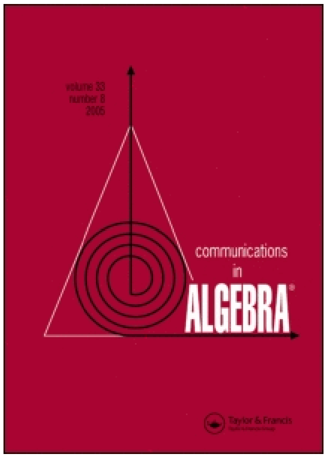

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Orders in rings without identity

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ORDERS IN RINGS WITHOUT IDENTITY

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1 Introduction

This paper is the first of a series of three papers on orders in rings which need not have an identity. Our second paper [6] is concerned with straight orders in (von Neumann) regular rings,the third one [7] is devoted to characterising orders in semiprime rings with minimal condition for principal right ideals, while this one presents the basic definitions and general theory.

The idea of a (classical) ring of left quotients Q of a ring R is a familiar one. The ring Q has an identity, its elements can be written as 'left fractions' $a^{-1}b$ with a, b in R and those elements of R which are not zero divisors in R are units in Q.

If Q is a ring without identity, then, of course, no element of Q has an inverse. However, some elements may have 'generalised inverses' [2]. Among the various kinds of generalised inverse we single out the group inverse. An element b of Q is a *group inverse* of an element a if $aba = a, bab = b$ and $ab = ba$. It is well known that if a has a group inverse, then it has only one which we shall denote by $a^\sharp$.

A necessary condition for an element r of a ring R to have an inverse in some overring of R is that r is not a zero divisor. An analogous necessary condition for an element r of a ring R to have a group inverse in some overring of R is that r be 'square–cancellable' (see Section 2 for the definition). Thus in our generalised definition of a ring of left quotients Q of a ring R we insist that each square–cancellable element of R has a group inverse in Q and each element of Q can be written as $a^{\sharp}b$ for some elements a, b of R. We also say that R is a left order in Q. The ring Q need not have an identity; if it does then, as we show in Section 3, it is also a ring of left quotients of R in the classical sense.

In the case of a classical ring of left quotients Q of a ring R there is no guarantee that square–cancellable elements of R will have group inverses in Q. They do in all the familiar situations such as when Q is regular or left or right perfect because in these cases the group of units of Q dominates the multiplicative structure of Q. There do exist, however, rings in which all non–zero divisors are units but which have square–cancellable elements which do not have group inverses.

Our generalised definition is taken from semigroup theory and in Section 2, after giving the definition, we recall some of the basic ideas about semigroups which are relevant to our purpose. We then consider the question of when a weak left order in a ring Q is necessarily a left order in Q where by a weak left order in Q we mean a subring R such that every element of Q can be written as $a^{\sharp}b$ for some a, b in R. We conclude Section 2 by showing that the ring of all endomorphisms of finite rank of a free module of infinite rank over a principal ideal domain is an order in a ring without identity.

Section 3 is devoted to a comparison of the old and new concepts of order in rings with identity. We have already mentioned that a left order in the new sense is a left order (in the same ring) in the old sense. The converse is not true but for the two-sided case we have that an order in a ring Q in the old sense is an order in Q in the new sense if Q is regular, left perfect or right perfect. In the one-sided case we find that a left order in a directly finite regular ring Q in the old sense is also a left order in Q in the new sense.

We prove a 'common denominator theorem' in Section 4 and use it to relate the left ideals in a left order to those in its ring of left quotients. We make heavy use of it again in Section 5 where we show that if a ring is a left order in regular rings Q_1 and Q_2, then these two rings are isomorphic.

We hope that we have written the paper in such a way that it is intelligible both to readers whose main interest is ring theory and to readers whose main interest is semigroup theory. This has led us to give some definitions and explanations of ideas which are regarded as standard by one of these sets of readers but not the other. Further details of the basic ideas of semigroup theory may be found in [3] and [9] and we mention [1],[8] and [10] as useful references for ring theory. The book [13] has ideas from both subjects.

To avoid any risk of confusion we mention now that by the term 'regular element' of a ring or semigroup S we mean an element a such that $a = axa$ for some x in S; we never use this term to describe an element which is not a zero divisor. Also we use the notation aS^1, S^1a, S^1aS^1 in the semigroup sense so that $aS^1 = aS \cup \{a\}$, etc.. We use the notation $|\, a)$ for the principal right ideal of a ring R generated by the element a. Of course, if R has an identity or if R is regular, then $|\, a) = aR^1 = aR$ but in general $|\, a)$ and aR^1 may be different.

2 Definitions

We recall that a ring Q is a *(classical) ring of left quotients* of its subring R, or that R is a *left order* in Q, if the following three conditions are satisfied :

(i) Q has an identity,

(ii) every element of R which is not a zero divisor is a unit of Q,

(iii)each element q of Q can be written as $q = a^{-1}b$ where a, b are elements of R and a^{-1} is the inverse of a in Q.

There are corresponding notions of *right order* and *ring of right quotients* and if R is both a left and a right order in Q we say simply that it is an *order* in Q or that Q is a *ring of quotients* of R.

We now turn to orders in semigroups or rings which need not have an identity, beginning with a discussion of the concepts used in the definition. By a subgroup of a semigroup or ring S we simply mean a (multiplicative) subsemigroup which is a group. If e is an idempotent in S, then the set

$$H_e = \{a \in S : ea = a = ae \text{ and for some } b \in S, ab = e = ba\}$$

is a subgroup of S which contains all those subgroups which have identity e. A *group inverse* of an element a of S is an element x which satisfies $axa = a, xax = x$ and $ax = xa$. If a has a group inverse, it has only one which will be denoted by $a^{\sharp}$. It is easy to see that a has a group inverse if and only if it is in a subgroup of S.

An element a of S is *left square-cancellable* if for all $x, y \in S^1, a^2x = a^2y$ implies $ax = ay$. Similarly, *right square-cancellable* is defined and *square-cancellable* means both left and right square-cancellable. We point out that being square-cancellable is a necessary condition for an element of S to have a group inverse in some oversemigroup (or overring) of S. The set of square cancellable elements of S will be denoted by $\mathcal{S}(S)$.

When S is a ring we denote the right annihilator of $a \in S$ by $r_S(a)$ and the left annihilator by $\ell_S(a)$; where there is no danger of ambiguity we write simply $r(a)$ and $\ell(a)$. Clearly, if $r(a) = 0$, then a is left square–cancellable and so elements which are not zero divisors are square–cancellable.

We now define a subsemigroup (or subring) S of a semigroup (or ring) Q to be a *left order* in Q when the following two conditions hold :

(i)each square–cancellable element of S has a group inverse in Q,

(ii)each element q of Q can be written as $q = a^\sharp b$ where a, b are elements of S and $a^\sharp$ is the group inverse of a in Q.

We also say that Q is a *semigroup* (or *ring) of left quotients* of S. Similarly we define *right order* and *semigroup* (or *ring) of right quotients.* If S is both a left and a right order in Q, then we say simply that S is an *order* in Q and that Q is a *semigroup* (or *ring) of quotients* of S.

If S is a subsemigroup (or subring) of a semigroup (or ring) Q for which condition (ii) above holds, then we say that S is a *weak left order* in Q. It is clear what is meant by *weak right order* and *weak order.*

The result of the following very easy lemma will be used constantly.

Lemma 2.1 *Let S be a weak left order in a semigroup Q. Every element q of Q can be written as $q = a^\sharp b$ where $aa^\sharp b = b, a \in \mathcal{S}(S), b \in S$.*

Proof Certainly q can be written as $c^\sharp d$ where $c \in \mathcal{S}(S), d \in S$. Now $c^2 \in \mathcal{S}(S)$ and $c^\sharp = (c^2)^\sharp c$; so putting $a = c^2$ and $b = cd$ gives the lemma.

From now on when S is a weak left order in Q and we write an element q of Q as $q = a^\sharp b$ we shall assume that $aa^\sharp b = b$.

If Q is a ring with identity, we now have two meanings for the phrase 'order in Q'. In fact, an order in Q in the new sense is an order in Q in the traditional sense but the converse is not true. We shall see, however, that in most cases of interest the two concepts do coincide and so there is little danger of ambiguity.

Before proceeding it will be helpful to recall some notation and terminology from semigroup theory (which, of course, can equally be applied to rings). We start with Green's relations. In a semigroup S the elements a, b are *$\mathcal{L}$-related* if $a = b$ or there are elements x, y in S such that $xa = b, yb = a$. In semigroup terms, this says that a and b generate the same principal left ideal. This is also the case in rings with 1 and regular rings ; there are, however, rings with elements which generate the same (ring) principal left ideal but which are not $\mathcal{L}$-related.

The relation $\mathcal{L}$ is a right congruence ; its left–right dual $\mathcal{R}$ is a left congruence. The join of these two equivalences is denoted by $\mathcal{D}$ and as $\mathcal{L}$ and $\mathcal{R}$ commute, $\mathcal{D} = \mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L}$. The intersection of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{H}$ and the maximal subgroups of S are those $\mathcal{H}$-classes which contain idempotents.

The $\mathcal{L}$-class of an element a of S is denoted by L_a and similar notation is used for $\mathcal{R}$-classes,$\mathcal{H}$-classes and $\mathcal{D}$-classes.

Associated with the equivalence $\mathcal{L}$ we have a preorder relation $\leq_{\mathcal{L}}$ defined as follows :

$a \leq_{\mathcal{L}} b$ if and only if $a \in S^1 b$.

Of course, $a\mathcal{L}b$ if and only if $a \leq_{\mathcal{L}} b$ and $b \leq_{\mathcal{L}} a$. Similarly, there is a preorder $\leq_{\mathcal{R}}$ associated with $\mathcal{R}$.

If a, b are elements of a subsemigroup T of S and if $a \leq_{\mathcal{L}} b$ (or $a \leq_{\mathcal{R}} b$) in T, then obviously $a \leq_{\mathcal{L}} b$ (or $a \leq_{\mathcal{R}} b$) in S. The converse is not generally true but we do have the following well known result.

Lemma 2.2 *Let T be a regular subsemigroup of a semigroup S. If a, b are elements of T such that $a \leq_{\mathcal{L}} b$ (or $a \leq_{\mathcal{R}} b$) in S, then $a \leq_{\mathcal{L}} b$ (or $a \leq_{\mathcal{R}} b$) in T.*

We note that this applies to the particular case when S is regular and T is an ideal of S.

We now introduce some relations closely connected with Green's relations. The relation $\mathcal{L}^*$ is defined on S by the rule that $a\mathcal{L}^*b$ if and only if the elements a, b of S are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually ; $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$. The following lemma from [11] and [12] provides useful alternative characterisations of $\mathcal{L}^*$.

Lemma 2.3 *The following conditions are equivalent for elements a, b of a semigroup S :*

(1) $a\mathcal{L}^*b$,

(2) for all $x, y \in S^1, ax = ay$ if and only if $bx = by$,

(3) there is an S^1-isomorphism $\phi : aS^1 \to bS^1$ with $a\phi = b$.

It is now easily seen that $\mathcal{L}^*$ is a right congruence and that $\mathcal{R}^*$ is a left congruence. Also we obviously have $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$.

There is a preorder $\leq_{\mathcal{L}^*}$ associated with $\mathcal{L}^*$ and defined by the rule :

$a \leq_{\mathcal{L}^*} b$ if and only if for all $x, y \in S^1, bx = by$ implies $ax = ay$.

In view of Lemma 2.3, $a\mathcal{L}^*b$ if and only if $a \leq_{\mathcal{L}^*} b$ and $b \leq_{\mathcal{L}^*} a$. Similarly, we have a preorder $\leq_{\mathcal{R}^*}$ associated with $\mathcal{R}^*$. It is clear that $\leq_{\mathcal{L}} \subseteq \leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}} \subseteq \leq_{\mathcal{R}^*}$ for any semigroup S. If S is regular, then it is easy to see that $\leq_{\mathcal{L}} = \leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}} = \leq_{\mathcal{R}^*}$, so that in this case $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$. Further details of these relations can be found in [5].

In rings there are some closely related preorders defined in terms of annihilators. For a ring R we define a relation $\preceq_\ell$ by the rule:

$a \preceq_\ell b$ if and only if $r(b) \subseteq r(a)$.

Certainly $\preceq_\ell$ is a preorder and there is a dually defined preorder $\preceq_r$. Let $\equiv_\ell$ and $\equiv_r$ be the associated equivalence relations.

A ring R is *left* (or *right) faithful* if it is faithful as a left (or right) module over itself, that is, if $rR = 0$ implies $r = 0$ (or $Rr = 0$ implies $r = 0$). We say that R is *faithful* if it is both left and right faithful. We emphasise that rings with identity, semiprime rings and regular rings are all faithful.

Lemma 2.4 *In a ring R,*

(1) $\leq_{\mathcal{L}^*} \subseteq \preceq_\ell$.

(2) $\leq_{\mathcal{L}^*} = \preceq_\ell$ *if and only if R is left faithful.*

Proof (1) If $a, b \in R, a \leq_{\mathcal{L}^*} b$ and $x \in r(b)$, then $bx = 0 = b0$ so that $ax = a0 = 0$ and so $r(b) \subseteq r(a)$.

(2) If R is left faithful, $r(b) \subseteq r(a)$ and $bx = by$ where $x, y \in R^1$, then for any c in R, $xc - yc$ is in $r(b)$ so that $(ax - ay)c = a(xc - yc) = 0$ for all c in R. Thus $ax = ay$ and so $a \leq_{\mathcal{L}^*} b$.

If the two preorders are equal, indeed, if $\mathcal{L}^* = \equiv_\ell$, and $aR = 0$, then $r(0) = r(a)$ so that $a\mathcal{L}^*0$. Now $01 = 00$ so that $a1 = a0$, that is, $a = 0$. Thus R is left faithful.

Corollary 2.5 *In a ring R,*

(1) $\mathcal{L}^* \subseteq \equiv_\ell$.

(2) $\mathcal{L}^* = \equiv_\ell$ *if and only if R is left faithful.*

Some of the significance of these relations for our concerns can be seen by observing that from Lemma 2.2 we have that an element a of a semigroup S is left square–cancellable if and only if $a\mathcal{L}^*a^2$. Putting this together with its dual we obtain that $a \in \mathcal{S}(S)$ if and only if $a\mathcal{H}^*a^2$. Also, by Corollary 2.5, in a left faithful ring an element a is left square–cancellable if and only if $r(a) = r(a^2)$.

A standard result of semigroup theory is that an element a in a semigroup S is in a subgroup of S if and only if $a\mathcal{H}a^2$ ([9], Theorem II.2.5). Thus condition (i) in the definition of a left order S in a semigroup Q is equivalent to :

$a\mathcal{H}^* a^2$ in S if and only if $a\mathcal{H}a^2$ in Q.

Since $\leq_{\mathcal{L}^*} = \leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}^*} = \leq_{\mathcal{R}}$ on a regular ring R we see that for elements a, b of R we have $aR \subseteq bR$ if and only if $\ell(b) \subseteq \ell(a)$ and a corresponding result for principal left ideals.

We now move towards the question of when weak left orders in rings are necessarily left orders. We begin by recalling that a ring Q with identity is called a *quotient ring* if every element in Q which is not a zero divisor is a unit in Q. It is well known that regular rings with identity and left perfect rings (i.e., rings with 1 which satisfy the minimal condition for principal right ideals) are quotient rings. We shall say that a semigroup or ring Q is a *q–semigroup* or *q–ring* if all square–cancellable elements of Q are in subgroups of Q. Clearly, every q–ring with an identity is a quotient ring. We shall see later that the converse is not true. It is evident that every regular semigroup (or ring) is a q–semigroup (or q–ring) . The proposition below helps us to find more. We say that a semigroup (or ring) satisfies the condition M_L or M_R if it satisfies the minimal condition on principal left or principal right ideals respectively.

Proposition 2.6 *If Q is a semigroup or ring which satisfies M_R, then all left square–cancellable elements of Q have group inverses in Q.*

Proof First consider the case of a semigroup Q. The descending chain $aQ^1 \supseteq a^2Q^1 \supseteq \ldots$ must terminate. Let n be the smallest number such that $a^nQ^1 = a^{n+1}Q^1$. If $n \geq 2$, then with the convention that $a^0 = 1$ we have

$$a^2a^{n-2} = a^n = a^{n+1}x = a^2a^{n-1}x$$

for some x in Q^1. Since a^2 is left square–cancellable we obtain $aa^{n-2} = aa^{n-1}x$ so that $a^{n-1} \in a^nQ^1$ contradicting the choice of n. Hence $n = 1$ and $a\mathcal{R}a^2$.

Thus we have $a = a^2x$ for some x in Q so that $a^2 = a^2xa$ and as a is left square–cancellable, this gives $a = axa$, that is, a is regular. Also, using

$a = a^2x$ we get $a = axa = (a^2x)(xa) = a^2x^2a$ and so $a^2 = a^2x^2a^2$ so that a^2 is also regular. Now $a\mathcal{L}^*a^2$ and from the regularity of a and a^2 we obtain $a\mathcal{L}a^2$.

We now have $a\mathcal{H}a^2$, that is, a is in a subgroup of Q.

In the case when Q is a ring the principal right ideal $(r\mid$ is the set

$$\{kr + rq : k \in \mathbf{Z}, q \in Q\}.$$

Now, for $n \geq 2, a^n \in (a^{n+1}\mid$ gives

$$a^n = ka^{n+1} + a^{n+1}q = a^2(ka^{n-1} + a^{n-1}q)$$

so that we can argue as above to get $a \in (a^2\mid$. This gives $a = ka^2 + a^2q$ so that $a^2 = ka^3 + a^3q$ and, in fact, $a = k(ka^3 + a^3q) + a^2q \in a^2Q$. Hence the above argument applies also to the case of rings.

Lemma 2.7 *Let R be a subring of a right faithful ring Q. Suppose that R is a weak left order in Q. Then the following are equivalent for elements a, b of R :*

(1) $a \leq_{\mathcal{R}^*} b$ in R,

(2) $\ell_R(b) \subseteq \ell_R(a)$,

(3) $\ell_Q(b) \subseteq \ell_Q(a)$,

(4) $a \leq_{\mathcal{R}^*} b$ in Q.

Proof That (3) and (4) are equivalent is simply the dual of Lemma 2.4.

If $x \in r_R(R)$ and $q \in Q$, then since $q = a^\sharp b$ for some $a \in \mathcal{S}(R), b \in R$ we have $qx = a^\sharp bx = 0$ so that $x \in r_Q(Q)$. Hence $x = 0$ and $r_R(R) = 0$. So the dual of Lemma 2.4 applies again to show that (1) and (2) are equivalent.

It is immediate that (2) follows from (3). If (2) holds and $q \in \ell_Q(b)$, then $q = x^\sharp y$ for some $x \in \mathcal{S}(R), y \in R$ with $xx^\sharp y = y$ and so $yb = xx^\sharp yb = xqb = 0$. Hence $ya = 0$ and consequently $qa = 0$ so that (3) holds.

Corollary 2.8 *Let R be a subring of a right faithful ring Q. Suppose that R is a weak left order in Q. Then the following are equivalent for elements a, b of R :*

(1) $a\mathcal{R}^*b$ in R,

(2) $\ell_R(a) = \ell_R(b)$,

(3) $\ell_Q(a) = \ell_Q(b)$,

(4) $a\mathcal{R}^*b$ in Q.

Corollary 2.9 *Let R be a subring of a faithful q–ring Q. If R is a weak order in Q, then R is an order in Q.*

Proof If $a \in \mathcal{S}(R)$, then $a\mathcal{H}^*a^2$ in R so that by Corollary 2.8 and its dual, $a\mathcal{H}^*a^2$ in Q, that is, $a \in \mathcal{S}(Q)$. But Q is a q–ring and so the result follows.

Proposition 2.10 *Let Q be a right faithful ring which satisfies M_L. If R is a weak left order in Q, then R is a left order in Q.*

Proof Let $a \in \mathcal{S}(R)$ so that certainly $a\mathcal{R}^*a^2$ in R. By Corollary 2.8, $a\mathcal{R}^*a^2$ in Q. By the dual of Proposition 2.6, this gives that a has a group inverse in Q and so R is a left order in Q.

We conclude this section with an example of an order in a ring without identity.

Example 2.11 .

Let D be a commutative principal ideal domain and let K be its field of fractions. Let M be a free D–module of infinite rank and let X be a set of free generators for M. Let V be the K-vector space with basis X so that M is a D-submodule of V.

Let R be the subring of $End_D(M)$ consisting of those endomorphisms whose image is finitely generated. Let Q be the subset of $End_K(V)$ consisting of those linear maps q which have finite rank and are such that $d_q(qM) \subseteq M$ for some non-zero element d_q of D. It is readily verified that Q is a subring of $End_K(V)$. Further, each element of R has a unique extension to an element of $End_K(V)$ which clearly lies in Q and so we may regard R as a subring of Q.

We claim that Q is a dense ring of linear maps of V. To see this, let $v_1, ..., v_n, w_1, ..., w_n \in V$ with $v_1, ..., v_n$ linearly independent. There is a finite subset Y of X such that $v_1, ..., v_n$ belong to the subspace $< Y >$ generated by Y. Let Z be a set of vectors such that $\{v_1, ..., v_n\} \cup Z$ is a basis for $< Y >$. Then $\{v_1, ..., v_n\} \cup Z \cup (X \setminus Y)$ is a basis for V and we can define $q \in End_K(V)$ by putting $qv_i = w_i$ for $i = 1, ..., n$ and $qv = 0$ for $v \in Z \cup (X \setminus Y)$. Since $qx \neq 0$ for only finitely many vectors x in X there is a non-zero element d_q of D such that $d_q(qx) \in M$ for all x in X. Thus $d_q(qM) \subseteq M$ and as q certainly has finite rank, we have $q \in Q$. Thus Q is dense.

In particular, Q is regular so that in view of Corollary 2.9, to show that R is an order in Q it is enough to prove that it is a weak order in Q.

Let $q \in Q$ and put $A = Imq \cap M$. It is easy to see that A is a finitely generated D-submodule of M and a direct summand of M, say $M = A \oplus B$. Let $d \in D, d \neq 0$ be such that $d(qM) \subseteq M$. Putting $s = dq$ we have $s \in R$ and $q = r^\sharp s$ where $r \in End_D(M)$ is given by $r(a + b) = da$ for $a \in A, b \in B$. Certainly $r \in R$; also it is clear that $r\mathcal{H}r^2$ in $End_K(V)$ so that by Lemma 2.2, $r\mathcal{H}r^2$ in Q and hence $r^\sharp$ exists in Q. Thus R is a weak left order in Q.

Now let U be a subspace of V such that $V = U \oplus Kerq$. Then U is finite dimensional and we may choose a basis $u_1, ..., u_n$ for U with $u_1, ..., u_n$ in M.

Let d and s be defined as in the previous paragraph. Defining $t \in End_K(V)$ by putting $tu_i = du_i$ for $i = 1, ..., n$ and $tv = 0$ for $v \in Ker q$ we see that $tM \subseteq M$ so that we may regard t as an element of R. As $t\mathcal{H}t^2$ in $End_K(V)$ we have $t\mathcal{H}t^2$ in Q so that $t^\sharp$ exists in Q. Noting that $q = st^\sharp$ we see that R is a weak right order in Q.

Thus R is an order in Q.

3 The old and the new

This section is devoted to comparing the traditional and new concepts of order in rings with identity. We show that in a ring Q with identity, a left order in Q in the new sense is a left order in Q in the traditional sense. The converse is not true and so we investigate conditions which allow a converse result. For the two–sided case we find that in all situations which have been studied in ring theory we do have a converse. To be more specific, we show that if Q is regular or left or right perfect, then a traditional order in Q is an order in Q according to the new definition. For the one–sided case we have a corresponding result when Q is regular and directly finite. We begin with an example of a quotient ring Q which is not a q–ring. Of course, Q is an order in itself in the traditional sense but not in the new sense.

Example 3.1 .

Let F be a field and let A be an F–algebra which its own Jacobson radical and has no zero divisors. Such algebras do exist ; for example, the subring of the field of fractions of $F[x]$ consisting of all elements of the form $xf/(1 + xg)$ where $f, g \in F[x]$ is such a ring . Let $B = \oplus\{A_i : i \in I\}$ where I is an infinite set and $A_i = A$ for each $i \in I$. Clearly every element of B is a zero divisor and B is its own Jacobson radical. Now adjoin an identity to B by forming $Q = F \times B$ and defining addition and multiplication by the rules :

$$(\alpha_1, b_1) + (\alpha_2, b_2) = (\alpha_1 + \alpha_2, b_1 + b_2)$$

$$(\alpha_1, b_1)(\alpha_2, b_2) = (\alpha_1\alpha_2, \alpha_1 b_2 + \alpha_2 b_1 + b_1 b_2).$$

For each element b of B there is an element b' in B such that $b + b' + bb' = 0 = b + b' + b'b$. Hence, if α is a non-zero element of F, the element (α, b) of Q is a unit with inverse $(\alpha^{-1}, \alpha^{-1}(\alpha^{-1}b)')$. All elements of Q with zero first coordinate are zero divisors, so Q is a quotient ring. As every element of the form $(0, b)$ is square-cancellable but the non-zero ones do not have group inverses, Q is not a q-ring.

Our next objective is to show that left orders in the new sense are left orders in the traditional sense. To avoid confusion we introduce the following notation for this section. If R is a ring, then MR denotes the multiplicative semigroup of R. We will write 'R is a (left,right) order in Q' to mean that R is a (left,right) order in Q in the classical sense ; we write 'MR is a (left,right) order in MQ' to mean that R is a (left,right) order in Q in the new sense.

Lemma 3.2 *Let e, f be idempotents in a ring Q with $ef = e = fe$. If the element a of Q is in the group $\mathcal{H}$-class H_e, then there is an element u in the $\mathcal{H}$-class H_f such that $ue = a = eu$.*

Proof Since $a \in H_e$, a has a group inverse $a^\sharp$ such that $aa^\sharp = e = a^\sharp a$. Put $u = f - e + a$ and $v = f - e + a^\sharp$. Then $eu = a = ue, uv = f = vu$ and $uf = u = fu$ so that $u \in H_f$.

Corollary 3.3 *Let Q be a ring with identity. If $a \in Q$ is $\mathcal{H}$-related to an idempotent e, then there is a unit u of Q such that $ue = a = eu$.*

Theorem 3.4 *Let R be a subring of a ring Q with identity. If MR is a left order in MQ, then R is a left order in Q.*

Proof If $q \in Q$, then $q = x^\sharp y$ for some elements $x \in \mathcal{S}(R), y \in R$ with $xx^\sharp y = y$. By Corollary 3.3, $x^\sharp = uxx^\sharp = xx^\sharp u$ for some unit u of Q and so $q = uxx^\sharp y = uy$. Now $u = a^\sharp d$ where $a \in \mathcal{S}(R), d \in R$ and $aa^\sharp d = d$. Hence $a^\sharp au = a^\sharp aa^\sharp d = a^\sharp d = u$ so that $aa^\sharp = 1$, a is a unit and $a^\sharp = a^{-1}$. So putting $b = dy$ we have $a, b \in R$ and $q = a^{-1}b$.

If a is an element of R which is not a zero divisor, then a is square–cancellable so that a has a group inverse $a^\sharp$ in Q. Now $(1 - aa^\sharp)a = 0$ and from above we have that $1 - aa^\sharp = c^{-1}d$ for some $c, d \in R$ so that $da = 0$, giving that $1 - aa^\sharp = c^{-1}0 = 0$ and a is a unit in Q.

We have seen that the converse of Theorem 3.4 is not true. There is, however, a partial converse which covers all cases of interest in the two–sided case.

Theorem 3.5 *Let Q be a q–ring with identity. If R is an order in Q, then MR is an order in MQ.*

Proof All we have to do is show that if a is in $\mathcal{S}(R)$, then a has a group inverse in Q. Since Q has an identity, Lemma 2.7 and its dual apply from which it follows that if a is in $\mathcal{S}(R)$,then a is in $\mathcal{S}(Q)$. As Q is a q–ring, this means that a has a group inverse in Q.

The following corollary is an immediate consequence of the theorem and the facts that regular rings are q–rings and by Proposition 2.6, left perfect rings and right perfect rings are q–rings.

Corollary 3.6 *Let R be an order in a ring Q with identity. If Q is regular, left perfect or right perfect, then MR is an order in MQ.*

Before obtaining a result for the one–sided case we must consider directly finite regular rings. In general, a ring R with 1 is *directly finite* if for all $x, y \in R$, $yx = 1$ follows from $xy = 1$. An R–module A is *directly finite* if A is not isomorphic to a proper direct summand of itself. By Lemma 5.1 of [8], R is directly finite if and only if the right R–module R_R is directly finite and this is so if and only if the left R–module ${}_RR$ is directly finite. Every direct summand of a directly finite module is directly finite and so R is directly finite if and only if every left ideal of the form Re where $e = e^2$ is directly finite ; there is a corresponding equivalent condition for idempotent generated principal right ideals.

In view of these comments, if R is any regular ring (with or without identity) we can define R to be *directly finite* when for every idempotent e in R, the left ideal Re is not isomorphic to a proper direct summand of itself. Using the lemma below which follows from Proposition 4.10 of [7], it is easy to see that this condition is equivalent to the corresponding one for right ideals.

Lemma 3.7 *The following conditions are equivalent for elements x, y of a regular ring R :*

(1) $xR \cong yR$,

(2) $x\mathcal{D}y$,

(3) $Rx \cong Ry$.

We have already mentioned Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$. We now recall the fifth one, $\mathcal{J}$,which is defined by the rule that for elements a, b of a semigroup S, $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$. We write $J(a)$ for S^1aS^1

which is the principal *semigroup* ideal generated by a. We define $I(a)$ to be the set $\{b \in J(a) : J(b) \subset J(a)\}$. Either $I(a)$ is empty or $I(a)$ is a semigroup ideal of S. If $I(a)$ is not empty we can form the *Rees quotient* semigroup $J(a)/I(a)$ which identifies all the elements of $I(a)$. The semigroups $J(a)/I(a)$ (and $J(a)$ if $I(a) = \emptyset$) are called the ***principal factors*** of S. A semigroup S is *simple* if S is the only ideal of itself ; if S has a zero, then S is 0–*simple* if $S^2 \neq 0$ and $S, \{0\}$ are the only ideals of S. If S is a regular semigroup, then each principal factor is 0–simple (simple if $I(a) = \emptyset$). If S has a zero, then $\{0\}$ is a principal factor; we shall refer to principal factors other than $\{0\}$ as the *non–trivial* principal factors.

The idempotents of a semigroup are partially ordered by the relation $\leq$ defined by $e \leq f$ if and only if $fe = e = ef$. A non–zero idempotent e is *primitive* if $f \leq e, f \neq 0$ implies $f = e$. In a ring, by Theorem VII.2 of [1], a non–zero idempotent is primitive if and only if it cannot be written as the sum of two non–zero orthogonal idempotents. A 0–simple semigroup is *completely 0–simple* if it contains a primitive idempotent in which case all its non–zero idempotents are primitive. Similarly, a *completely simple* semigroup is a simple semigroup with a primitive idempotent.

Theorem 3.8 *A regular ring R is directly finite if and only if each non–trivial principal factor of R is completely 0–simple.*

Proof Suppose that R is directly finite and let J/I be a non–trivial principal factor of R. Since R is regular, J/I is 0–simple and regular so that J/I contains a non–zero idempotent e. If J/I is not completely 0–simple, then by Theorem 2.54 of [3], J/I contains a bicyclic subsemigroup B having e as identity element. Now B contains an infinite chain $e > e_1 > e_2 \ldots$ of idempotents which are $\mathcal{D}$–related in B, hence are $\mathcal{D}$–related in J/I and consequently are $\mathcal{D}$–related in

R. By Lemma 3.7, $Re \cong Re_1 \cong Re_2 \cong \ldots$ and as $Re = R(e - e_1) \oplus Re_1$ we have a contradiction. Hence J/I is completely 0–simple.

Conversely, suppose that each non–trivial principal factor of R is completely 0–simple. If R is not directly finite, then for some idempotent e, $Re = A \oplus B$ for some left ideals A, B with $B \cong Re, B \neq Re$. If follows that $A = Rh, b = Rf$ where h, f are orthogonal idempotents with $e = h + f$. So $e \neq f, f \leq e$ but $e\mathcal{D}f$ by Lemma 3.7. Now $\mathcal{D} \subseteq \mathcal{J}$ so that e, f are in the same $\mathcal{J}$–class and hence they are non–zero idempotents in a principal factor. Thus e is a non–primitive idempotent in its principal factor, a contradiction. So R is directly finite.

The following corollary is now immediate by Theorem 6.45 of [3].

Corollary 3.9 *If R is a regular ring which satisfies M_L and M_R, then R is directly finite.*

We denote the minimal condition on principal semigroup ideals by M_J. From Exercise 8 of Section 6.6 of [3] we have the following result .

Corollary 3.10 *If a directly finite regular ring satisfies any one of M_L, M_R or M_J, then it satisfies the other two.*

Theorem 3.11 *Let R be a left order in directly finite regular ring (with identity) Q. Then MR is a left order in MQ.*

Proof Let $a \in \mathcal{S}(R)$; if $a = 0$, clearly a lies in a subgroup of Q. If $a \neq 0$, then by Lemma 2.7, $a\mathcal{R}^*a^2$ in Q so that $a\mathcal{R}a^2$ in Q as Q is regular. Thus a^2 is a non–zero element of the principal factor $J(a)/I(a)$ of Q. By Theorem 3.8, $J(a)/I(a)$ is completely 0–simple and it follows that $a\mathcal{H}a^2$ in $J(a)/I(a)$. Consequently, $a\mathcal{H}a^2$ in Q, that is, a is in a subgroup of Q.

4 A common left denominator theorem

Throughout this section we consider a regular ring Q and a subring R of Q which is a left order in Q. Our first objective is to prove a 'common left denominator' theorem which we use to investigate the relationship between left ideals in R and in Q. We begin by quoting Lemma 2 of [4] which proves to be very useful for us.

Lemma 4.1 *If $e_1, ..., e_n$ are idempotents of Q, then there is an idempotent k in Q such that $e_1, ..., e_n \in kQk$.*

Lemma 4.2 *Let $q_1, ..., q_n$ be elements of Q. Then any element q of Q can be written as $q = x^\sharp y$ where $x \in \mathcal{S}(R), y \in R, xx^\sharp y = y$ and $xx^\sharp q_i = q_i$ for $i = 1, ..., n$.*

Proof Let $q = a^\sharp b$ where $a \in \mathcal{S}(R), b \in R$ and $aa^\sharp b = b$. Let $e_1, ..., e_n$ be idempotents in Q with $e_iQ = q_iQ$ for $i = 1, ..., n$. Let $e_i = a_i^\sharp b_i$ and put $e = a^\sharp a$. By Lemma 4.1, there is an idempotent k in Q such that $e, e_1, ..., e_n \in kQk$. By Lemma 3.2, there is an element u is the group $\mathcal{H}$-class H_k with $eu = ue = a^\sharp$. Now $u = x^\sharp z$ for some x in $\mathcal{S}(R)$, z in R with $xx^\sharp z = z$. Since, for $i = 1, ..., n$,

$$q_iQ = e_iQ \subseteq kQ = uQ \subseteq x^\sharp Q,$$

we have $xx^\sharp q_i = q_i$. Also,

$$q = a^\sharp b = ueb = uaa^\sharp b = ub = x^\sharp zb$$

so that putting $y = zb$ we have $q = x^\sharp y$ with $xx^\sharp y = y$.

Theorem 4.3 *Let $a_1^\sharp b_1, ..., a_n^\sharp b_n$ be elements of Q with $a_i^\sharp a_i b_i = b_i$ for $i = 1, ..., n$. Then there are elements $c, d_1, ..., d_n$ of R with $c \in \mathcal{S}(R)$ such that for $i = 1, ..., n$, we have $d_i \in Rb_i$ and*

$$c^\sharp d_i = a_i^\sharp b_i, c^\sharp cd_i = d_i, c^\sharp ca_i = a_i.$$

Proof Clearly, it suffices to show that we can find elements $c, t_1, ..., t_n$ in R with $a_i^\sharp = c^\sharp t_i$ and $cc^\sharp t_i = t_i$ for $i = 1, ..., n$, for then we can take $d_i = t_i b_i$.

If $n = 1$, we take c to be a_1^2 and t_1 to be a_1. Suppose now that we have elements $u, v_1, ..., v_{n-1}$ of R with $u \in \mathcal{S}(R), u^\sharp v_i = a_i^\sharp$ and $uu^\sharp v_i = v_i$ for $i = 1, ..., n-1$. By Lemma 4.2, we can write $u^\sharp$ as $x^\sharp y$ where $x \in \mathcal{S}(R), y \in R, xx^\sharp y = y$ and $x^\sharp x a_n^\sharp = xx^\sharp a_n^\sharp = a_n^\sharp$. Again by Lemma 4.2, we can write $xa_n^\sharp$ as $r^\sharp s$ for some $r \in \mathcal{S}(R), s \in R$ with $rr^\sharp s = s$ and $rr^\sharp yv_i = yv_i$ for $i = 1, ..., n-1$. Now let $x^\sharp r^\sharp = c^\sharp z$ where $c \in \mathcal{S}(R), z \in R$ and $cc^\sharp z = z$. We now have

$$a_n^\sharp = x^\sharp x a_n^\sharp = x^\sharp r^\sharp s = c^\sharp zs$$

and for $i = 1, ..., n-1$ we have

$$a_i^\sharp = u^\sharp v_i = x^\sharp yv_i = x^\sharp rr^\sharp yv_i = x^\sharp r^\sharp ryv_i = c^\sharp zryv_i.$$

Put $t_n = zs$ and $t_i = zryv_i$ for $i = 1, ..., n-1$. Since $cc^\sharp z = z$ we have the result by induction.

Proposition 4.4 *Let I be a left ideal of Q and J be a left ideal of R. Then*

(1) $I \cap R$ is a left ideal of R and $I = Q(I \cap R)$,

(2) QJ is a left ideal of Q and

$$QJ = \{a^\sharp j : a \in \mathcal{S}(R), j \in J\} = \{a^\sharp j : a \in \mathcal{S}(R), j \in J, aa^\sharp j = j\},$$

(3) $QJ \cap R = \{r \in R : ar \in J$ for some $a \in \mathcal{S}(R)$ with $a^\sharp ar = r\}$.

Proof (1) This is straightforward.

(2) It is clear that QJ is a left ideal of Q and that for $j \in J$, every element of the form $a^\sharp j$ is in QJ . Now let q be an element of QJ so that $q = q_1 j_1 + ... + q_n j_n$ for some $j_1, ..., j_n \in J, q_1, ..., q_n \in Q$. By Theorem 4.3, we can write $q_i = a^\sharp b_i$ for some $a \in \mathcal{S}(R), b_1, ..., b_n \in R$. Putting $j = b_1 j_1 + ... + b_n j_n$, we have $j \in J$ and $q = a^\sharp j$.

(3) This is straightforward.

Corollary 4.5 *Let J be a left ideal of R. Then $J = I \cap R$ for some left ideal I of Q if and only if for elements a, b of R with $a \in \mathcal{S}(R)$ and $aa^\sharp b = b$, we have $ab \in J$ implies $b \in J$.*

Proposition 4.6 *Let $J_1, ..., J_n$ be independent left ideals of R, that is, their sum is direct. Then $QJ_1, ..., QJ_n$ are independent left ideals of Q.*

Proof Suppose that $0 = q_1 + ... + q_n$ where $q_i \in QJ_i$. By Proposition 4.4, $q_i = a_i^\sharp j_i$ for some $j_i \in J_i$ and $a_i \in \mathcal{S}(R)$ with $a_i a_i^\sharp j_i = j_i$. By Theorem 4.3, we can write each $a_i^\sharp j_i$ as $c^\sharp d_i$ for some c in $\mathcal{S}(R)$,d_i in Rj_i and $cc^\sharp d_i = d_i$. Thus $d_i \in J_i$ and

$$0 = c0 = c(c^\sharp d_1 + ... + c^\sharp d_n) = d_1 + ... + d_n$$

so that $d_1 = ... = d_n = 0$ by the independence of $J_1, ..., J_n$. Hence each q_i is zero and $QJ_1, ..., QJ_n$ are independent.

5 Uniqueness

In this section we again consider a left order R in a regular ring Q. Our aim is to show that any regular ring of left quotients of R is isomorphic to Q. We begin by examining the restrictions to R of the preorders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ on Q.

Lemma 5.1 *For elements a, b of R, $aQ \subseteq bQ$ if and only if $\ell_R(b) \subseteq \ell_R(a)$, that is, if and only if $a \leq_{\mathcal{R}^*} b$ in R.*

Proof Regular rings are faithful and so by Lemma 2.7, $\ell_R(b) \subseteq \ell_R(a)$ if and only if $a \leq_{\mathcal{R}^*} b$ in R if and only in $a \leq_{\mathcal{R}^*} b$ in Q. But this is equivalent to $a \leq_{\mathcal{R}} b$ in Q, that is, $aQ \subseteq bQ$.

Lemma 5.2 *For elements a, b of R, $Qa \subseteq Qb$ if and only if there are elements h in $\mathcal{S}(R)$, k in R with $ha = kb$ and $\ell_R(h) \subseteq \ell_R(a)$.*

Proof In view of Lemma 5.1 we know that $\ell_R(h) \subseteq \ell_R(a)$ if and only if $aQ \subseteq hQ$. Thus, if appropriate elements h, k exist, then $h^\sharp$ exists in Q and $a = h^\sharp ha = h^\sharp kb$ so that $Qa \subseteq Qb$.

Conversely, if $a = qb$ for some $q \in Q$, then $q = h^\sharp k$ for some $h \in \mathcal{S}(R), k \in R$ with $hh^\sharp k = k$. So $ha = kb$ and $aQ \subseteq h^\sharp Q = hQ$ as required.

Lemma 5.3 *Let $a^\sharp b, x^\sharp y$ be elements of Q with $a, b, x, y \in R, aa^\sharp b = b, xx^\sharp y = y$ and $a \in xQ$. Then $a^\sharp b = x^\sharp y$ if and only if there are elements $u \in \mathcal{S}(R), v \in R$ with $y, xa, v \in uQ, v \in Qa$ and such that $uxa = va^2$ and $vb = uy$.*

Proof Suppose first that $a^\sharp b = x^\sharp y$. Since $xa^\sharp \in Q$, we have $xa^\sharp = u^\sharp v$ for some u in $\mathcal{S}(R)$, v in R with $uu^\sharp v = v$, that is, $v \in uQ$. Now $xa = xa^\sharp a^2 = u^\sharp va^2 = u(u^\sharp)^2 va^2 \in uQ$ and $y = xx^\sharp y = xa^\sharp b \in uQ$. Also $v = uxa^\sharp \in Qa$ and

$$uxa = u(xa^\sharp)a^2 = u(u^\sharp v)a^2 = va^2.$$

Finally, $uy = uxa^\sharp b = uu^\sharp vb = vb$.

Conversely, from the given conditions we obtain $xa^\sharp = u^\sharp v$ and as $a \in xQ$, it follows from $y = u^\sharp vb = xa^\sharp b$ that $x^\sharp y = a^\sharp b$ as required.

Lemma 5.4 *Let $a^\sharp b, x^\sharp y \in Q$ with $a, b, x, y \in R, aa^\sharp b = b, xx^\sharp y = y$ and $a \in xQ$. Let $c \in \mathcal{S}(R)$ be such that $y \in Qc$. Then $a^\sharp bc^\sharp = x^\sharp y$ if and only if there are elements $m \in \mathcal{S}(R), n \in R$ with $n, xa, yc^2 \in mQ, n \in Qa$ and such that $mxa = na^2$ and $nbc = myc^2$.*

Proof Suppose that $a^\sharp bc^\sharp = x^\sharp y$. We have $xa^\sharp = m^\sharp n$ for some $m \in \mathcal{S}(R), n \in R$ with $mm^\sharp n = n$, that is, $n \in mQ$. Note that $n = mxa^\sharp$ so that $n \in Qa$ and $mxa = na^2$. Also $xa = m^\sharp na^2 \in mQ$. Furthermore,

$$nbc = mxa^\sharp bc = mxa^\sharp bc^\sharp c^2 = mxx^\sharp yc^2 = myc^2$$

and

$$yc^2 = xx^\sharp yc^2 = xa^\sharp bc^\sharp c^2 = m^\sharp nbc \in mQ.$$

Conversely, if the given conditions hold, then from $mxa = na^2$ and $xa \in mQ, n \in Qa$ it follows that $xa^\sharp = m^\sharp n$. Hence, using $yc^2 \in mQ$ and $nbc = myc^2$ we get $yc^2 = m^\sharp nbc = xa^\sharp bc$. Since $a \in xQ$ and $y \in Qc$ we now obtain $a^\sharp bc^\sharp = x^\sharp y$.

Theorem 5.5 *Let T be a regular ring and $\psi : R \to T$ be an embedding such that $S = \psi(R)$ is a left order in T. Then there is a unique isomorphism $\theta : Q \to T$ which extends ψ.*

Proof For elements a, b of R we claim that

$$bQ \subseteq aQ \text{ if and only if } \psi(b)T \subseteq \psi(a)T \tag{1}$$

and

$$Qb \subseteq Qa \text{ if and only if } T\psi(b) \subseteq T\psi(a). \tag{2}$$

Since ψ is an isomorphism from R onto S, we have $x \in \mathcal{S}(R)$ if and only if $\psi(x) \in \mathcal{S}(S)$; thus (1) follows from Lemma 5.1 and (2) follows from Lemma 5.2.

We define θ by specifying its value on elements of Q written as $a^\sharp b$ with $a \in \mathcal{S}(R), b \in R$ and $aa^\sharp b = b$ by the rule :

$$\theta(a^\sharp b) = \psi(a)^\sharp \psi(b).$$

To see that θ is well-defined, let $q = a^\sharp b = c^\sharp d$. By Lemma 4.2, we can write $q = x^\sharp y$ where $x \in \mathcal{S}(R), y \in R$ and $xx^\sharp y = y, xx^\sharp a = a, xx^\sharp c = c$. As we have $a \in xQ$ and $c \in xQ$ we can now apply Lemma 5.3 to $a^\sharp b = x^\sharp y$ and to $c^\sharp d = x^\sharp y$; in view of (1) and (2), the conditions we get are carried over to T by ψ to yield (by Lemma 5.3 again), $\psi(a)^\sharp\psi(b) = \psi(x)^\sharp\psi(y)$ and $\psi(c)^\sharp\psi(d) = \psi(x)^\sharp\psi(y)$. It follows that θ is well-defined.

A similar argument shows that θ is one–one and clearly, θ is onto.

Now let $q_1, q_2 \in Q$ with $q_1 = a^\sharp b, q_2 = c^\sharp d$ where $a, c \in \mathcal{S}(R), b, d \in R$ and $aa^\sharp b = b, cc^\sharp d = d$. Consider $a^\sharp bc^\sharp$; by Lemma 4.2, we can write $a^\sharp bc^\sharp$ as $x^\sharp y$ for some $x \in \mathcal{S}(R), y \in R$ with $xx^\sharp y = y$ and $xx^\sharp a = a$. Also $y = xa^\sharp bc^\sharp \in Qc$ so that the conditions of Lemma 5.4 hold and by virtue of (1) and (2), applying ψ gives the corresponding conditions in T so that by Lemma 5.4 again, we get $\psi(a)^\sharp\psi(b)\psi(c)^\sharp = \psi(x)^\sharp\psi(y)$. Thus

$$\theta(q_1q_2) = \theta(a^\sharp bc^\sharp d) = \theta(x^\sharp yd) = \psi(x)^\sharp\psi(yd) = \psi(x)^\sharp\psi(y)\psi(d)$$

$$= \psi(a)^\sharp\psi(b)\psi(c)^\sharp\psi(d) = \theta(a^\sharp b)\theta(c^\sharp d) = \theta(q_1)\theta(q_2).$$

By Theorem 4.3, there are elements z, t, u in R with $z \in \mathcal{S}(R)$, $zz^\sharp t = t, zz^\sharp u = u, q_1 = z^\sharp t$ and $q_2 = z^\sharp u$. Thus

$$\theta(q_1) + \theta(q_2) = \theta(z^\sharp t) + \theta(z^\sharp u) = \psi(z)^\sharp\psi(t) + \psi(z)^\sharp\psi(u)$$

$$= \psi(z)^\sharp(\psi(t) + \psi(u)) = \psi(z)^\sharp\psi(t+u)$$

$$= \theta(z^\sharp(t+u)) = \theta(z^\sharp t + z^\sharp u) = \theta(q_1 + q_2).$$

Thus θ is an isomorphism and further, it is not difficult to see that θ extends ψ. If $\eta : Q \to T$ is also an isomorphism which extends ψ, then for $q = a^\sharp b \in Q$ where $a \in \mathcal{S}(R), b \in R$ and $aa^\sharp b = b$, we have

$$\eta(q) = \eta(a^\sharp b) = \eta(a^\sharp)\eta(b) = \eta(a)^\sharp\eta(b) = \psi(a)^\sharp\psi(b) = \theta(a^\sharp b) = \theta(q)$$

so that $\theta = \eta$ and θ is unique.

The following corollary is an immediate consequence of the theorem.

Corollary 5.6 *If R is a left order in regular rings Q_1 and Q_2, then Q_1 is isomorphic to Q_2 via an isomorphism which restricts to the identity on R.*

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