy to establish that $S = \mathcal{M}^0(T; I, \Lambda; P)$ is a left order in $Q = \mathcal{M}^0(G; I, \Lambda; P)$.

We claim that S is a left order in completely 0-simple semigroups Q_ℓ and Q_r such that there is *no* completely 0-simple semigroup of left quotients of S which is a homomorphic image of both Q_ℓ and Q_r under homomorphisms which restrict to the identity on S.

We first show that $\{0\}$ and $S \setminus \{0\}$ are the $\mathcal{R}^*$-classes of S. Since $\{0\}$ is always an $\mathcal{R}^*$-class, this amounts to showing that if $a, b \in S \setminus \{0\}$ then $a\mathcal{R}^*b$. It is easy to see that non-zero elements of S are right cancellative so that if $x, y \in S$ then it is certainly true that $xa = ya$ if and only if $xb = yb$. Suppose now that $x \in S$ and $xa = a$. Putting $a = (j, c, \lambda)$, x must have the form $x = (j, d, \lambda)$. This gives that $dp_{\lambda j}c = c$ so that $d = p_{\lambda j}^{-1}$. Either $p_{\lambda j} = h$ or k and in either case the statement $hT \cap kT = \emptyset$ is contradicted. Thus no such x exists and it follows that $a\mathcal{R}^*b$.

Let $\mathcal{R}_\ell$ be the equivalence relation on S with classes $\{0\}, \{(1, t, \lambda), (2, t, \lambda) : t \in T\}, \{(3, t, \lambda) : t \in T\}$. Clearly, $\mathcal{R}_\ell$ is a left congruence and we have $\rho^t \subseteq \mathcal{R}_\ell \subseteq \mathcal{R}^*$ on S. Suppose now that $a, b, c \in S$ where $a\mathcal{R}_\ell b$ and

$ca = cb \neq 0$. Then a, b and c are non-zero and we know that if $a\mathcal{R}b$ in Q then $a = b$. If $a = (1, d, \lambda), b = (2, e, \lambda)$ and $c = (i, f, \lambda)$ for some $d, e, f \in T$ and $i \in \{1, 2, 3\}$ then $fp_{\lambda 1}d = fp_{\lambda 2}e$ giving $hd = ke$, contradicting $hT \cap kT = \emptyset$. We must therefore have that $a\mathcal{R}b$ in Q and so (I) holds. By Theorem 3.6, S is a left order in a completely 0-simple semigroup Q_ℓ such that $\mathcal{R}^{Q_\ell} \cap (S \times S) = \mathcal{R}_\ell$.

Let $\mathcal{R}_r$ be the equivalence relation on S with classes $\{0\}, \{(1, t, \lambda) : t \in T\}, \{(2, t, \lambda), (3, t, \lambda) : t \in T\}$. Dually to the above we have that S is a left order in a completely 0-simple semigroup Q_r such that $\mathcal{R}^{Q_r} \cap (S \times S) = \mathcal{R}_r$.

Suppose that S is also a left order in P, where P is a completely 0-simple semigroup and suppose in addition that there are onto homomorphisms $\phi_\ell : Q_\ell \to P$ and $\phi_r : Q_r \to P$ which restrict to the identity on S. Let $\mathcal{R}' = \mathcal{R}^P \cap (S \times S)$. We have $\mathcal{R}_\ell \subseteq \mathcal{R}'$ and $\mathcal{R}_r \subseteq \mathcal{R}'$ so that $\mathcal{R}_\ell \vee \mathcal{R}_r \subseteq \mathcal{R}'$, giving that $S \setminus \{0\}$ and $\{0\}$ are the classes of $\mathcal{R}'$. Now $(1, h, \lambda), (3, h, \lambda) \in S$ and $(1, h, \lambda)\mathcal{R}'(3, h, \lambda)$. Also

$$(1, h, \lambda)(1, h, \lambda) = (1, h, \lambda)(3, h, \lambda) \neq 0$$

but $(1, h, \lambda) \neq (3, h, \lambda)$. This contradicts condition (I). Thus no such P exists.

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