Boolean inverse semigroups

by

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"Philosophy"

- Need a class of semigroups that is as wide as possible, but not so wide that no interesting theorems can be proved.
- Need good examples.
- Would like natural connections with other parts of Mathematics.

Approach

Generalize the commutative structures
of lattice theory

frames, distributive lattices, Boolean algebras

to a non-commutative setting

pseudogroups, distributive inverse semigroups

Boolean inverse semigroups.

In this talk, I will only deal with

Boolean inverse <u>semigroups</u>.

These are non-commutative generalizations of <u>generalized</u> Boolean algebras.

Boolean inverse <u>monoids</u>

generalize Boolean algebras.

A generalized Boolean algebra is a distributive lattice with zero equipped with a binary operation $\setminus$ satisfying

$$0 = y \wedge (x \setminus y) \quad \text{and} \quad x = (x \wedge y) \vee (x \setminus y).$$

A Boolean algebra is a unital generalized Boolean algebra. This is equipped with a unary operation, the complement, given by $\overline{e} = 1 \setminus e$.

Ideas / notation from inverse semigroups

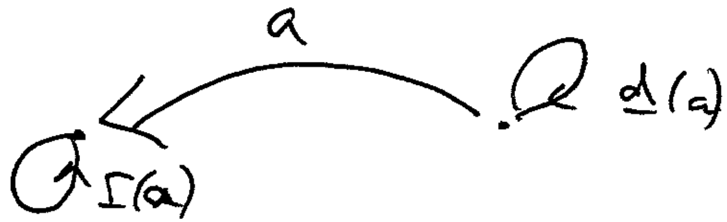
$\leq$ the natural partial order

$\sim$ the compatibility relation

$\underline{E}(S)$ semilattice of idempotents

$\underline{U}(S)$ group of units (when S is a monoid)

$\underline{d}(a) = a^{-1}a$, $\underline{r}(a) = aa^{-1}$



Reduced product $a \cdot b$ $(= ab)$ defined when $\underline{d}(a) = \underline{r}(b)$.

Boolean inverse semigroup

- Inverse semigroup S.
- $x \sim y \Rightarrow \exists\, x \vee y$ in S.
- If $\exists\, x \vee y$ then for any $z \in S$,

  $\exists\, zx \vee zy$ and $\exists\, xz \vee yz$ and

  $z(x \vee y) = zx \vee zy$ and $(x \vee y)z = xz \vee yz$.
- $\underline{E}(S)$ is a generalized Boolean algebra w.r.t. $\leq$ as its order.

Notation in Boolean inverse semigroups

- $\underline{e}(x) = \underline{d}(x) \vee \underline{r}(y)$, the *extent of x*.

- $x \perp y$, x is *orthogonal to* y, if $\underline{d}(x)\underline{d}(y) = 0$ and $\underline{r}(x)\underline{r}(y) = 0$. Their join is denoted by $x \oplus y$

- If $y \leq x$ define $x \setminus y = x(\underline{d}(x) \setminus \underline{d}(y))$.

Observe that $x = y \oplus (x \setminus y)$.

First examples

1. $I(X)$ symmetric inverse monoids are Boolean inverse monoids. Denote by I_n the symmetric inverse monoid on n letters.

2. $I^{fin}(X)$ partial bijections with a finite domain. These are Boolean inverse semigroups.

Important definitions

An inverse semigroup is <u>fundamental</u> if the only elements commuting with all idempotents are idempotents.

An <u>additive ideal</u> in a Boolean inverse semigroup is a semigroup ideal closed under binary joins.

A Boolean inverse semigroup is <u>0-simplifying</u> if it has only trivial additive ideals.

A Boolean inverse semigroup is <u>simple</u> if it is 0-simplifying and fundamental.

In this talk I shall focus on Boolean inverse monoids to simplify the presentation.

Finite Boolean inverse monoids

Remarkably, these can be described explicitly.

Rook matrices

Let S be a Boolean inverse monoid. An $n \times n$ rook matrix over S is such that elements of S in the same row have orthogonal ranges, and elements of S in the same column have orthogonal domains.

Denote by $M_n(S)$ the set of all $n \times n$ rook matrices over S.

[Goes back to Peter Hines' thesis].

<u>Proposition</u> $M_n(S)$ is a Boolean inverse monoid. Multiplication is like matrix multiplication except addition is replaced by join.
If $A \in M_n(S)$ then its inverse A^* is defined by

$$A^* = (a_{ji}^{-1}).$$

The idempotents of $M_n(S)$ are the diagonal matrices whose diagonal entries are idempotents.

Theorem.

Let S be a finite Boolean inverse monoid. Then there are finite groups $G_1, \ldots, G_r$ and natural numbers $n_1, \ldots, n_r$ s.t.

$$S \cong M_{n_1}(G_1^0) \times \ldots \times M_{n_r}(G_r^0).$$

S is fundamental $\Leftrightarrow$ $S \cong I_{n_1} \times \ldots \times I_{n_r}$.

S is simple $\Leftrightarrow$ $S \cong I_n$ for some finite n.

<u>Proof</u> Let S be a finite Boolean inverse monoid. Let G be the set of atoms of S. W.r.t. the reduced product G is a groupoid. Now use standard groupoid theorem. The proof generalizes the classical proof giving the structure of finite Boolean algebras. $\blacksquare$

The *Tarski algebra* is the unique countable atomless Boolean algebra

A Boolean inverse monoid will be called a *Tarski monoid* if it is countably infinite and its semilattice of idempotents is a Tarski algebra.

The Dichotomy Theorem

Let S be a simple, countable Boolean inverse monoid. Then

(i) $S \cong I_n$ for some finite n.

OR

(ii) S is a Tarski monoid.

Under non-commutative Stone duality,

Tarski monoids correspond to

second-countable étale topological groupoids with a Cantor space of identities; these groupoids are also minimal and effective. They are Hausdorff precisely when the Tarski monoid has all binary meets.

Such groupoids are studied by Hiroki Matui and are of interest to C^*-algebra theorists.

The groups of units of Tarski monoids are countably infinite analogues of finite symmetric groups.

<u>Example</u> We show that the classical Thompson groups arise in this way.

Let A_n^* be the free monoid on n ($n \geq 2$) generators.

Let R_1 and R_2 be right ideals of A_n^*. A *morphism* $\theta : R_1 \to R_2$ is a function such that $\theta(xa) = \theta(x)a$ $(\forall x \in R_1)$ $(\forall a \in A_n^*)$.

The *polycyclic inverse monoid* P_n is the inverse semigroup of all bijective morphisms between principal right ideals of A_n^*.

The Lenz equivalence relation $\equiv$.

Let S be a distributive inverse $\wedge$-semigroup.

$s \equiv t$ for $s, t \in S$ iff

- If $0 < a \leq s$ then $t \wedge a \neq 0$.
- If $0 < y \leq t$ then $s \wedge y \neq 0$.

<u>Theorem.</u> Let S be the inverse monoid of all bijective morphisms between the f.g right ideals of A_n^*.

Then $S/\equiv$ is a Boolean inverse monoid whose group of units is isomorphic to the Thompson group G_n.

[When $n=2$, $G_2 = V$].

The above theorem can be generalized when free monoids are replaced by suitable "higher-rank graphs".

Ongoing work with Aidan Sims and Alina Vdovina.

[Fundamental $\Rightarrow$ group is a subgroup of the group of self-homeomorphisms of the Cantor space.]