MAXIMAL ORDERS IN SEMIGROUPS[1]

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Introduction

Inspired by the ring theory concepts of orders and classical rings of quotients, Fountain and Petrich introduced the notion of a completely 0-simple semigroup of quotients in [18]. This was generalised to a much wider class of semigroups by Gould in [19]. The notion extends the well known concept of group of quotients [8]. To give the definition we first have to explain what is meant by a square-cancellable element in a semigroup. Let a be an element of a semigroup S. We say that a is *square-cancellable* in S if for all $x, y \in S^1$,

$$xa^2 = ya^2 \text{ implies that } xa = ya$$

and

$$a^2x = a^2y \text{ implies that } ax = ay.$$

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It is clear that all cancellable elements are square-cancellable but the converse is not true because it is equally clear that any element which lies in a subgroup of S is also square-cancellable. In fact, it is easy to see that being square-cancellable is a necessary condition for an element of S to be a member of a subgroup of some oversemigroup of S.

We now define a semigroup Q to be a *semigroup of left quotients* of a semigroup S if S is a subsemigroup of Q satisfying

(i) every square-cancellable element of S lies in a subgroup of Q, and

(ii) every element of Q can be written as $a^\sharp b$ for some elements a, b of S where a is square-cancellable and $a^\sharp$ is the inverse of a in a subgroup of Q.

We also say that S is a *left order* in Q.

Semigroups of right quotients and *right orders* are defined dually. If S is both a left order and a right order in Q, then we say that S is an *order* in Q and that Q is a *semigroup of quotients* of S.

In ring theory the classical ring of (one-sided) quotients has an identity but this is not appropriate in the context of semigroups. Our definition gives equal status to all the maximal subgroups of the semigroup. When applied to rings with identity it coincides in many cases with the classical ring of quotients [14].

Much research has been devoted to characterising orders or left orders in various classes of semigroups and surveys of the early work of this type may be found in [10, 11, 22]. More recent papers on this aspect of the theory are [1, 2].

The definition of left order in a semigroup Q can also be applied to the case when Q is a ring, where in addition we require the left order to be a subring of Q. As already mentioned, for several important classes of rings with identity, our definition gives the classical notion of left order. However, our definition can also be applied when Q is a ring *without* identity and this idea has been explored in a series of papers by the first two authors [14, 17, 15, 16], Ánh and Márki [3, 4, 6, 5] and López, Rus and Campos [9].

The present paper studies orders from a different perspective, again inspired by ring theory. To say that S is a left order in a semigroup or ring Q tells us something about the way in which S sits in Q and also ties the structure of S very closely to that of Q. For a given semigroup or ring Q there may, of course, be many different orders in Q and we can ask how the orders in a fixed Q might be classified. In the case when Q is a ring with identity this question has been extensively investigated. If R and S are (classical) left orders in such a ring Q, then R and S are said to be *equivalent* if there are units $a, b, c, d \in Q$ such that $aRb \subseteq S$

and $cSd \subseteq R$. A left order is then said to be a *maximal* left order in Q if it is maximal (under inclusion) in its equivalence class. If R is an order in Q and S is a left order equivalent to R, then S is also an order in Q so that the equivalence class of an order consists entirely of orders and there is no ambiguity in the notion of maximal order. The concept of maximal order was introduced by Asano [7] and generalises the notion of completely integrally closed commutative integral domain. We refer the reader to [25] and Chapters 3 and 5 of [26] for further details.

Our aim in this paper is to develop corresponding notions of equivalence and maximality within the context of semigroups of quotients. We restrict our attention to regular semigroups of quotients and, in fact, work with what we call weak straight left orders (see Section 1) in a regular semigroup Q. In Section 1 we introduce two equivalence relations, $\sim$ and $\equiv$, the latter being defined in terms of the former. Two weak straight left orders are said to be equivalent if they are related by $\equiv$. Maximal weak straight left orders are those which are maximal in their $\equiv$-classes. Although the definition of equivalence is rather complicated we show that it can be used effectively in the remaining sections of the paper. In Section 2 we show that the maximal orders in commutative groups are precisely those commutative cancellative monoids which can be thought of as analogues of completely integrally closed commutative integral domains. We also provide examples to illustrate our concepts of equivalence and maximality and consider the relationship between them and the established notions in ring theory.

We investigate maximal left orders in Clifford semigroups in Section 3 and obtain a characterisation for a class of Clifford semigroups which includes the class of commutative regular semigroups. In Section 4 we investigate when weak straight (left) orders are $\sim$-maximal, that is, maximal in their $\sim$-classes. For the one-sided case this leads us to introduce the notion of fractional ideal by analogy with ring theory.

Some results of this paper are contained in the thesis of the third author [27] and together with the results of a sequel by the first two authors they were reported in [12].

1. Equivalence

Throughout the paper, Q denotes a regular semigroup. A subset U of Q is said to be *large* if it has non-empty intersection with each group $\mathcal{H}$-class of Q. We denote the set of all large subsemigroups of Q by $\mathcal{LS}_Q$.

Following [14] we define a *weak left order* in Q to be a subsemigroup S of Q such that every element q of Q can be written as $q = a^\sharp b$ for some $a, b \in S$.

Thus we have dropped condition (i) in the definition of left order so that square-cancellable elements of S are not required to lie in subgroups of Q. Clearly we can formulate corresponding notions of *weak right order* and *weak order*.

We point out that for some semigroups Q a weak left order is automatically a left order. For example, this is clearly the case if Q is completely regular or completely 0-simple.

We say that a (weak) left order S in Q is *straight* if every element of Q can be written as $a^\sharp b$ where $a, b \in S$ and $a \,\mathcal{R}\, b$ in Q. *(Weak) straight right orders* and *(weak) straight orders* are defined in the obvious way. Straight left orders in semigroups are studied in [23] and straight left orders in rings in [17]. Examples of (weak) straight left orders are plentiful in view of the following result from [21]. It was given there for left orders but the proof is valid for weak left orders.

Proposition 1.1. *If Q is a regular semigroup on which $\mathcal{H}$ is a congruence, then any weak left order in Q is straight.*

We make extensive use of another result from the same paper.

Proposition 1.2. *Let S be a weak straight left order in a semigroup Q. If $q \in Q$ and $q = a^\sharp b$ where $a, b \in S$ with a square-cancellable and if $a \,\mathcal{R}\, b$ in Q, then $q \,\mathcal{H}\, b$ in Q. Hence S has non-empty intersection with every $\mathcal{H}$-class of Q. Further, if H is a group $\mathcal{H}$-class of Q, then $S \cap H$ is a left order in H.*

The latter property is referred to in [21] as *locality* of S in Q. Let S, T be subsets of Q and let Π index the group $\mathcal{H}$-classes of Q. We define the relation $\sim$ by the rule that $S \sim T$ if and only if

for all $\sigma \in \Pi$, there are elements $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in H_\sigma$ such that for all $\pi, \theta \in \Pi$, $a_\pi S b_\theta \subseteq T$ and $c_\pi T d_\theta \subseteq S$.

We are concerned with the restriction of $\sim$ to $\mathcal{LS}_Q$ and we have the following easy lemma.

Lemma 1.3. *On $\mathcal{LS}_Q$, the relation $\sim$ is an equivalence.*

Proof If $S \in \mathcal{LS}_Q$, then, for all $\pi, \theta \in \Pi$, we have $S \cap H_\pi \neq \emptyset$ and $S \cap H_\theta \neq \emptyset$. If $a_\pi \in S \cap H_\pi$ and $a_\theta \in S \cap H_\theta$, then certainly $a_\pi S a_\theta \subseteq S$ and so $\sim$ is reflexive.

Clearly $\sim$ is symmetric.

If $S, T, R \in \mathcal{LS}_Q$ with $S \sim T, T \sim R$, then for all $\pi, \theta \in \Pi$ there are elements $a_\pi, a'_\pi, c_\pi, c'_\pi$ in H_π and $b_\theta, b'_\theta, d_\theta, d'_\theta$ in H_θ such that $a_\pi S b_\theta \subseteq$

$T, c_\pi T d_\theta \subseteq S, a'_\pi T b'_\theta \subseteq R$ and $c'_\pi R d'_\theta \subseteq T$. Hence $(a'_\pi a_\pi) S (b_\theta b'_\theta) \subseteq R$ and $(c_\pi c'_\pi) R (d'_\theta d_\theta) \subseteq S$ so that $S \sim R$ since H_π and H_θ are groups.

It is an immediate consequence of the following result that if $[S]$ denotes the $\sim$-equivalence class of $S \in \mathcal{LS}_Q$, then either $[S]$ consists entirely of weak straight left orders in Q or contains no such subsemigroups.

Proposition 1.4. *Let S be a weak straight left order in Q and suppose that $S \sim T$ for some $T \subseteq Q$. Then any element q of Q can be written as $q = u^\sharp v$ for some element $u, v \in T$ with $u \mathcal{R} v$ in Q. Thus if T is a subsemigroup of Q, then T is a weak straight left order in Q.*

Proof Let $q \in Q$ and let e, f be idempotents such that $e \mathcal{R} q \mathcal{L} f$ and suppose that $e \in H_\pi, f \in H_\theta$ for $\pi, \theta \in \Pi$. For any $\sigma \in \Pi$, there are elements $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in H_\sigma$ satisfying the definition of $S \sim T$. From $b_\pi \mathcal{H} e \mathcal{R} q \mathcal{L} f \mathcal{H} b_\theta^\sharp$ and the fact that $\mathcal{R}$ (respectively $\mathcal{L}$) is a left (respectively right) congruence, we deduce that $q \mathcal{H} b_\pi q b_\theta^\sharp$. Since S is a weak straight left order in Q, there are elements $h, k \in S$ with $h \mathcal{R} k$ in Q and $b_\pi q b_\theta^\sharp = h^\sharp k$ so that by Proposition 1.2, $b_\pi q b_\theta^\sharp \mathcal{H} k$ and hence $q \mathcal{H} k$.

Consequently $h \mathcal{H} h^2 \mathcal{R} b_\pi$ and $h^\sharp \mathcal{H} (h^\sharp)^2 \mathcal{R} a_\pi^\sharp$. Since $\mathcal{R}$ is a left congruence, it follows that $hb_\pi \mathcal{H} b_\pi$ and $h^\sharp a_\pi^\sharp \mathcal{H} a_\pi^\sharp$. Thus $a_\pi h b_\pi, b_\pi^\sharp h^\sharp a_\pi^\sharp \in H_\pi$ and we obtain $(a_\pi h b_\pi)^\sharp = b_\pi^\sharp h^\sharp a_\pi^\sharp$. Now

$$\begin{aligned} q = eqf &= (b_\pi^\sharp b_\pi) q (b_\theta^\sharp b_\theta) = b_\pi^\sharp (b_\pi q b_\theta^\sharp) b_\theta \\ &= b_\pi^\sharp h^\sharp k b_\theta = b_\pi^\sharp h^\sharp a_\pi^\sharp a_\pi k b_\theta = (a_\pi h b_\pi)^\sharp a_\pi k b_\theta \end{aligned}$$

and certainly $u = a_\pi h b_\pi, v = a_\pi k b_\theta \in T$. Finally, since

$$a_\pi \mathcal{H} e \mathcal{R} k \mathcal{L} f \mathcal{H} b_\theta,$$

it follows that $a_\pi k b_\theta \mathcal{H} k$ and so $u \mathcal{H} e \mathcal{R} q \mathcal{H} k \mathcal{H} v$.

In general, $\sim$ is not the relation we want to define equivalence of orders but we now use it to construct the appropriate relation. First, for a principal factor J/I of Q and subsemigroup S of Q, put $S_I = (S \cap (J \setminus I)) \cup \{I\}$. The proof of the next lemma is straightforward.

Lemma 1.5. *If S is a large subsemigroup of Q, then S_I is a large subsemigroup of J/I. Furthermore, if S is a weak straight left order in Q, then S_I is a weak straight left order in J/I.*

We now define a relation $\equiv$ on $\mathcal{LS}_Q$ by the rule that
$S \equiv T$ if and only if $S_I \sim T_I$ in J/I for each principal factor J/I of Q.
The following result is immediate from Lemmas 1.3 and 1.5.

Corollary 1.6. *The relation $\equiv$ is an equivalence relation on $\mathcal{LS}_Q$.*

Lemma 1.7. *If S is a weak straight left order in Q and $S \equiv T$ for some subsemigroup T of Q, then T is also a weak straight left order in Q.*

Proof If $S \equiv T$ and J/I is any principal factor of Q, then S_I is a weak straight left order in J/I, T_I is a subsemigroup of J/I and $S_I \sim T_I$ so that by Proposition 1.4, T_I is a weak straight left order in J/I. It is now routine to verify that T is a weak straight left order in Q.

We now define weak straight left orders S and T to be *equivalent* if $S \equiv T$. We also say that a weak straight left order is *maximal* if it is a maximal member (under inclusion) of its $\equiv$-equivalence class in $\mathcal{LS}_Q$.

The relations $\sim$ and $\equiv$ are, of course, closely related and clearly, when Q is simple or 0-simple so that we have only one non-trivial principal factor, then $\sim$ and $\equiv$ coincide on $\mathcal{LS}_Q$. In general, we have the following result.

Lemma 1.8. *If $S, T \in \mathcal{LS}_Q$ and $S \sim T$, then $S \equiv T$.*

Proof This follows easily from the definitions of $\sim$ and $\equiv$ because in any principal factor J/I of Q a non-trivial group $\mathcal{H}$-class of J/I is actually a group $\mathcal{H}$-class of Q.

In general the relations $\sim$ and $\equiv$ do not coincide as we see from the simple example below.

Example 1.9.

Let $Q = G_1 \cup G_0$ be a chain of two infinite cyclic groups G_1, G_0 with generators a, b respectively and multiplication determined by the trivial homomorphism $G_1 \to G_0$.

Put $C_1 = \{a^k : k \geq 0\}, C_0 = \{b^k : k \geq 0\}, S_0 = \{b^k : k \geq 2\}$ and let $M = C_1 \cup C_0, N = C_1 \cup S_0$. Then it is easy to see that M and N are both straight orders in Q.

Now $S_0 \subseteq C_0$ and $bC_0b \subseteq S_0$ so that $S_0 \sim C_0$ in the principal factor G_0 and certainly $C_1 \cup \{0\} \sim C_1 \cup \{0\}$ in the principal factor $G_1 \cup \{0\}$. Hence $M \equiv N$.

On the other hand, $C_0 \subseteq a^h M a^k$ for all $h, k \in \mathbb{Z}$ so that $a^h M a^k \not\subseteq N$ and M and N are not $\sim$-related.

Suppose that M is a weak straight left order in a regular semigroup P and $M \subseteq N$ for some subsemigroup N of P. Clearly N is also a weak straight left order in P. Let Θ index the group $\mathcal{H}$-classes of P.

Since M is large in P we can choose $m_\theta \in M \cap H_\theta$, for any $\theta \in \Theta$. Clearly $m_\pi M m_\phi \subseteq N$ for any $\pi, \phi \in \Theta$.

This elementary observation simplifies what we have to do to prove that a weak straight left order is maximal.

Lemma 1.10. *Let S be a weak straight left order in Q and, for any principal factor J/I of Q, let the group $\mathcal{H}$-classes of J/I be indexed by Π_J. Then S is a maximal order in Q if and only if the following condition holds:-*

if $T \in \mathcal{LS}_Q, S \subseteq T$ and for every principal factor J/I of Q and all $\pi, \theta \in \Pi_J$ there are elements $a_\pi \in H_\pi, b_\theta \in H_\theta$ such that $a_\pi T_I b_\theta \subseteq S_I$, then $S = T$.

2. Examples and special cases

The definitions of the previous section might seem unwieldy and difficult to use. To persuade the reader that they are natural and potentially useful we provide a number of examples and also some results which show that in several special cases of interest the definitions simplify and become easy to apply.

Proposition 2.1. *A commutative cancellative semigroup C is a maximal order in its group of quotients G if and only if C satisfies the following condition:-*

(A) if $a \in C, g \in G$ are such that $ag^n \in C$ for all $n \geq 1$, then $g \in C$.

Proof If C is maximal and $a \in C, g \in G$ are such that $ag^n \in C$ for all $n \geq 1$, then
$$D = \bigcup \{C^1 g^k : k \geq 0\}$$
is clearly a subsemigroup of G which contains C and hence D is an order in G. Further, $aD1 \subseteq C$ and hence C and D are equivalent. Since C is maximal we have $C = D$ and hence $g \in C$.

Conversely, if condition (A) is satisfied, and if B is an order in G equivalent to and containing C, then B^1 also has these properties and so $aB^1 \subseteq C$ for some element a of G. It follows that $a \in C$ and so if $g \in B$, then $g^n \in B$ for all $n \geq 1$ so that $ag^n \in C$ for all $n \geq 1$ and hence $g \in C$ by condition (A). Thus $B = C$ and C is maximal.

We remark that when condition (A) is applied to the non-zero elements of a commutative integral domain we get a completely integrally closed domain. Thus the proposition is a precise analogue of the ring theory result and, indeed, the proof is essentially the same.

Example 2.2. *In an infinite cyclic group G with generator a there are, excluding G itself, two equivalence classes of orders in G each* *containing just one maximal order. The two maximal orders in G other than G itself are $C = \{a^k : k \geq 0\}$ and $D = \{a^k : k \leq 0\}$.*

Proof Any subsemigroup which contains both positive and negative powers of a is a subgroup, and so all orders other than G are contained in either C or D. It is easy to see that any order contained in C (respectively D) is equivalent to C (respectively D) and that neither C nor D is equivalent to G.

Now let T be a left order in a group G and let P be a $\Lambda \times I$ matrix over $T \cup \{0\}$. If every column of P contains a non-zero entry, then the Rees matrix semigroup $S = \mathcal{M}^0(T; I, \Lambda; P)$ is a left order in $Q = \mathcal{M}^0(G; I, \Lambda; P)$. Even if we assume that every row of P contains a non-zero element so that Q is completely 0-simple we can still have T maximal in G but S not maximal in Q as the following example demonstrates.

Example 2.3. *Let G be the infinite cyclic group generated by a. Let $T = \{a^k : k \geq 0\}$ and $P = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then S is not maximal in Q.*

Proof Put $R = S \cup \{(2, a^{-1}, 1)\}$. Then it is easy to verify that R is a left order in Q. Choosing the element $(1, a, 1), (2, a, 2)$ from the two non-zero group $\mathcal{H}$-classes of Q, we see that $(i, a, i)R(j, a, j) \subseteq S$ for $i, j \in \{1, 2\}$ and so R is equivalent to S. Hence S is not maximal.

For a positive result we have to consider a sandwich matrix whose rows and columns contain units of T.

Proposition 2.4. *If each row and each column of P contains a unit of T and if T is maximal in G, then S is maximal in Q.*

Proof Suppose that R is a left order in Q containing S and equivalent to S. Let $i \in I$ and $\lambda \in \Lambda$ and put $R_{i\lambda} = \{g \in G : (i, g, \lambda) \in R\}$. For some $\mu \in \Lambda, j \in I$ the elements $p_{\mu i}, p_{\lambda j}$ are units in T so that if $g_1, g_2 \in R_{i\lambda}$, then

$$(i, g_1, \lambda)(j, p_{\lambda j}^{-1} p_{\mu i}^{-1}, \mu)(i, g_2, \lambda) \in R$$

and hence $g_1 g_2 \in R_{i\lambda}$. Now $T \subseteq R_{i\lambda}$ since $S \subseteq R$ and so $R_{i\lambda}$ is a left order in G.

As R and S are equivalent, there are elements $(i, a, \mu), (j, b, \lambda)$ in Q such that $(i, a, \mu)R(j, b, \lambda) \subseteq S$ and so $a p_{\mu i} g p_{\lambda j} b \in T$ for any $g \in R_{i\lambda}$. Hence $a p_{\mu i} R_{i\lambda} p_{\lambda j} b \subseteq T$ and so $R_{i\lambda}$ is equivalent to T. But T is maximal in G so that $R_{i\lambda} = T$ and hence $R = S$, that is, S is a maximal left order in Q.

Recall that if S is a semigroup and θ is an endomorphism of S, then we can form the Bruck-Reilly extension $BR(S,\theta)$ with underlying set $\mathbb{N} \times S \times \mathbb{N}$ and multiplication defined by

$$(m,a,n)(p,b,q) = (m-n+t, a\theta^{t-n}b\theta^{t-p}, q-p+t)$$

where $t = \max\{n,p\}$. Now suppose that T is a left order in a group G and that θ is an endomorphism of T with an extension $\overline{\theta} : G \to G$. It is observed in [19] that $B = BR(T,\theta)$ is a left order in $Q = BR(G,\overline{\theta})$.

Proposition 2.5. *If T is maximal in G, then B is maximal in Q.*

Proof Let S be a left order in Q containing B and equivalent to B. For $m,n \in \mathbb{N}$, put $S_{(m,n)} = \{g \in G : (m,g,n) \in S\}$. Since T is maximal in G, $1 \in T$ so that $(n,1,m) \in B$ for all $m,n \in \mathbb{N}$. It is then easy to see that $S_{(m,n)}$ is a submonoid of G and, in fact, it is a left order in G since $T \subseteq S_{(m,n)}$.

Now B is equivalent to S and so there are elements $(m,a,m),(n,b,n)$ such that $(m,a,m)S(n,b,n) \subseteq B$. It follows that $aS_{(m,n)}b \subseteq T$ and hence that $S_{(m,n)}$ is equivalent to T. But T is maximal so that $T = S_{(m,n)}$ and hence $B = S$.

When Q is a regular monoid the relation $\sim$ can be defined in a much simpler way than for the general case of a regular semigroup.

Proposition 2.6. *If Q is a regular monoid and S,T are weak straight left orders in Q, then $S \sim T$ if and only if there are invertible elements u,v,x,y in Q such that $uSv \subseteq T$ and $xTy \subseteq S$.*

The proof rests on the following lemma.

Lemma 2.7. *Let S,T be subsemigroups of a regular semigroup Q which meet every $\mathcal{H}$-class of Q. Let I index the $\mathcal{R}$-classes and Λ index the $\mathcal{L}$-classes of Q. Then $S \sim T$ if and only if for all $i \in I$ and $\lambda \in \Lambda$ there are elements $a_i, c_i \in R_i, b_\lambda, d_\lambda \in L_\lambda$ such that $a_iSb_\lambda \subseteq T$ and $c_iTd_\lambda \subseteq S$.*

Proof If $S \sim T$, then it is easy to see that the condition holds.

Now suppose that the condition holds and that the group $\mathcal{H}$-classes are indexed by Π. For each $\pi \in \Pi$, let e_π be the identity of H_π and let $i = i_\pi \in I$ be such that $H_\pi \subseteq R_i$. Then $a_i \mathcal{R} e_\pi$ and so there is a *semigroup* inverse a_i' of a_i with $e_\pi = a_ia_i'$, and $a_i \mathcal{L} a_i'a_i \mathcal{R} a_i'$. Let $s_\pi \in S$ be such that $s_\pi \mathcal{H} a_i'$. Then $a_\pi = a_is_\pi \in H_\pi$. Similarly, if $\lambda = \lambda_\pi$ is such that $e_\pi \in L_\lambda$, then there is an element $t_\pi \in S$ with $b_\pi = t_\pi b_\lambda \in H_\pi$.

Then for all $\pi, \theta \in \Pi$, with $e_\theta \in L_\zeta$

$$a_\pi Sb_\theta = a_is_\pi St_\theta b_\zeta \subseteq a_iSb_\zeta \subseteq T.$$

Together with its dual, this gives $S \sim T$.

It is now easy to prove Proposition 2.6. For, suppose that u, v, x, y are units in Q with $uSv \subseteq T$ and $xTy \subseteq S$ and for each $i \in I, \lambda \in \Lambda$, choose elements $T_i \in T \cap R_i$ and $s_\lambda \in T \cap L_\lambda$. Then, since $1 \mathcal{R} u$ and $1 \mathcal{L} v$ we have that $t_i \mathcal{R} t_i u$ and $s_\lambda \mathcal{L} vs_\lambda$ so that for all i, λ,

$$(t_i u)S(vs_\lambda) \subseteq t_i T s_\lambda \subseteq T.$$

Together with its dual this gives $S \sim T$ by Lemma 2.7.

We conclude this section with a result which relates the ring theory notion of equivalence with our relation $\sim$. First we recall that if S is a subring of a regular ring Q with identity, then by Theorem 3.4 and Corollary 3.6 of [14], S is an order in Q in the ring theory sense if and only if S is an order in Q in the semigroup sense. Thus there is no ambiguity in the phrase 'S is an order in Q' and we may speak of straight orders in Q. The following result is now immediate from Proposition 2.6.

Corollary 2.8. *If Q is a regular ring with identity and the subrings S, T are straight orders in Q, then $S \sim T$ if and only if S and T are equivalent orders in the ring theory sense.*

3. Maximal orders in semilattices of groups

Let $Q = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α with linking homomorphisms $\theta_{\alpha,\beta}$ for $\alpha, \beta \in Y$ with $\alpha \geq \beta$. For each $\alpha \in Y$, let e_α be the identity of G_α.

If S is a left order in Q, then by [20], $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semilattice Y of right reversible, cancellative semigroups $S_\alpha = G_\alpha \cap S$ and for each $\alpha \in Y$, S_α is a left order in G_α. The question arises as to whether there is any connection between the semigroup S being a maximal left order in Q and each constituent S_α being a maximal left order in G_α. On the positive side we have:-

Proposition 3.1. *If for each $\alpha \in Y, S_\alpha$ is a maximal left order in G_α, then S is a maximal left order in Q.*

Proof Let T be a left order in Q which contains S and is equivalent to S. Then for $\alpha \in Y$, $S_\alpha = S \cap G_\alpha \subseteq T \cap G_\alpha$ and putting $T_\alpha = T \cap G_\alpha$ we have that T_α is a left order in G_α. Since $T \equiv S$ and $G_\alpha \cup \{0\}$ (or G_α if α is the zero of Y) is a principal factor of Q, we see that $T_\alpha \cup \{0\} \sim S_\alpha \cup \{0\}$. Hence $T_\alpha \sim S_\alpha$ and so $S_\alpha = T_\alpha$ since S_α is maximal in G_α. Thus $T = S$ and so S is maximal.

Whether the converse of Proposition 3.1 is true is an open question. We do, however, have a partial answer. First we need two straightforward lemmas. The first follows from the definition of the relation $\equiv$ and the fact that the principal factors of Q are the semigroups $G_\alpha \cup \{0\}$ for each non-zero $\alpha \in Y$ and G_ζ if Y has zero ζ.

Lemma 3.2. *Let S, T be left orders in Q and let $S_\alpha = S \cap G_\alpha, T_\alpha = T \cap G_\alpha$. Then*

$$S \equiv T \text{ if and only if } S_\alpha \sim T_\alpha \text{ for all } \alpha \in Y.$$

Lemma 3.3. *Let S be a left order in Q and let*

$$T = \langle \bigcup_{\alpha \in Y} (S_\alpha \cup \{e_\alpha\}) \rangle$$

be the subsemigroup of Q generated by S and $E(Q)$. Then T is a left order in Q and $S \equiv T$.

Proof That T is a left order in Q is immediate since $S \subseteq T$. By Lemma 3.2, to show that $S \equiv T$ we have to show that $S_\alpha \sim T_\alpha$ for each $\alpha \in Y$ and since $S_\alpha \subseteq T_\alpha$, we need only show that there are elements $u_\alpha, v_\alpha \in G_\alpha$ such that $u_\alpha T_\alpha v_\alpha \subseteq S_\alpha$. Choose u_α, v_α to be any two elements of S_α. Now $E(Q)$ is central in Q so that if $t \in T_\alpha$, then $t = e_\gamma s$ for some $e_\gamma \in E(Q), s \in S$. In fact, we must have $s \in S_\delta$ for some δ such that $\gamma\delta = \alpha$ and so $t = e_\alpha t = e_\alpha s$. Now $u_\alpha t v_\alpha = u_\alpha e_\alpha s v_\alpha = u_\alpha s v_\alpha \in S_\alpha$ since $u_\alpha, v_\alpha, s \in S$ and $\alpha \leq \delta$. Thus $u_\alpha T_\alpha v_\alpha \subseteq S_\alpha$ and so $T_\alpha \sim S_\alpha$.

Corollary 3.4. *If S is a maximal left order in Q, then $E(Q) \subseteq S$.*

Theorem 3.5. *Suppose that for all $\alpha, \beta \in Y$ with $\alpha > \beta$, Im $\theta_{\alpha,\beta} \subseteq Z(G_\beta)$, the centre of G_β. Then S is a maximal left order in Q if and only if each S_α is a maximal left order in G_α.*

Proof Suppose that S is a maximal left order in Q and let $\alpha \in Y$. Let T_α be a left order in G_α which contains and is equivalent to S_α. Then there are elements $p, q \in G_\alpha$ such that $pT_\alpha q \subseteq S_\alpha$.

Now let $T = \langle S \cup T_\alpha \rangle$ be the subsemigroup of Q generated by S and T_α. Certainly T is a left order in Q since $S \subseteq T$. Any element of $T \cap G_\alpha$ is a product of elements of T_α and elements of S. Such elements of S must be members of some S_γ's where $\alpha \leq \gamma$ and so in a product with members of T_α they can be replaced by their images under $\theta_{\gamma,\alpha}$ for the appropriate γ. But in view of Corollary 3.4 all such images are in S_α and hence in T_α. Thus $T \cap G_\alpha = T_\alpha$. Hence there is no ambiguity if for each $\gamma \in Y$ we put $T_\gamma = G_\gamma \cap T$ as usual.

If $\gamma \in Y$ and $\alpha < \gamma$ or if α and γ are incomparable, then clearly $T_\gamma = S_\gamma$. Suppose that $\gamma < \alpha$. Then any element of T_γ is a product of elements of S and elements $t\theta_{\alpha,\gamma}$ where $t \in T_\alpha$. But the latter are central in G_γ and so if $t_\gamma \in T_\gamma$, then $t_\gamma = s(t\theta_{\alpha,\gamma})$ for some $s \in S, t \in T_\alpha$. Hence $(p\theta_{\alpha,\gamma})t_\gamma(q\theta_{\alpha,\gamma}) = (p\theta_{\alpha,\gamma})s(t\theta_{\alpha,\gamma})(q\theta_{\alpha,\gamma}) = s((ptq)\theta_{\alpha,\gamma})$ since $p\theta_{\alpha,\gamma}$ is also central and $\theta_{\alpha,\gamma}$ is a homomorphism. But $ptq \in S_\alpha$ so that $(p\theta_{\alpha,\gamma})t_\gamma(q\theta_{\alpha,\gamma}) \in S_\gamma$, and $(p\theta_{\alpha,\gamma})T_\gamma(q\theta_{\alpha,\gamma}) \subseteq S_\gamma$. Thus we have seen that $T_\gamma \sim S_\gamma$ for all γ and so by Lemma 3.2, $S \equiv T$. Now S is maximal and $S \subseteq T$ so that $S = T$ and hence $S_\alpha = T_\alpha$, that is, S_α is a maximal order in G_α.

We have the following immediate corollaries.

Corollary 3.6. *Let $Q = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of abelian groups G_α. Then $S = \bigcup_{\alpha \in Y} S_\alpha$ is a maximal order in Q if and only if, for each $\alpha \in Y$, S_α is a maximal order in G_α.*

Corollary 3.7. *Let $Q = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y with trivial linking homomorphisms. Then $S = \bigcup_{\alpha \in Y} S_\alpha$ is a maximal left order in Q if and only if, for each $\alpha \in Y$, S_α is a maximal left order in G_α.*

4. Fractional ideals

Although the relations $\equiv$ and $\sim$ do not coincide in general, there are, as we have noted, several interesting special cases where they do. It is shown in [13] that this is also the case when Q is a simple ring with minimal one-sided ideals. There is some interest, therefore, in examining necessary and sufficient conditions for a weak straight order to be $\sim$-maximal, that is, maximal in its $\sim$-equivalence class.

Let Q be a regular semigroup, S a subsemigroup of Q and K an ideal of S. We say that K is a *large ideal* if it is large as a subset of Q. Using this concept we have our first characterisation of $\sim$-maximal weak straight orders.

Proposition 4.1. *Let S be a weak straight order in a regular semigroup Q. Then S is $\sim$-maximal if and only if for all large ideals I of S and elements q of Q,*

$$qI \subseteq I \text{ implies } q \in S \text{ and } Iq \subseteq I \text{ implies } q \in S.$$

Proof Suppose that the condition holds and let T be a weak order in Q which contains S and is such that $S \sim T$. Let Π index the group $\mathcal{H}$-classes of Q. Then for any $\pi, \theta \in \Pi$, there are elements $a_\pi \in H_\pi, b_\theta \in H_\theta$ such that $a_\pi T b_\theta \subseteq S$. By Proposition 1.2 we can write $a_\pi = u_\pi{}^\sharp v_\pi, b_\theta = w_\theta z_\theta{}^\sharp$ for some elements $u_\pi, v_\pi \in H_\pi \cap S$ and

$w_\theta, z_\theta \in H_\theta \cap S$. It follows that $v_\pi T w_\theta \subseteq S$ for all $\pi, \theta \in \Pi$ and hence that the set

$$I = \{s \in S : sTw_\theta \subseteq S \text{ for all } \theta \in \Pi\}$$

is large in Q. Since $S \subseteq T$ we have for each $\theta \in \Pi$ that

$$(S^1 I S^1) T w_\theta \subseteq S^1 I T w_\theta \subseteq S^1 S \subseteq S$$

so that $S^1 I S^1 \subseteq I$ and I is an ideal of S. Furthermore, for all $\pi, \psi \in \Pi$, we have $ITw_\pi \subseteq S$ and

$$(ITw_\pi)(Tw_\psi) = I(Tw_\pi T)w_\psi \subseteq ITw_\psi \subseteq S$$

since $w_\pi \in S \subseteq T$. Hence $ITw_\pi \subseteq I$ and so, by the condition, $Tw_\pi \subseteq S$.

Now let $K = \{s \in S : Ts \subseteq S\}$. Then $w_\pi \in K$ for all $\pi \in \Pi$ so that K is large in Q. It is easy to see that K is an ideal of S and since $T(TK) \subseteq TK \subseteq S$ it follows that $TK \subseteq K$. Hence $T \subseteq S$ by the given condition. Thus $T = S$ and S is $\sim$-maximal in Q.

Conversely suppose that S is $\sim$-maximal in Q and that I is a large ideal of S. Put

$$T = \{q \in Q : qI \subseteq I\}.$$

Certainly $S \subseteq T$ and T is a subsemigroup of Q. It follows that T is a weak straight order in Q. For any $\pi, \theta \in \Pi$ choose an element $a_\pi \in S \cap H_\pi$ and an element $b_\theta \in I \cap H_\theta$. Then $a_\pi T b_\theta \subseteq a_\pi I \subseteq I \subseteq S$ so that $S \sim T$ and by the maximality of S, $S = T$. Thus $qI \subseteq I$ implies $q \in S$. The dual condition is obtained in a similar manner.

Remark It is worth mentioning that the condition of the theorem is also, in fact, a necessary condition for S to be a maximal weak straight order in Q. A sufficient condition for S to be maximal can be obtained by imposing the conditions on each principal factor of Q rather than Q itself.

The study of equivalent and maximal orders in ring theory is facilitated by the notion of a fractional ideal. The following definitions provide semigroup analogues of this concept. We introduce them in order to obtain a result corresponding to Proposition 4.1 for weak straight *left* orders; it transpires that in the one-sided case they play the role that large ideals take in that proposition.

Let S be a large subsemigroup of a regular semigroup Q. A subset I of Q is a *left S-ideal* if

(i) $SI \subseteq I$, and

(ii) I is large in Q.

By replacing (i) by its dual we obtain the notion of a *right S-ideal* and if (i) and its dual both hold we have an *S-ideal*.

A left S-ideal I is a *left fractional left S-ideal* if

(iii) for every group $\mathcal{H}$-class H of Q there is an element $c \in H$ such that $Ic \subseteq S$.
By replacing (iii) by its dual we obtain the notion of a *right fractional left S-ideal* and if (iii) and its dual both hold we have a *fractional left S-ideal.*

When S is a weak straight left (or right) order in Q we have an alternative description of left fractional left S-ideals given by the following proposition.

Proposition 4.2. *Let S be a weak straight left (or right) order in a regular semigroup Q and let I be a subset of Q. Then I is a left fractional left S-ideal of Q if and only if the following conditions hold:-*
(i) $SI \subseteq I$
(ii) I meets every $\mathcal{L}$-class of Q
(iii) for every $\mathcal{R}$-class R of Q, there is an element $b \in R$ such that $Ib \subseteq S$.

Proof Suppose that I is a left fractional left S-ideal of Q. Then certainly condition (i) holds. Condition (ii) is immediate from the fact that I is large and condition (iii) is immediate from (iii) in the definition of left fractional left S-ideal.

Conversely suppose that the conditions of the proposition hold. We have condition (i) of the definition. Now let H be a group $\mathcal{H}$-class of Q with identity e. Then there is an element c in $L_e \cap I$. Let f be an idempotent in R_c, the $\mathcal{R}$-class of c. Then, by Proposition 1.2, there is an element a in $S \cap (R_e \cap L_f)$. Now $f^2 = f \in L_a \cap R_c$ and so, by Proposition 2.3.7 of [24], $ac \in R_a \cap L_c$, that is, $ac \in H$. Since $c \in I$ and $a \in S$, condition (i) gives $ac \in I$ and so $H \cap I \neq \emptyset$ and I is large.

Finally, we again let H be a group $\mathcal{H}$-class of Q with identity e and let $b \in R_e$ be such that $Ib \subseteq S$. Let f be an idempotent in the $\mathcal{L}$-class of b. Again, by Proposition 1.2, there is an element c in $S \cap (L_e \cap R_f)$ and again by Proposition 2.3.7 of [24], $bc \in H$. Now $Ibc \subseteq Sc \subseteq S$ and so I is a left fractional left S-ideal.

This allows us to obtain the following result for monoids.

Corollary 4.3. *Let S be a weak straight left (or right) order in a regular monoid Q and let I be a subset of Q. Then I is a left fractional left S-ideal if and only if the following conditions hold:-*
(i) $SI \subseteq I$,
(ii) I contains a unit of Q,
(iii) there is a unit v of Q such that $Iv \subseteq S$.

Proof It is clear from the definition that the conditions hold for a left fractional left S-ideal.

Suppose now that the conditions hold for a subset I and let L be an $\mathcal{L}$-class of Q. Let u be a unit of Q in I and let e be an idempotent in L. Then $e = zu$ for some $z \in Q$ and since, by Proposition 1.2, S meets every $\mathcal{H}$-class of Q, there is an element a in $S \cap H_z$. From $a\mathcal{L}z$ we obtain $au \,\mathcal{L}\, zu$. Now $au \in I$ since $SI \subseteq I$ and so $L \cap I \neq \emptyset$.

Now let v be a unit of Q such that $Iv \subseteq S$ and let R be an $\mathcal{R}$-class of Q and e an idempotent in R. Then $e = vy$ for some $y \in Y$ and S meets H_y. Let $a \in S \cap H_y$. Then $va \,\mathcal{R}\, vy$ since $a \,\mathcal{R}\, y$ and so $va \in R$. Also $Iva \subseteq Sa \subseteq S$.

Thus the conditions of Proposition 4.2 hold and hence I is a left fractional left S-ideal.

The conditions of Corollary 4.3 are precisely those used to define a (left) fractional left S-ideal in ring theory [26]. It follows from Corollary 3.6 of [14] and Proposition 3.10 of [17] that ring orders in simple artinian rings are straight left and right orders in our sense and so our concept of fractional ideal coincides with the standard one in this case.

We conclude with a result which gives a criterion for one-sided straight weak orders in a regular semigroup to be $\sim$-maximal. It is similar to the condition in Proposition 4.1 for two-sided orders but uses fractional S-ideals instead of large ideals.

Proposition 4.4. *Let S be a weak straight left order in Q. Then S is $\sim$-maximal if and only if for all left (respectively right) fractional S-ideals I, if $Iq \subseteq I$ (respectively $qI \subseteq I$), then $q \in S$.*

Proof Let Π index the group $\mathcal{H}$-classes of Q.

Suppose that S is $\sim$-maximal. Let I be a left fractional S-ideal and put $T = \{q \in Q : Iq \subseteq I\}$. Clearly, $S \subseteq T$ and T is a subsemigroup of Q so that T is a weak straight left order in Q. Now I meets every group $\mathcal{H}$-class of Q; say $a_\pi \in I \cap H_\pi$ for $\pi \in \Pi$. Also for all $\theta \in \Pi$, there is an element $c_\theta \in H_\theta$ with $Ic_\theta \subseteq S$ so that for all $\pi, \theta \in \Pi$, $a_\pi T c_\theta \subseteq Ic_\theta \subseteq S$. Then $S \sim T$ and since $S \subseteq T$ and S is $\sim$-maximal, $S = T$, that is, if $Iq \subseteq I$, then $q \in S$.

A similar argument gives the condition for right fractional S-ideals.

Conversely, suppose that the conditions hold and let T be a weak straight left order in Q which contains S and is such that $S \sim T$. Then for all $\pi, \theta \in \Pi$, there are elements $a_\pi \in H_\pi, b_\theta \in H_\theta$ with $a_\pi T b_\theta \subseteq S$. By Proposition 1.2, $a_\pi = u_\pi^{-1} v_\pi$ for some $u_\pi, v_\pi \in S \cap H_\pi$ so that $v_\pi T b_\theta \subseteq u_\pi S \subseteq S$. Let

$$I = \{x \in Q : xTb_\theta \subseteq S \text{ for all } \theta \in \Pi\}.$$

Then for any $\pi \in \Pi, v_\pi \in I$ so that I is large in Q. If $d_\pi \in S \cap H_\pi$, then $Id_\pi b_\pi \subseteq ISb_\pi \subseteq ITb_\pi \subseteq S$. Also, if $x \in I, \theta \in \Pi$, then

$$S^1 x S^1 T b_\theta \subseteq S^1 x T b_\theta \subseteq S^1 S \subseteq S$$

and hence $S^1 I S^1 \subseteq I$ so that I is a left fractional S-ideal.

Let $J = \{s \in S : Ts \subseteq S\}$. Let $\pi, \theta \in \Pi$ and d_π, b_θ be as above. Then

$$ITd_\pi T b_\theta \subseteq ITb_\theta \subseteq S$$

so that $ITd_\pi \subseteq I$ and hence $Td_\pi \subseteq S$ by the assumed conditions. Thus $J \neq \emptyset$. If $s \in J$, then clearly

$$TS^1 s S^1 \subseteq TsS^1 \subseteq SS^1 \subseteq S$$

so that $S^1 J S^1 \subseteq J$. If $\pi \in \Pi$, then $d_\pi \in J \cap H_\pi$ and $d_\pi J \subseteq S$. Hence J is a right fractional S-ideal. But $TJ \subseteq S$ and $T(TJ) \subseteq TJ \subseteq S$ so that $TJ \subseteq J$ and by the assumed conditions, $T \subseteq S$. Hence $T = S$ and S is $\sim$-maximal.

Remark As with Proposition 4.1, it is not difficult to modify the proof to show that the conditions are necessary for S to be a maximal weak straight left order in Q. Also, a sufficient condition for S to be maximal can be obtained by imposing the conditions on each principal factor of Q rather than Q itself.

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