Congruence Lattices of Partition Monoids

Nik Ruškuc
nik.ruskuc@st-andrews.ac.uk

School of Mathematics and Statistics, University of St Andrews

York, 8 February 2017
Aim and credits

- Describe the congruence lattice of the partition monoid $P_n$ and its various important submonoids.
- By way of introduction: congruence lattices of symmetric groups and full transformation monoids.
- Plus a quick introduction to partition monoids.
- Joint work with: James East, James Mitchell and Michael Torpey.
Normal subgroups of the symmetric group

Theorem
The alternating group $\mathcal{A}_n$ is the only proper normal subgroup of $S_n$ ($n \neq 1, 2, 4$).

Remark
- Exceptions: $S_1, S_2$ (too small) and $S_4$ (because of the Klein 4-group $K_4$).
- The normal subgroups of any group form a (modular) lattice.
- $\text{Norm}(G) \cong \text{Cong}(G)$.
Theorem (A.I. Mal’cev 1952)
Cong(\(T_n\)) is the chain shown on the right.
Green’s structure of $\mathcal{T}_n$

The following are well known:

- $\alpha \mathcal{L} \beta \iff \text{im} \alpha = \text{im} \beta$.
- $\alpha \mathcal{R} \beta \iff \ker \alpha = \ker \beta$.
- $\alpha \mathcal{J} \beta \iff \text{rank} \alpha = \text{rank} \beta$.
- All $\mathcal{J}(=\mathcal{D})$-classes are regular.
- The maximal subgroups corresponding to the idempotents of rank $r$ are all isomorphic to $S_r$. 
Every ideal of $T_n$ has the form
\[ I_r = \{ \alpha \in T_n : \text{rank } \alpha \leq r \}. \]
All ideals are principal, and they form a chain.
To every ideal $I_r$ there corresponds a (Rees) congruence
\[ R_r = \Delta \cup (I_r \times I_r). \]
Consider a typical $\mathcal{J}$-class $J_r = \{ \alpha \in \mathcal{T}_n : \text{rank } \alpha = r \}$.

Let $\mathcal{J}_r$ be the corresponding principal factor.

$\mathcal{J}_r \cong \mathcal{M}^0[S_r; K, L; P]$ – a Rees matrix semigroup.

For every $N \subseteq S_r$, the semigroup $\mathcal{M}^0[S_r/N; K, L; P/N]$ is a quotient of $\mathcal{J}_r$.

Let $\nu_N$ be the corresponding relation on $J_r$.

$R_N = \Delta \cup \nu_N \cup (I_{r-1} \times I_{r-1})$ is a congruence on $\mathcal{T}_n$.

Intuitively $R_N$: collapses $I_{r-1}$ to a single element (zero); collapses each $S_r$ in $J_r$ to $S_r/N$, and correspondingly collapses the non-group $\mathcal{H}$-classes; leaves the rest of $\mathcal{T}_n$ intact.
Proof outline of Mal’cev’s Theorem

- Verify that all the congruences $R_r$ and $R_N$ form a chain.
- This relies on the fact that the ideals form a chain, and that congruences on each $S_r$ form a chain.
- It turns out that all these congruences are principal.
- For every pair $(\alpha, \beta) \in \mathcal{T}_n \times \mathcal{T}_n$, determine the congruence $(\alpha, \beta)^\#$ generated by it, and verify it is one of the listed congruences.
- Since every congruence is a join of principal congruences, conclude that there are no further congruences on $\mathcal{T}_n$. 
Further remarks on $\text{Cong}(\mathcal{T}_n)$

- Mal’cev also describes $\text{Cong}(\mathcal{T}_X)$, $X$ infinite.
- Analogous results have been proved for:
  - full matrix semigroups (Mal’cev 1953);
  - symmetric inverse monoids (Liber 1953);
  - and many others.
- In all instances, $\text{Cong}(S)$ is a chain.
From transformations to partitions

View mappings graphically, e.g:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 2 & 3 & 4 & 4
\end{pmatrix}
\]

Composition:
Partition monoid $\mathcal{P}_n$

Partition = a set partition of $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$.
For example: $\alpha = \{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\}\} 
\in \mathcal{P}_6$.

Some useful parameters:

$\text{dom } \alpha = \{1, 3, 5, 6\}$  $\text{ker } \alpha = \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$
$\text{codom } \alpha = \{1', 4', 6'\}$  $\text{coker } \alpha = \{\{1', 6'\}, \{2', 3'\}, \{4'\}, \{5'\}\}$

$\text{rank } \alpha = 2$. 
Partition monoid $\mathcal{P}_n$: some remarks

- $\mathcal{P}_n$ contains $S_n$, $T_n$, $I_n$, $O_n$ as submonoids.
- It also contains: Brauer monoid, Motzkin monoid, Temperely–Lieb (Jones) monoid.
- They form a basis from which their name-sakes algebras are built – connections with Mathematical Physics, Representation Theory and Topology.
- Elements of $\mathcal{P}_n$ can be viewed as partial bijections between quotients of $\{1, \ldots, n\}$.
Green’s relations on $\mathcal{P}_n$

- $\alpha R \beta \iff \ker \alpha = \ker \beta \ \& \ \text{dom} \ \alpha = \text{dom} \ \beta$.
- $\alpha L \beta \iff \text{coker} \ \alpha = \text{coker} \ \beta \ \& \ \text{codom} \ \alpha = \text{codom} \ \beta$.
- $\alpha J \beta \iff \text{rank} \ \alpha = \text{rank} \ \beta$.
- All $J (=D)$-classes are regular.
- The maximal subgroups corresponding to the idempotents of rank $r$ are all isomorphic to $S_r$. 
Ideals of $\mathcal{P}_n$, and congruences arising

- Every ideal of $\mathcal{P}_n$ has the form $I_r = \{ \alpha \in \mathcal{P}_n : \text{rank } \alpha \leq r \}$.
- All ideals are principal, and they form a chain.
- To every ideal $I_r$ there corresponds a congruence $R_r = \Delta \cup (I_r \times I_r)$.
- Analogous to $\mathcal{T}_n$, we also have congruences $R_N$ for $N \trianglelefteq S_r$.
- One difference though: The minimal ideal of $\mathcal{P}_n$ (partitions of rank 0) is a proper rectangular band.
- (As opposed to a right zero semigroup of constant mappings in $\mathcal{T}_n$.)
Cong\((\mathcal{P}_n)\)

Theorem

[J. East, J.D. Mitchell, NR, M. Torpey]

Cong\((\mathcal{P}_n)\) is the lattice shown on the right.
\( R \) and \( L \) on the minimal ideal

**Theorem (Folklore?)**

Let \( S \) be a finite monoid with the minimal ideal \( M \). The relations \( \rho_0 = \Delta \cup R|_M \) and \( \lambda_0 = \Delta \cup L|_M \) are congruences on \( S \).
Retractions

A (computational) inspection of the congruence $\mu_1$ yields:

$$\mu_1 = \{ (\alpha, \beta) \in I_1 \times I_1 : \ker \alpha = \ker \beta, \ coker \alpha = coker \beta \} \cup \Delta$$

It is a congruence, because the following mapping is a retraction:

$$l_1 \to l_0, \ \alpha \mapsto \hat{\alpha} \in l_0, \ \ker \alpha = \ker \hat{\alpha}, \ coker \alpha = coker \hat{\alpha}.$$  

**Definition**

Let $S$ be a semigroup and $T \leq S$. A homomorphism $f : S \to T$ with $f|_T = 1_T$ is called a **retraction**.
Definition
Let $S$ be a finite monoid with minimal ideal $M$. A triple $\mathcal{T} = (I, f, N)$ is a congruence triple if:

- $I$ is an ideal;
- $f : I \to M$ is a retraction;
- $N$ is a normal subgroup of a maximal subgroup in a $\mathcal{J}$-class ‘just above’ $I$;
- All elements of $N$ act the same way on $M$, i.e. $|xN| = |Nx| = 1 \ (x \in M)$. 

A family of congruences

Definition
To every congruence triple $\mathcal{T}$ associate three relations:

$\lambda_{\mathcal{T}} = \Delta \cup \nu_{N} \cup \{(x, y) \in I \times I : f(x) \mathcal{L} f(y)\};$

$\rho_{\mathcal{T}} = \Delta \cup \nu_{N} \cup \{(x, y) \in I \times I : f(x) \mathcal{R} f(y)\};$

$\mu_{\mathcal{T}} = \Delta \cup \nu_{N} \cup \{(x, y) \in I \times I : f(x) \mathcal{H} f(y)\}.$

Theorem
$\lambda_{\mathcal{T}}, \rho_{\mathcal{T}}$ and $\mu_{\mathcal{T}}$ are congruences.

Theorem
The congruences $\lambda_{\mathcal{T}}, \rho_{\mathcal{T}}$ and $\mu_{\mathcal{T}}$, together with $R_{N}$, form a diamond lattice.
Cong\( (\mathcal{P}_n) \) explained

- Key fact: \( (I_1, \alpha \mapsto \hat{\alpha}, S_2) \) is a congruence triple on \( \mathcal{P}_n \).

- It induces two ‘smaller’ congruence triples \( (I_1, \alpha \mapsto \hat{\alpha}, \{1\}) \) and \( (I_0, 1, \{1\}) \).

- The rest is the same as for \( \mathcal{T}_n \).

- But: not all congruences are principal!
Planar partition monoid

- Planar partition: can be drawn without edges crossing.
- Edges need not be straight, but have to be contained within the rectangle with corners $1, 1', n, n'$.

\[
\begin{align*}
\lambda_1 & \quad \mu_1 & \quad R_0 \\
\lambda_0 & \quad \mu_0 & \quad R_0 \\
\mu_0 & \quad \Delta & \\
\end{align*}
\]

VS
Brauer monoid $\mathcal{B}_n$

$\mathcal{B}_n = \text{partitions with blocks of size 2}.$

$n$ odd

$n$ even
An $\alpha \in \mathcal{B}_n$ with rank $\alpha = 2$ has precisely two transversal blocks $\{i, j'\}$, $\{k, l'\}$.

Let $\hat{\alpha} \in l_0$ be obtained from $\alpha$ by replacing those two blocks by $\{i, k\}$, $\{j, l\}$.

$(l_2, \alpha \mapsto \hat{\alpha}, K \unlhd S_4)$ is a congruence triple.

Three further derived triples: $(l_2, \alpha \mapsto \hat{\alpha}, \{1\})$, $(l_0, 1, S_2 \leq S_2)$, $(l_0, 1, \{1\})$. 
Concluding remarks

- Congruence lattices determined for all partition monoids shown in the diagram.
- Work was crucially informed by computational evidence obtained using GAP package Semigroups (J.D. Mitchell et al.)
- All the congruences are instances of the construction(s) described here.
- The work to determine the principal congruences is still case-specific.
- Related work: J. Araújo, W. Bentz, G.M.S. Gomes, Congruences on direct products of transformation and matrix monoids.
Some speculations about future work...

- Develop a general theory of generators for the congruences introduced here.
- For example: Under which genereral conditions are the congruences $R_N$, $\rho_T$, $\lambda_T$ and $\mu_T$ principal?
- The answer is likely to be couched in terms of groups, Rees matrix semigroups, and the actions on $R$- and $L$-classes.
- To what extent does this point to a general approach towards computing (and understanding) congruence lattices of arbitrary semigroups?
- What are families of semigroups to which one could turn next, in search of interesting behaviours and patterns?

THANK YOU FOR LISTENING!