Short presentations for some transformation-like semigroups

York Semigroup

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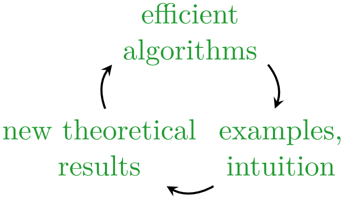
1st November 2023

Overview

Interested in presentations for various semigroups and monoids, with few relations

Three related strands: small; irredundant; minimum

Using computational tools to inspire



Presentations

Presentations consist of *generators* and *relations*. Relations specify where two words over the generators represent the same element.

$\langle a, b \mid ab = ba \rangle$ – or $\langle A \mid R \rangle$ with $A = \{a, b\}$ and $R = \{(ab, ba)\}$

$\langle a \mid a^5 = \varepsilon \rangle$

$\langle a \mid a^6 = a^5 \rangle$

Elementary sequences derive consequences of relations.

If $w = \alpha_1, \alpha_2, \ldots, \alpha_m = u$ is an elementary sequence (w.r.t. R), then $w = u$ in $\langle A \mid R \rangle$.

$$aabba, \quad ababa, \quad abaab$$

Redundancy

$\langle a, b \mid ab = ba,\ aabba = abaab \rangle$ has a redundant relation.

$\langle a, b \mid a^2b = ba,\ b^3 = a^2,\ (ab)^2 = bab \rangle$ has a redundant relation.

Definition (Irredundant)

A generating set for a semigroup/monoid is *irredundant* if it has no proper subsets which are generating sets.

If $\langle A \mid R \rangle$ is a presentation, we say that a relation $(u, v) \in R$ is *redundant* if $\langle A \mid R \setminus \{(u, v)\} \rangle$ defines the same semigroup.

If $\langle A \mid R \rangle$ contains no redundant relations, and A corresponds to an irredundant generating set for the semigroup defined, we say the presentation $\langle A \mid R \rangle$ is *irredundant*.

Some preliminaries

Any generating set for $\mathcal{T}_n$, $\mathcal{I}_n$ or $\mathcal{PT}_n$ must contain a generating set for S_n; similarly, any sets of defining relations must contain defining relations for S_n.

The *rank* of a transformation f is $|\operatorname{im} f|$.

The *kernel* of a transformation is the equivalence $(x, y) \in \ker f \Leftrightarrow (x)f = (y)f$.

Generating sets: for $\mathcal{T}_n$ and $\mathcal{I}_n$, we need a rank $n-1$ element. For $\mathcal{PT}_n$, we need a rank $n-1$ full transformation and a rank $n-1$ partial bijection.

$$\varepsilon_{1,2} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 4 & 5 \end{pmatrix} \qquad \varepsilon_{(1)} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & 2 & 3 & 4 & 5 \end{pmatrix}$$

Some preliminaries

Our usual generating set for S_n is $\{(1\ 2), (1\ 3), \ldots, (1\ n)\}$. Symbol π_i will correspond to $(1\ i)$.

> We always have defining relations for S_n. We can think of words representing permutations as permutations, and multiply them accordingly.

Example: instead of $\pi_2\pi_3\pi_2\varepsilon_{1,2}\pi_2$, we may write $(2\ 3)\varepsilon_{1,2}(1\ 2)$; or $(1\ 2\ 3)(1\ 2)\varepsilon_{1,2}(1\ 2)$, and so on...

If σ is a word representing a permutation s, then σ^{-1} is a word representing the permutation s^{-1}.

Often won't distinguish between symbols from alphabet and monoid elements: usually clear from context.

Theorem (Mitchell + W.)

There is a presentation for $\mathcal{T}_n$ with 5 non-S_n relations.

Aizenstat (1958) gives a presentation with 7 additional relations.

Theorem (Mitchell + W.)

There is a presentation for $\mathcal{I}_n$ with 3 non-S_n relations.

Multiple independent authors (inc. East, Popova) give a presentation with 5 additional relations.

Theorem (East, 2007)

There is a presentation for $\mathcal{PT}_n$ with 12 non-S_n relations.

Using our $\mathcal{T}_n$ work, we can reduce this to 11. We can also prove irredundancy.

Examples of some relations

Define $\varepsilon_{i,j}$ and $\varepsilon_{(i)}$ analogously to $\varepsilon_{1,2}$ and $\varepsilon_{(1)}$.

$$\pi_3\pi_4\pi_3\varepsilon_{1,2}\varepsilon_{3,2} = \pi_3\pi_4\pi_3\varepsilon_{1,2} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 1 & 4 & 3 & 5 & \cdots & n \end{pmatrix}$$

$$\varepsilon_{3,4}\varepsilon_{2,3}\varepsilon_{1,2} = \varepsilon_{1,2}\varepsilon_{1,3}\varepsilon_{1,4} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 1 & 1 & 1 & 5 & \cdots & n \end{pmatrix}$$

$$\varepsilon_{1,2}\varepsilon_{(1)} = \varepsilon_{(1)}\varepsilon_{(2)} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ - & - & 3 & 4 & \cdots & n \end{pmatrix}$$

If $\sigma \in S_n$ has $(i)\sigma = k$ and $(j)\sigma = l$, then $\sigma^{-1}\varepsilon_{i,j}\sigma = \varepsilon_{k,l}$.

Similarly, if $\sigma \in S_n$ has $(i)\sigma = j$, then $\sigma^{-1}\varepsilon_{(i)}\sigma = \varepsilon_{(j)}$.

A (big) presentation for $\mathcal{T}_n$

Theorem (Iwahori + Iwahori, 1974)

Let A consist of symbols t_{ij} representing all transpositions $(i\ j)$, and $\langle A \mid R \rangle$ be a presentation for S_n. Form A' from A by adding all symbols of the form $\varepsilon_{i,j}$ with $i \neq j$, and R' from R by adding all following relations for distinct i, j, k and l:

(a) $t_{kl}\varepsilon_{i,j}t_{kl} = \varepsilon_{i,j}$

(b) $t_{jk}\varepsilon_{i,j}t_{jk} = \varepsilon_{i,k}$

(c) $t_{ki}\varepsilon_{i,j}t_{ki} = \varepsilon_{k,j}$

(d) $t_{ij}\varepsilon_{i,j}t_{ij} = \varepsilon_{j,i}$

(e) $\varepsilon_{i,j}\varepsilon_{k,l} = \varepsilon_{k,l}\varepsilon_{i,j}$

(f) $\varepsilon_{i,j}\varepsilon_{i,k} = \varepsilon_{i,k}\varepsilon_{i,j} = \varepsilon_{k,j}\varepsilon_{i,k}$

(g) $\varepsilon_{i,j}\varepsilon_{j,k} = t_{jk}\varepsilon_{i,k}$

(h) $\varepsilon_{i,j}\varepsilon_{k,j} = \varepsilon_{i,j}$

(i) $\varepsilon_{i,j}\varepsilon_{i,j} = \varepsilon_{i,j}$

(j) $\varepsilon_{i,j}\varepsilon_{j,i} = \varepsilon_{j,i}$

(k) $t_{ij}\varepsilon_{i,j} = \varepsilon_{i,j}$.

Then $\langle A' \mid R' \rangle$ is a presentation for $\mathcal{T}_n$, where $n \geq 4$.

A (small) presentation for $\mathcal{T}_n$

Theorem (Aizenstat, 1958)

Where $\langle a, b \mid R \rangle$ is a presentation for S_n with a representing $(1\ 2)$ and b representing $(1\ 2\ \dots\ n)$, the following is a presentation for $\mathcal{T}_n$:

$$\Big\langle a, b, t \mid R,\ at = b^{n-2}ab^2tb^{n-2}ab^2 = bab^{n-1}abtb^{n-1}abab^{n-1} = \left(tbab^{n-1}\right)^2 = t, \left(b^{n-1}abt\right)^2 = tb^{n-1}abt = \left(tb^{n-1}ab\right)^2, \left(tbab^{n-2}ab\right)^2 = \left(bab^{n-2}abt\right)^2 \Big\rangle$$

Several relations correspond to specific instances of the previous presentation's families

A (smaller) presentation for $\mathcal{T}_n$

Starting point: redundant relations in Iwahori and Iwahori.

(Theorem, Mitchell + W.)

Let $A = \{\pi_2, \pi_3, \ldots, \pi_n\}$, and $\langle A \mid R \rangle$ a presentation for S_n with $\pi_i = (1\ i)$. Let $A' = A \cup \{\varepsilon_{1,2}\}$, and form R' from R by adding the following relations:

(i) $\varepsilon_{1,2}\pi_3\varepsilon_{1,2}\pi_3 = \varepsilon_{1,2}$;

(ii) $\varepsilon_{1,2}\pi_2\pi_3\pi_2\varepsilon_{1,2}\pi_2\pi_3\pi_2 = \pi_3\pi_2\varepsilon_{1,2}\pi_2\pi_3\varepsilon_{1,2}$;

(iii) $\varepsilon_{1,2}\pi_4\pi_2\pi_3\pi_2\varepsilon_{1,2}\pi_2\pi_3\pi_2\pi_4 = \pi_4\pi_2\pi_3\pi_2\varepsilon_{1,2}\pi_2\pi_3\pi_2\pi_4\varepsilon_{1,2}$;

(iv) $\pi_3\pi_4\pi_3\varepsilon_{1,2} = \varepsilon_{1,2}\pi_3\pi_4\pi_3$;

(v) $\pi_3\pi_4\pi_5\cdots\pi_n\pi_3\pi_2\varepsilon_{1,2} = \varepsilon_{1,2}\pi_3\pi_4\pi_5\cdots\pi_n\pi_3$.

Then $\langle A' \mid R' \rangle$ is a presentation for $\mathcal{T}_n$ for all $n \geq 4$, and the relations (i) to (v) are irredundant.

A (smaller) presentation for $\mathcal{T}_n$

Starting point: redundant relations in Iwahori and Iwahori.

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Let $A = \{\pi_2, \pi_3, \ldots, \pi_n\}$, and $\langle A \mid R \rangle$ a presentation for S_n with $\pi_i = (1\ i)$. Let $A' = A \cup \{\varepsilon_{1,2}\}$, and form R' from R by adding the following relations:

(i) $\varepsilon_{1,2}(1\ 3)\varepsilon_{1,2}(1\ 3) = \varepsilon_{1,2}$;

(ii) $\varepsilon_{1,2}(2\ 3)\varepsilon_{1,2}(2\ 3) = (1\ 2\ 3)^{-1}\varepsilon_{1,2}(1\ 2\ 3)\varepsilon_{1,2}$;

(iii) $\varepsilon_{1,2}(1\ 4)(2\ 3)\varepsilon_{1,2}(2\ 3)(1\ 4) =$
$(1\ 4)(2\ 3)\varepsilon_{1,2}(2\ 3)(1\ 4)\varepsilon_{1,2}$;

(iv) $(3\ 4)\varepsilon_{1,2} = \varepsilon_{1,2}(3\ 4)$;

(v) $(3\ 4 \cdots n)(1\ 2)\varepsilon_{1,2} = \varepsilon_{1,2}(3\ 4\ \cdots n)$.

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Word graphs

Definition (Word graph)

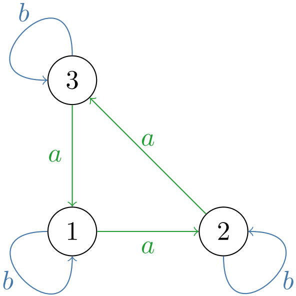
Let A be a set. A *word graph* is a digraph with nodes N, edges $E = \{(x, a, y) \mid x, y \in N,\ a \in A\}$ (and some 'initial' node). We say that a word graph (N, E) is:

(i) *deterministic* if there are no two distinct edges with equal source and label;

(ii) *complete* if every node is the source of an edge with every label from A;

(iii) *compatible with the relation* $u = v$ with $u, v \in A^*$ if it is deterministic, and for each $x \in N$, the paths starting at x labelled by u and v respectively have the same endpoint (should both endpoints exist).

Word graphs

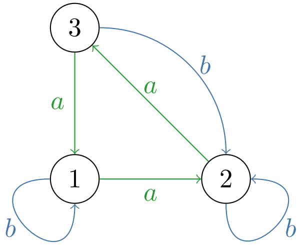


Complete

Deterministic

Compatible
with $aba = a^2$

Word graphs



Complete

Deterministic

Not compatible
with $aba = a^2$

Irredundancy via word graphs

Theorem (not mine)

If S is defined by $\langle A \mid R \rangle$, there is a one-to-one correspondence between right congruences of S and (standard) complete words graphs compatible with all the relations of R, which have a node from which all others are reachable.

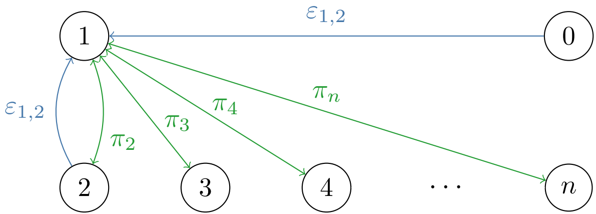
Let $\langle A \mid R \rangle$ be a presentation, and $u = v$ a relation of R. Take $R' = R \setminus \{(u, v)\}$.

Suppose there is a word graph compatible with each relation in R', but *not* $u = v$. The monoids defined by R and R' have different right congruences.

Hard as a human to construct – many nodes, many relations, many paths.

We can efficiently compute word graphs.

Useful to consider S_n relations separately. Note $\varepsilon_{i,j}$ does not have its $\mathcal{T}_n$ meaning.

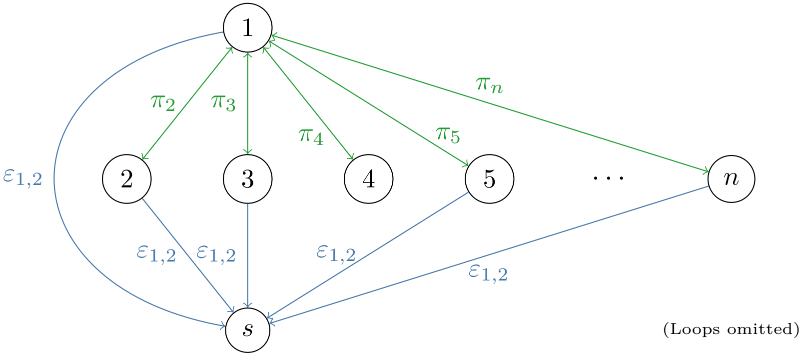


(Loops omitted)

This word graph is compatible with each of our $\mathcal{T}_n$ relations, *apart from*

$$\varepsilon_{1,2}\pi_4\pi_2\pi_3\pi_2\varepsilon_{1,2}\pi_2\pi_3\pi_2\pi_4 = \pi_4\pi_2\pi_3\pi_2\varepsilon_{1,2}\pi_2\pi_3\pi_2\pi_4\varepsilon_{1,2}.$$

For example, it is compatible with $\varepsilon_{1,2}\pi_3\varepsilon_{1,2}\pi_3 = \varepsilon_{1,2}$.

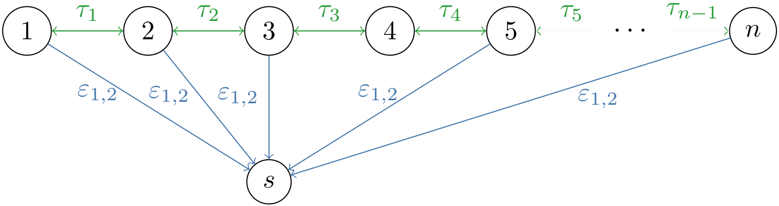


This word graph is compatible with all of our $\mathcal{T}_n$ relations, except for $\varepsilon_{1,2}\pi_3\pi_4\pi_3 = \pi_3\pi_4\pi_3\varepsilon_{1,2}$.

Previous compatibility example $\varepsilon_{1,2}\pi_3\varepsilon_{1,2}\pi_3 = \varepsilon_{1,2}$ is rather more straightforward...

If we change generators, paths may become easier or harder to follow.

Change S_n generators from $\pi_i = (1 \ \ i)$ to $\tau_i = (i \ \ i+1)$.



This word graph is compatible with all of our $\mathcal{T}_n$ relations, except for the (re-expressed) $\varepsilon_{1,2}\tau_3 = \tau_3\varepsilon_{1,2}$.

Word graph approach sadly is fruitless for the other relation.

Minimum sized presentations

Natural to ask about the smallest number of relations required in a presentation.

> Where M is $\mathcal{T}_n$, $\mathcal{I}_n$ or $\mathcal{PT}_n$, and $\langle A \mid R \rangle$ defines the symmetric group, what is the minimum $|R'|$ such that $\langle A \cup A' \mid R \cup R' \rangle$ defines M?

The lowest previous upper bounds I can find are: 7, 5 and 12 respectively, for $\mathcal{T}_n$, $\mathcal{I}_n$ and $\mathcal{PT}_n$, respectively.

Not much known about lower bounds.

The sets $R_k = \{f \in \mathcal{I}_n \mid \operatorname{rank} f \leq k\}$ are ideals of $\mathcal{I}_n$.

If there is an elementary sequence starting at some word u, we cannot use any relation of lower rank than the transformation u represents.

rank n	S_n presentation
rank $n-1$	two relations
rank $n-2$	one relation

Similar kind of picture for $\mathcal{T}_n$ and $\mathcal{PT}_n$.

Some terminology

The *leading permutation* of a word w is the permutation corresponding to the largest prefix of w representing a permutation.

The leading permutation of $\pi_2\pi_3\varepsilon_{(1)}\pi_3$ is $(1\ 2\ 3)$.

The *trailing permutation* is defined analogously.

The *rank of a word* u is the rank of the transformation it represents. The *rank of a relation* $u = v$ is the rank of u (alternatively, v).

The rank of $\pi_2\pi_3\pi_2\varepsilon_{(1)} = \varepsilon_{(1)}\pi_2\pi_3\pi_2\varepsilon_{(1)}$ is $n - 1$.

Rank $n-1$ in $\mathcal{I}_n$

What can we say about the leading permutation of equivalent words?

Where $\sigma, \tau \in S_n$, if $\sigma\varepsilon_{(1)}u = \tau\varepsilon_{(1)}v$ in $\mathcal{I}_n$, we immediately see $\sigma^{-1}\tau\varepsilon_{(1)}v = \varepsilon_{(1)}u$.

In rank $n-1$ specifically, this means $\sigma^{-1}\tau \in \mathrm{Fix}(1)$. Must be possible to change the leading permutation of a word, within the same coset.

There must be a collection of relations of the form $\sigma_i\alpha_i\varepsilon_{(1)}w_i = \sigma_i\varepsilon_{(1)}u_i$, where $\mathrm{Fix}(1) = \langle \alpha_1, \alpha_2 \ldots, \alpha_k \rangle$.

So, at least two relations in rank $n-1$.

Rank $n-2$ in $\mathcal{I}_n$

Rank n relations: leading permutation of a word is invariant.

Rank $n-1$ relations: left coset in Fix(1) of leading permutation is invariant.

In rank $n-2$, coset of leading permutation is not invariant, e.g. $(1\ 2)\varepsilon_{(1)}(1\ 2)\varepsilon_{(1)} = \varepsilon_{(1)}(1\ 2)\varepsilon_{(1)}$ ...

...so there must be a rank $n-2$ relation, where the leading permutations of the relation words lie in different cosets.

Any presentation for $\mathcal{I}_n$ requires at least three non-S_n relations.

Attacking the upper bound for $\mathcal{I}_n$

Theorem (various indep.)

Alongside a presentation for S_n, the following five 'relations' define I_n:

(i) $\varepsilon_{(1)}^2 = \varepsilon_{(1)}$;

(ii) $(2\,3 \cdots n)\varepsilon_{(1)} = \varepsilon_{(1)}(2\,3 \cdots n)$;

(iii) $(2\,3)\varepsilon_{(1)} = \varepsilon_{(1)}(2\,3)$;

(iv) $(1\,2)\varepsilon_{(1)}(1\,2)\varepsilon_{(1)} = \varepsilon_{(1)}(1\,2)\varepsilon_{(1)}(1\,2)$; and

(v) $(1\,2)\varepsilon_{(1)}(1\,2)\varepsilon_{(1)} = \varepsilon_{(1)}(1\,2)\varepsilon_{(1)}$.

Via word graphs: can prove these relations are not redundant.

Attempting to prove five is the minimum number was very helpful in finding a smaller presentation

Attacking the upper bound for $\mathcal{I}_n$

Theorem (various indep.)

Alongside a presentation for S_n, the following five 'relations' define I_n:

(i) $\varepsilon_{(1)}^2 = \varepsilon_{(1)}$;

(ii) $(2\ 3 \cdots n)\varepsilon_{(1)} = \varepsilon_{(1)}(2\ 3 \cdots n)$;

(iii) $(2\ 3)\varepsilon_{(1)} = \varepsilon_{(1)}(2\ 3)$;

(iv) $(1\ 2)\varepsilon_{(1)}(1\ 2)\varepsilon_{(1)} = \varepsilon_{(1)}(1\ 2)\varepsilon_{(1)}(1\ 2)$; and

(v) $(1\ 2)\varepsilon_{(1)}(1\ 2)\varepsilon_{(1)} = \varepsilon_{(1)}(1\ 2)\varepsilon_{(1)}$.

Must be able to: reduce number of $\varepsilon_{(1)}$'s to 1; change odd/evenness of the product of the permutations in each word; change cosets in $\mathrm{Fix}(1)$ of trailing and leading permutations.

Defining $\mathcal{I}_n$ in minimum extra relations

Theorem (Mitchell + W.)

Alongside a presentation for S_n, the following three relations define $\mathcal{I}_n$:

(i) $(2\ 3 \cdots n)\varepsilon_{(1)} = \varepsilon_{(1)}^2(2\ 3 \cdots n)$;

(ii) $(2\ 3)\varepsilon_{(1)} = \varepsilon_{(1)}(2\ 3)$;

(iii) $(1\ 2)\varepsilon_{(1)}(1\ 2)\varepsilon_{(1)}(1\ 2)\varepsilon_{(1)}(1\ 2) = \varepsilon_{(1)}(1\ 2)\varepsilon_{(1)}$.

Answer for $\mathcal{I}_n$ is three.

Showing that $\varepsilon_{(1)}^2 = \varepsilon_{(1)}$ can be brought into the Fix(1) relations is more difficult than the rank $n-2$ work.

Rank $n-1$ in $\mathcal{T}_n$

We use a similar technique as in $\mathcal{I}_n$

Suppose $\sigma\varepsilon_{1,2}u = \tau\varepsilon_{1,2}v$. Then $\sigma^{-1}\tau\varepsilon_{1,2}v = \varepsilon_{1,2}u$.

If this is in rank $n-1$, the only non-singleton kernel class is $\{1,2\}$.

If $j \notin \{1,2\}$ had $(j)\sigma^{-1}\tau \in \{1,2\}$, then j would be in a kernel class of size 2. This means that $\sigma^{-1}\tau \in \mathrm{Fix}(\{1,2\})$.

There must be a collection of relations of the form $\sigma_i\alpha_i\varepsilon_{1,2}w_i = \sigma_i\varepsilon_{1,2}u_i$, where $\mathrm{Fix}(\{1,2\}) = \langle \alpha_1, \alpha_2 \ldots, \alpha_k \rangle$.

So, at least two relations in rank $n-1$.

Rank $n-2$ in $\mathcal{T}_n$

Rank n relations can't change the leading permutation of a word; and rank $n-1$ relations can only change the left coset in $\mathrm{Fix}(\{1,2\})$.

There must be a rank $n-2$ relation $u = v$ whose words' leading permutations have different cosets in $\mathrm{Fix}(\{1,2\})$.

If a relation $u = v$ is used to change the leading permutation of a word $w = \sigma\varepsilon_{1,2}t$, then $u = \sigma'\varepsilon_{1,2}t'$, where σ' is a suffix of σ and t' a prefix of t.

In rank $n-2$, the words u and w must have the same 'kernel shape'. There are two kernel shapes in rank $n-2$, so we require at least two rank $n-2$ relations.

In $\mathcal{T}_n$, the additional relations in our presentation are:

(i) $\varepsilon_{1,2}(1\ 3)\varepsilon_{1,2}(1\ 3) = \varepsilon_{1,2}$;

(ii) $(3\ 4)\varepsilon_{1,2} = \varepsilon_{1,2}(3\ 4)$;

(iii) $(3\ 4 \cdots n)(1\ 2)\varepsilon_{1,2} = \varepsilon_{1,2}(3\ 4\ \cdots n)$.

(iv) $\varepsilon_{1,2}(2\ 3)\varepsilon_{1,2}(2\ 3) = (1\ 2\ 3)^{-1}\varepsilon_{1,2}(1\ 2\ 3)\varepsilon_{1,2}$;

(v) $\varepsilon_{1,2}(1\ 4)(2\ 3)\varepsilon_{1,2}(2\ 3)(1\ 4) =$
$(1\ 4)(2\ 3)\varepsilon_{1,2}(2\ 3)(1\ 4)\varepsilon_{1,2}$;

Relation (iii): can get both $(1\ 2)\varepsilon_{1,2} = \varepsilon_{1,2}$ and $(3\ 4 \cdots n)\varepsilon_{1,2} = \varepsilon_{1,2}(3\ 4 \cdots n)$.

Working on finding two rank $n-1$ relations which give the three above. Have a construction which works for degree up to 7.

Answers to the 'minimum' question

Theorem (Mitchell + W.)

The minimum number of non-S_n relations in any presentation for $\mathcal{I}_n$ is 3.

Working on proving lower bound of 4 for $\mathcal{T}_n$ is realised:

For $\mathcal{T}_n$, the minimum number is either 4 or 5. For each $4 \leq n \leq 7$, there is a presentation with only 4 non-S_n relations.

For any presentation of $\mathcal{PT}_n$, the minimum number m of non-S_n relations has $8 \leq m \leq 11$. For each $4 \leq n \leq 7$, there is a presentation with only 9 non-S_n relations.

Thanks