STRUCTURE OF LEFT ADEQUATE AND LEFT EHRESMANN MONOIDS

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Abstract. This is the first of two articles studying the structure of left adequate and, more generally, of left Ehresmann monoids. Motivated by a careful analysis of normal forms, we introduce here a concept of *proper* for a left adequate monoid M. In fact, our notion is that of T-*proper*, where T is a submonoid of M. We show that any left adequate monoid M has an X^*-proper cover for some set X, that is, there is a left adequate monoid $\widehat{M}$ that is X^*-proper, and an idempotent separating epimorphism $\theta : \widehat{M} \to M$ of the appropriate type. Given this result, we may deduce that the free left adequate monoid on any set X is X^*-proper.

In a subsequent paper, we show how to construct T-proper left adequate monoids from any monoid T acting via order preserving maps on a semilattice with identity, and prove that the free left adequate monoid is of this form. An alternative description of the free left adequate monoid appears in a recent preprint of Kambites. We show how to obtain the labelled trees appearing in his result from our structure theorem.

Our results apply to the wider class of left Ehresmann monoids, and we give them in full generality. We also indicate how to obtain some of the analogous results in the two-sided case. This paper and its sequel, and the two of Kambites on free (left) adequate semigroups, demonstrate the rich but accessible structure of (left) adequate semigroups and monoids, introduced with startling insight by Fountain some 30 years ago.

Introduction

Left adequate monoids were introduced by Fountain in [3] as monoids M for which every principal left ideal is projective as a left M-act, and such that the set $E(M)$ of idempotents forms a semilattice. The former condition is equivalent to every $\mathcal{R}^*$-class of M containing an idempotent; the latter guarantees that this idempotent is unique. Denoting by a^+ the (unique) idempotent in the $\mathcal{R}^*$-class

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of $a \in M$, it is easy to see that the class of left adequate monoids forms a quasi-variety of algebras of type $(2, 1, 0)$ (that is, possessing the binary and nullary monoid operations, and the unary operation of $^+$, as basic operations), but *not* a variety.

If M is an inverse monoid, then $\mathcal{R}^* = \mathcal{R}$ on M and certainly M is left adequate. The structure of the free inverse monoid $\mathcal{FIM}(X)$ on a set X was discovered by Scheiblich [23] and Munn [22]. Certainly $\mathcal{FIM}(X)$ is E-unitary, which for an inverse monoid is equivalent to being *proper*, that is, $\mathcal{R} \cap \sigma = \iota$. Here σ is the least congruence on a monoid M identifying all the idempotents, so that if M is inverse, σ is the least group congruence. The powerful results of McAlister [20, 21] show that proper inverse monoids are ubiquitous in the sense that *any* inverse monoid M is closely related to a proper inverse monoid $\widehat{M}$ (its 'cover') and moreover, any proper inverse monoid P can be constructed from a group G acting by order automorphisms on a partially ordered set X with subsemilattice Y (P is isomorphic to a 'P-semigroup' $\mathcal{P} = \mathcal{P}(G, X, Y)$). In the case $X = Y$, the semigroup $\mathcal{P}$ becomes a semidirect product.

Naturally, one would wish for similar theory for left adequate monoids. It was rapidly realised, however, that this was overambitious, and to succeed one would need to specialise to *left ample monoids* (formerly, left type A), that is, left adequate monoids satisfying $xy^+ = (xy^+)^+x$. This identity, which ensures some control over the position of idempotents in products, is what enables the free left ample monoid $\mathcal{FLA}m\mathcal{M}(X)$ on X to be embedded in $\mathcal{FIM}(X)$ [6]. Further, $\mathcal{FLA}m\mathcal{M}(X)$ is *proper* in the sense that $\mathcal{R}^* \cap \sigma = \iota$, where σ is now the least right cancellative congruence. A theory analogous to that of McAlister has been developed for left ample monoids, initially by Fountain in [4].

Until recently, little was known of the structure of left adequate monoids in general. We aim to address this issue in the current article and its sequel [10]. After Section 1 of preliminaries, we take a simple minded approach to the structure of left ample monoids in Section 2. The purpose of our analysis is to focus on the role of normal forms in left ample monoids, which enables us enroute to make some comments on the notion of factorisability, and, more significantly, leads us in Section 3 to develop a theory of *T-normal forms*, and a concept of being *T-proper*, where T is a submonoid of a left adequate monoid M. In Section 4 we show that every left adequate monoid has an X^*-proper cover. Finally in Section 5, without fully determining at this stage the structure of the free left adequate monoid $\mathcal{FLA}d\mathcal{M}(X)$ on a set X, we show that $\mathcal{FLA}d\mathcal{M}(X)$ is X^*-proper.

In the subsequent article [10], we develop a 'recipe' for constructing a T-proper left adequate monoid $\mathcal{P}(T, Y)$ from a right cancellative monoid T acting by order-preserving maps on a semilattice Y with identity, that is in a loose sense an analogue of a semidirect product. Our construction is inspired by that of the free left h-adequate monoid given in [5], where it occurs in the very special case of T being a free monoid. Left h-adequate monoids need not be left ample, but neither is every left adequate monoid left h-adequate [3]. We also show that a left adequate monoid M has uniqueness of T-normal forms if and only if it is isomorphic to $\mathcal{P}(T, E(M))$ and further, every left adequate monoid has a proper cover of the form $\mathcal{P}(X^*, E(M))$. We then use our recipe to provide a description of $\mathcal{FLAdM}(X)$. An alternative description appears in the preprint [15] of Kambites, which arises from his consideration of the free adequate monoid in [16].

All of our results are given in the more general setting of left Ehresmann monoids. Such monoids have been championed by Lawson [18]; they are the *variety* generated by the quasi-variety of left adequate monoids (see [10] and [15]). We also remark that we concentrate on monoids rather than semigroups. For technical reasons this makes some of our arguments more straightforward; the free left adequate monoid is the free left adequate semigroup with an identity adjoined (see [15]), so there is no significant loss in generality.

1. Preliminaries

In this section we give the basic definitions and results needed for the rest of the article. We first define twenty one related quasi-varieties of algebras. Since some of our results apply to *all* of these quasi-varieties, the effort is worthwhile. Further details may be found in the notes [12].

The relation $\mathcal{R}^*$ is defined on a monoid M by the rule that for any $a, b \in M$, $a \,\mathcal{R}^*\, b$ if and only if for all $x, y \in M$,

$$xa = ya \text{ if and only if } xb = yb.$$

It is easy to see that $\mathcal{R}^*$ is a left congruence, $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{R} = \mathcal{R}^*$ if M is regular. In general, however, the inclusion can be strict. Let M be a monoid and suppose now that $E \subseteq E(M)$ and E forms a commutative subsemigroup of M; we will say simply that E is a *semilattice* in M.

Definition 1.1. A monoid M is *left E-adequate* if E is a semilattice in M, and every $\mathcal{R}^*$-class contains an idempotent of E. If $E = E(M)$ then we say that M is *left adequate.*

In further definitions, where $E = E(M)$, we may drop explicit mention of E, as in Definition 1.1. From the commutativity of idempotents it is clear that any $\mathcal{R}^*$-class contains at most one idempotent of E. Where it exists we denote the (unique) idempotent of E in the $\mathcal{R}^*$-class of a by a^+. If *every* $\mathcal{R}^*$-class contains an idempotent of E, then $^+$ is a unary operation on M and we may regard M as an algebra of type $(2, 1, 0)$; as such, morphisms must preserve the unary operation of $^+$ (and hence the relation $\mathcal{R}^*$). We may refer to such morphisms as '$(2, 1, 0)$-morphisms' if there is danger of ambiguity. Of course, any semigroup isomorphism must preserve the additional operations. Similarly, if X is a set of generators of a left E-adequate monoid as an algebra with the augmented signature, then we say that X is a set of $(2,1,0)$-*generators* and write $M = \langle X \rangle_{(2,1,0)}$ for emphasis. We remark here that if M is inverse and $E = E(M)$, then $a^+ = aa^{-1}$ for all $a \in M$.

Definition 1.2. A left adequate monoid M is *left ample* if the *left ample* identity (AL) holds:

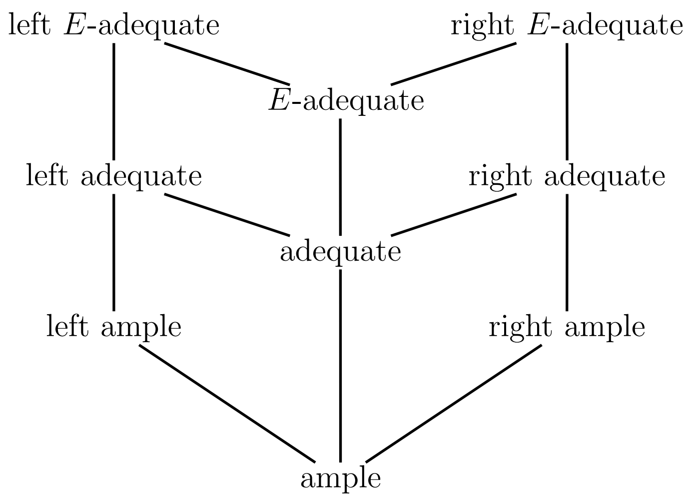
$$xy^+ = (xy^+)^+x \quad \text{(AL)}.$$

We observe that there is no need to explicity define and discuss 'left E-ample monoids', since if a left E-adequate monoid satisfies (AL), E is forced to be $E(S)$.

Remark 1.3. *The class of left E-adequate monoids forms a quasi-variety of algebras of type* $(2,1,0)$ *with sub-quasi-varieties the classes of left adequate and left ample monoids.*

Left ample monoids have a nice representation theory: they are precisely the submonoids of symmetric inverse monoids closed under $^+$ (see, for example [12]).

The relation $\mathcal{L}^*$ is the dual of $\mathcal{R}^*$ and may be used to give an abstract characterisation of right (E-)adequate and right ample monoids. We denote the unique idempotent in the $\mathcal{L}^*$-class of a, where it exists, by a^*. Observe that if M is inverse, then $a^* = a^{-1}a$ for all $a \in M$. The right ample identity (AR) states that $b^*a = a(b^*a)^*$ for all $a, b \in M$. A monoid is *E-adequate* if it is both left and right E-adequate *with respect to the same semilattice* E, and *adequate (ample)* if it is both left and right adequate (ample). The class of E-adequate monoids therefore forms a quasi-variety of algebras of type $(2,1,1,0)$, with sub-quasi-varieties the quasi-varieties of adequate and ample monoids.



We remark that as any inverse monoid is certainly ample, any submonoid of an inverse monoid that is closed under $^+$ and * is ample. On the other hand it is undecidable whether a finite ample monoid embeds as a $(2,1,1,0)$-algebra into an inverse monoid [13].

We now turn our attention to classes defined by certain relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. Again, let E be a semilattice in a monoid M. The relation $\widetilde{\mathcal{R}}_E$ on M is defined by the rule that for any $a, b \in M$, $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if for all $e \in E$,

$$ea = a \text{ if and only if } eb = b,$$

that is, a and b have the same set of left identities from E. Dually, we define $\widetilde{\mathcal{L}}_E$. It is easy to see that for any monoid M, we have $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$, with both inclusions equalities if M is regular and $E = E(M)$; in general, however, these inclusions can be strict. The relation $\widetilde{\mathcal{R}}_E$ is certainly an equivalence; however, unlike the case for $\mathcal{R}$ and $\mathcal{R}^*$, it need not be left compatible, not even when $E = E(M)$. As a guide, the adjective 'weakly' in front of any of the classes appearing in the above diagram denotes the correponding class obtained by replacing $\mathcal{R}^*$ ($\mathcal{L}^*$) in the definition by $\widetilde{\mathcal{R}}_E$ ($\widetilde{\mathcal{L}}_E$), with the addition of the condition that $\widetilde{\mathcal{R}}_E$ ($\widetilde{\mathcal{L}}_E$) be a left (right) congruence.

It is clear that any $\widetilde{\mathcal{R}}_E$-class contains at most one idempotent from E. If every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E, we again have a unary operation $a \mapsto a^+$, where a^+ is now the (unique) idempotent of E in the $\widetilde{\mathcal{R}}_E$-class of a. We may then consider M as an algebra of type $(2, 1, 0)$. Notice that a^+ is the least element in the set of left identities of a lying in E, with respect to the natural partial order on E. In the case that $E = E(M)$, we continue to drop the 'E' from notation and terminology, for example, we write $\widetilde{\mathcal{R}}_{E(M)}$ more simply as $\widetilde{\mathcal{R}}$.

Definition 1.4. A monoid M with semilattice E is *left Ehresmann* (with distinguished semilattice E) if every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E and $\widetilde{\mathcal{R}}_E$ is a left congruence. In addition, if $E = E(M)$, we say that M is *weakly left adequate*; if M satisfies (AL), then M is *left restriction* and if $E = E(M)$ and M satisfies (AL), then M is *weakly left ample*.

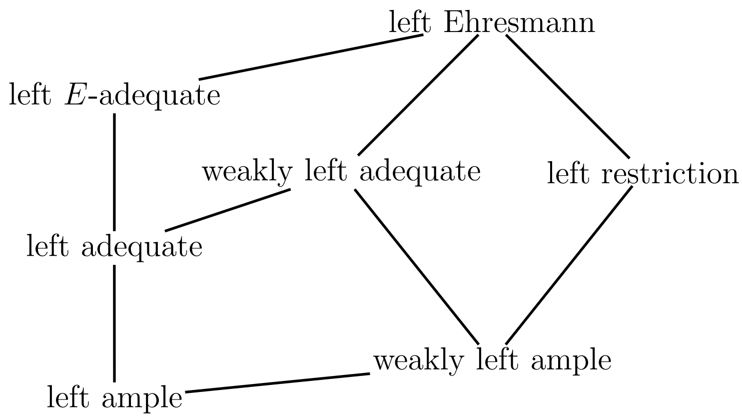
According to our convention, left Ehresmann monoids may also be referred to as *weakly left E-adequate* monoids. Left restriction monoids have arisen in a number of contexts (see [12]) and have received various names, in particular that of *weakly left E-ample* (see, for example, [7]). They are precisely those submonoids of partial transformation monoids closed under $^+$, where α^+ is the identity map in the domain of α.

Remark 1.5. *The class of left Ehresmann monoids is a variety of algebras of type* $(2, 1, 0)$*, with the class of left restriction monoids being a subvariety, and the classes of weakly left adequate (ample) monoids being sub-quasi-varieties.*

It is worth making the remark that if M is a left Ehresmann monoid, then $E = \{a^+ : a \in M\}$. Moreover, the identity of M must lie in E, for we must have that $1^+ = 1$.

Right Ehresmann, *right restriction*, weakly right adequate and weakly right ample monoids may be defined in terms of $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_E$. In each case, we denote the dual of the operation $^+$ by *. A monoid is *Ehresmann (restriction)* if it is both left and right Ehresmann (restriction) with respect to the same semilattice E, in this case $E = \{a^+ : a \in M\} = \{a^* : a \in M\}$. Similarly, a monoid is *weakly adequate (ample)* if it is both left and right weakly adequate (ample). Ehresmann monoids form a variety of algebras of type $(2, 1, 1, 0)$, with subvariety the variety of restriction monoids, and sub-quasi-varieties the quasi-varieties of weakly adequate and of weakly ample monoids.

The following diagram depicts some of the inclusion relation between the classes discussed.



After a discussion of left ample monoids and left restriction monoids in Section 2, largely by way of illustration and motivation, the paper will focus on left Ehresmann monoids i.e. *we dispense with Condition (AL).*

We now give a technical result which will be useful in the subsequent sections. It follows immediately from the fact that in a left Ehresmann monoid, $\widetilde{\mathcal{R}}_E$ is a left congruence. The relation $\leq$ appearing in its statement is the natural partial order on E.

Lemma 1.6. *Let M be a left Ehresmann monoid. Then for any $a, b \in M$ and $e \in E$, $(ab)^+ = (ab^+)^+$, $(ea)^+ = ea^+$ and $(ab)^+ \leq a^+$.*

The following lemma is folklore, but we include its proof here for completeness, since the underlying idea is central to our approach to the structure of left Ehresmann monoids.

Lemma 1.7. *Let M be a left Ehresmann monoid and let T be a submonoid of M. Then T acts on E on the left via order preserving maps by $(t, e) \mapsto t \cdot e = (te)^+$. If M is left restriction, then the action is by morphisms of E.*

Proof. For any $e \in E$, we have that $1 \cdot e = (1e)^+ = e^+ = e$, and for any $s, t \in T$,

$$st \cdot e = (ste)^+ = (s(te)^+)^+ = s \cdot (te)^+ = s \cdot (t \cdot e),$$

by Lemma 1.6. Hence T acts on E.

If $e, f \in E$ and $e \leq f$, then for any $s \in T$

$$(sf)^+se = (sf)^+(sf)e = sfe = se,$$

so that $(se)^+ \leq (sf)^+$, since $(se)^+$ is the minimum left identity of se in E. Thus $\cdot$ is an order-preserving action.

Suppose now that M is left restriction, so that $ae = (ae)^+a$ for all $a \in M$ and $e \in E$. Then for any $t \in T$ and $e, f \in E$ we have

$$t \cdot (ef) = (tef)^+ = ((te)^+tf)^+ = ((te)^+(tf)^+)^+ = (te)^+(tf)^+ = (t \cdot e)(t \cdot f),$$

so that T acts by morphisms as required. □

We recall that a left Ehresmann monoid M is said to be *hedged* [9] if $\cdot$ is an action by morphisms; in particular, if the left ample identity holds, then M is hedged.

Let S be a semigroup and suppose that $E \subseteq E(S)$. We define the relation σ_E to be the *semigroup congruence on S generated by $E \times E$*; that is, for any $a, b \in S$ we have that $a \,\sigma_E\, b$ if and only if $a = b$ or there exists a sequence

$$a = c_1e_1d_1, c_1f_1d_1 = c_2e_2f_2, \ldots, c_nf_nd_n = b,$$

where $c_1, d_1, \ldots, c_n, d_n \in S^1$ and $(e_1, f_1), \ldots, (e_n, f_n) \in E \times E$. Notice that in a left Ehresmann monoid M, for any $a, b \in M$, $a^+ \,\sigma_E\, b^+$ (whether or not $a \,\sigma_E\, b$), giving us the following.

Lemma 1.8. *Let M be a left Ehresmman monoid with distinguished semilattice E. Then E is contained in a σ_E-class and σ_E is a $(2,1,0)$-congruence.*

With the addition of (AL), we have a closed formula for σ_E.

Lemma 1.9. [7, Proposition 2.5]. *Let S be a left restriction semigroup with distinguished semilattice of idempotents E. Then $a \,\sigma_E\, b$ if and only if $ea = eb$ for some $e \in E$.*

If $E = E(M)$ then we write σ for $\sigma_{E(M)}$. From [4, 11, 12, 14, 17] we have the following.

Proposition 1.10. *Let M be a monoid and $E \subseteq E(M)$ a semilattice:*

(i) if M is left restriction with $E = E(M)$, then σ is the least unipotent congruence on M;

(ii) if M is left ample, then σ is the least right cancellative congruence on M;

(iii) if M is ample, then σ is the least cancellative congruence on M;

(iv) if M is inverse, then σ is the least group congruence on M.

Considerations of duality now tell us that if M is right restriction, then $a \,\sigma_E\, b$ if and only if $af = bf$ for some $f \in E$.

It is well known that an inverse monoid is E-unitary if and only if it is proper, where here proper means that $\mathcal{R} \cap \sigma = \iota$ or equivalently, $\mathcal{L} \cap \sigma = \iota$. Analogously, we say that a left ample monoid is *proper* if $\mathcal{R}^* \cap \sigma = \iota$, and a left restriction monoid is *proper* if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$. Since $\mathcal{R}^* = \widetilde{\mathcal{R}}$ for a left ample monoid (and so certainly for an inverse monoid), there is little danger of ambiguity. In the two sided case (where in general we do not have the natural duality guaranteed by the existence of the involution $^{-1}$ in the inverse case), we say that an ample monoid is *proper* if $\mathcal{R}^* \cap \sigma = \mathcal{L}^* \cap \sigma = \iota$, with the obvious alteration in the restriction case. Proper left ample monoids are E-unitary, but the converse is not true [4].

In this article we require care with signatures. To this end we give a technical but straightforward result, the proof of which we omit.

Lemma 1.11. *(i) Let M be a left Ehresmann monoid with distinguished semilattice E, and let X be a subset of M. Put $T = \langle X\rangle_{(2,0)}$. Then*

$$\langle E \cup X\rangle_{(2,1,0)} = \langle E \cup X\rangle_{(2)} = \langle E \cup T\rangle_{(2)} = \langle E \cup T\rangle_{(2,1,0)}.$$

(ii) Let M be an Ehresmann monoid with distinguished semilattice E, and let X be a subset of M. Put $T = \langle X\rangle_{(2,0)}$. Then

$$\langle E \cup X\rangle_{(2,1,1,0)} = \langle E \cup X\rangle_{(2)} = \langle E \cup T\rangle_{(2)} = \langle E \cup T\rangle_{(2,1,1,0)}.$$

Corollary 1.12. *(i) Let M be a left Ehresmann monoid with distinguished semilattice E, and suppose that $M = \langle X\rangle_{(2,1,0)}$. Put $T = \langle X\rangle_{(2,0)}$. Then $M = \langle E \cup T\rangle_{(2)}$.*

(ii) Let M be an Ehresmann monoid with distinguished semilattice E, and suppose that $M = \langle X\rangle_{(2,1,1,0)}$. Put $T = \langle X\rangle_{(2,0)}$. Then $M = \langle E \cup T\rangle_{(2)}$.

Proof. For (i) We have that

$$M = \langle X\rangle_{(2,1,0)} = \langle E \cup X\rangle_{(2,1,0)} = \langle E \cup T\rangle_{(2)},$$

from Lemma 1.11. The proof of (ii) is virtually identical. $\square$

For convenience we define a list ($\mathscr{L}$) of the quasi-varieties and varieties of monoids we have discussed. More accurately, these are classes of algebras with an underlying monoid structure:

(i) left (or right, or two-sided) ample;
(ii) weakly left (or right, or two-sided) ample;
(iii) left (or right, or two-sided) restriction;
(iv) left (or right, or two-sided) adequate;
(v) left (or right, or two-sided) E-adequate;
(vi) weakly left (or right, or two-sided) adequate;
(vii) left (or right, or two-sided) Ehresmann.

In Section 5 we show that if M is free on X in any class in our list ($\mathscr{L}$), and if $T = \langle X\rangle_{(2,0)}$ as above, then $T \cong X^* \cong M/\sigma_E$.

2. Left restriction monoids

To set the scene for our investigation of left adequate and related monoids, we give a short discussion of the approach to proper covers in the case where the ample identity holds.

We recall that a left restriction monoid M with distinguished semilattice E is *proper* if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$. This tells us that M is a subdirect product *as a set* of $M/\widetilde{\mathcal{R}}_E$ and M/σ_E. Since every element of M is $\widetilde{\mathcal{R}}_E$-related to a unique idempotent of E, we may identify $M/\widetilde{\mathcal{R}}_E$ with E. Letting

$$P = \{(s^+, s\sigma_E) : s \in M\}$$

and defining $\theta : M \to P$ by $s\theta = (s^+, s\sigma_E)$, then $(P, *)$ is a semigroup isomorphic to M under

$$s\theta * t\theta = (st)\theta.$$

The aim of a 'P-theorem' is to show that there is an action of M/σ_E on a partially ordered set $\overline{E}$ containing E as a subsemilattice, such that the binary operation $*$ is given as in a semidirect product, that is, by

$$(e, s\sigma_E) * (f, t\sigma_E) = (e \wedge (s\sigma_E \cdot f), s\sigma_E t\sigma_E).$$

Let M be a left Ehresmann monoid, and let T be a submonoid of M such that $M = \langle E \cup T\rangle_{(2)}$ where E is the distinguished semilattice of M. We say that a pair $(e, t) \in E \times T$ is a *strong T-normal form* if $e \leq t^+$. If $m \in M$ and $m = et$ where (e, t) is a strong T-normal form, then we say that $m = et$ is *a factorisation of m in strong T-normal form.* Where there is no danger of ambiguity, we may say that et is, or is in, strong T-normal form, if $e \leq t^+$. Observe that, in this case, $m^+ = (et)^+ = et^+ = e$, so e is unique.

A left Ehresmann monoid M with distinguished semilattice E is said to be *factorisable by a submonoid T* if $M = ET$.

Lemma 2.1. *Let M be a left Ehresmann monoid with distinguished semilattice E. Suppose that T is a submonoid of M. Then:*

(a) If $M = \langle E \cup T\rangle_{(2)}$ and M is left restriction, then $M = ET$, i.e. M is factorisable by T.

(b) If M is factorisable by T, then

(i) any $m \in M$ can be written in strong T-normal form;

(ii) for any $[m] \in M/\sigma_E$ we have that $[m] = [a]$ for some $a \in T$;

(iii) if $m = ea$ and $n = fb$ where $e, f \in E$, $a, b \in T$ are factorisations in strong T-normal form, then

$$m\,(\widetilde{\mathcal{R}}_E \cap \sigma_E)\,n \text{ if and only if } e = f \text{ and } a\,\sigma_E\,b.$$

Proof. (a) Certainly $E \cup T \subseteq ET$, as $1 \in E \cap T$. If $ea, fb \in ET$ where $e, f \in E$ and $a, b \in T$, then

$$(ea)(fb) = e(af)b = e(af)^+ab \in ET.$$

It follows that $M = ET$.

(b) Suppose that M is factorisable by T.

(i) Simply note that if $e \in E$ and $a \in T$, then $ea = ea^+a$ and $ea^+ \leq a^+$.

(ii) Let $m \in M$ and write m as $m = ea$ for some $e \in E$ and $a \in T$; then clearly $m = ea\, \sigma_E\, a^+a = a$.

(iii) Let m, n, e, f, a and b be as given. Notice that $m^+ = e$, $n^+ = f$, $m\, \sigma_E\, a$ and $n\, \sigma_E\, b$, from which the result follows. □

If every element of M has a unique expression as a strong T-normal form, we say that M *has uniqueness of strong T-normal forms.*

Lemma 2.2. *Let M be a left restriction monoid with distinguished semilattice E such that M is factorisable by a submonoid T. Then M has uniqueness of strong T-normal forms if and only if $\nu = \nu_{\sigma_E}|_T : T \to M/\sigma_E$ is a monoid isomorphism.*

Proof. Suppose that M has uniqueness of strong T-normal forms. By Lemma 2.1 (b)(ii), the map ν is onto. If $s, t \in T$ and $s\nu = t\nu$, then $es = et$ for some $e \in E$, by Lemma 1.9. It follows that $(es^+t^+)s = (es^+t^+)t$, and as (es^+t^+, s) and (es^+t^+, t) are strong T-normal forms, we have that $s = t$ and ν is injective. Hence ν is an isomorphism.

Conversely, suppose that ν is an isomorphism, and $es = ft$ where (e, s) and (f, t) are strong T-normal forms. Then $s\nu = t\nu$, so that $s = t$. Moreover,

$$e = es^+ = (es)^+ = (ft)^+ = ft^+ = f.$$

Hence $(e, s) = (f, t)$ and M has uniqueness of strong T-normal forms. □

Lemma 2.3. *Let M be a left restriction monoid with distinguished semilattice E, factorisable by a submonoid T (or equivalently, $M = \langle E \cup T \rangle_{(2)}$). If M has uniqueness of strong T-normal forms, then $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$ so that M is proper.*

Proof. Let $m = es$ and $n = ft$ be elements of M in strong T-normal form, and suppose that $m\, (\widetilde{\mathcal{R}}_E \cap \sigma_E)\, n$. From Lemma 2.1, $e = f$ and $s\, \sigma_E\, t$. From Lemma 2.2 it follows that $s = t$ so that $m = n$ and M is proper. □

A left Ehresmann monoid Q with distinguished semilattice E' is said to be a *cover* of a left Ehresmann monoid M with distinguished semilattice E if there exists a $(2, 1, 0)$-morphism from Q onto M which is injective on E'.

Let M be a left restriction monoid with distinguished semilattice E, and submonoid T. From Lemma 1.7, we know that T acts on the left of E via morphisms. We may thus form the semidirect product $E * T$. From [7, Lemma 6.2]

$$E *_m T = \{(e, t) \in E \times T : e \leq t^+\}$$

is a proper left restriction monoid with $(e, t)^+ = (e, 1)$, identity $(1, 1)$ and distinguished semilattice

$$\overline{E} = \{(e, 1) : e \in E\}.$$

Proposition 2.4. *Let M be a left restriction monoid with distinguished semilattice E and submonoid T.*

(i) The function $\theta : E *_m T \to M$ given by $(e,t)\theta = et$ is an $\overline{E}$-separating morphism from $E *_m T$ to M.

(ii) The monoid is $E *_m T$ is a proper cover of M via the morphism θ if and only if $M = \langle E \cup T\rangle_{(2)}$ (if and only if M is factorisable by T).

(iii) If $M = \langle E \cup T\rangle_{(2)}$, then M has uniqueness of strong T-normal forms if and only if θ is an isomorphism.

Proof. Clearly θ preserves the identity; if $(e,s) \in E *_m T$, then

$$\big((e,s)\theta\big)^+ = (es)^+ = es^+ = e = (e,1)\theta = (e,s)^+\theta.$$

If in addition $(f,t) \in E *_m T$, then

$$\big((e,s)(f,t)\big)\theta = (e(sf)^+, st)\theta = e(sf)^+st = esft = (e,s)\theta(f,t)\theta,$$

so that θ is a morphism. Clearly θ is $\overline{E}$-separating.

If θ is onto, then $M = ET$, so that certainly $M = \langle E \cup T\rangle_{(2)}$. The converse follows by Lemma 2.1 $(b)(i)$. Hence (ii) holds.

If $M = \langle E \cup T\rangle_{(2)}$, then M has uniqueness of strong T-normal forms if and only if each $m \in M$ has a unique expression as $m = et$ for some $e \in E, t \in T$ with $e \leq t^+$. But this is exactly saying that for each $m \in M$, $m\theta^{-1}$ is a singleton, or equivalently, θ is one-one. $\square$

We pause to consider the application of the above results to the concept of factorisability for inverse monoids and their generalisations. Recall that an inverse monoid is certainly left restriction where $m^+ = mm^{-1}$ (so that $E = E(M)$). An inverse monoid is *factorisable* if $M = E(M)\mathcal{U}(M)$, where $\mathcal{U}(M)$ is the group of units of M.

The last proposition applied to the inverse case gives the well known result that an inverse monoid M is factorisable if and only if it admits the semidirect product $E(M) * \mathcal{U}(M)$, where the action is given by $g \cdot e = geg^{-1} = (ge)^+$, as proper cover, via the map $(e,g) \mapsto eg$.

We now demonstrate this in the broader setting of left restriction monoids. The concept of a factorisable right adequate monoid is due to El Qallali, and first appeared in [1]; the dual was considered by El Qallali and Fountain for left ample monoids in [2]. These concepts are discussed in detail by Szendrei and the second author in [8]. We say that a left restriction monoid M is *factorisable* if it is factorisable by $\widetilde{R}_1$, where $\widetilde{R}_1$ is the $\widetilde{\mathcal{R}}_E$-class of 1. That is, M is factorisable if $M = E\widetilde{R}_1$; this extends the definition in [8] given in the case $E = E(M)$.

Proposition 2.5. *Let M be a left restriction monoid. Then M is proper and factorisable if and only if M is isomorphic to the semidirect product $E * \widetilde{R}_1$ under the isomorphism $(e,m) \mapsto em$.*

Proof. We first note that $\widetilde{R}_1$ is certainly a submonoid of M, and as by very definition, $m^+ = 1$ for any $m \in \widetilde{R}_1$, we have that $E *_m \widetilde{R}_1 = E * \widetilde{R}_1$.

If $\theta : E * \widetilde{R}_1 \to M$ is an isomorphism, then by Proposition 2.4 and Lemma 2.3, M is proper and factorisable.

Conversely, if M is factorisable, then θ is onto and certainly $M = \langle E \cup \widetilde{R}_1 \rangle_{(2)}$. Suppose now that M is also proper and $eg = fh$, where $(e, g), (f, h) \in E * \widetilde{R}_1$. Then $g \, \sigma_E \, h$ and since $g^+ = 1 = h^+$, we have that $g = h$ as M is proper. Also, $e = (eg)^+ = (fh)^+ = f$, and therefore θ is an isomorphism. □

The above result may easily be adapted to give the characterisation of proper (i.e. E-unitary) factorisable inverse monoids as semidirect products of semilattices by groups (where we must take the submonoid $\mathcal{U}(M)$ of $\widetilde{R}_1$) [19].

We have shown how the concepts of proper and unique strong normal forms are intrinsically related in the world of left restriction monoids. The importance of the class of proper left restriction monoids is well established. In the next section we investigate the corresponding class of left Ehresmann monoids.

We remark that (in view of the next section) we could have defined 'strong T-normal form' in a slightly different way, by insisting that in a product ea we have $e = 1$ or $e < a^+$. However, the approach we have taken is easier to present.

3. Proper left Ehresmann monoids

Once we drop the 'ample' condition, moving from classes of left restriction monoids to classes of left Ehresmann monoids, the nice behaviour of generators as in Lemma 2.1 does not hold; consequently, we cannot describe the structure of these monoids using semidirect products.

The following result is obtained from a series of straightforward manoevres, but provides us with an important idea.

Lemma 3.1. *Let M be a left Ehresmann monoid. Suppose that $M = \langle E \cup T \rangle_{(2)}$ for some submonoid T of M. Then any $x \in M$ can be written as*

$$x = t_0 e_1 t_1 \ldots e_n t_n,$$

where $n \geq 0$, $e_1, \ldots, e_n \in E \setminus \{1\}$, $t_1, \ldots, t_{n-1} \in T \setminus \{1\}$, $t_0, t_n \in T$ and for $1 \leq i \leq n$,

$$e_i < (t_i e_{i+1} \ldots e_n t_n)^+.$$

Proof. Since M is generated as a semigroup by E and T, and bearing in mind that $1 \in T$, any element x of M can be written as $x = s_0 f_1 s_1 \ldots f_m s_m$ for some $s_0, \ldots, s_m \in T$ and $f_1, \ldots, f_m \in E$. We now give an algorithm for reducing m to an expression in the required form.

Step 1. Eliminate all f_i's such that $f_i(s_i \ldots f_m s_m)^+ = (s_i \ldots f_m s_m)^+$ to obtain $x = u_0 g_1 u_1 \ldots g_k u_k$ where $u_0, \ldots, u_k \in T$ and $g_1, \ldots, g_k \in E$ are such that $g_i(u_i \ldots g_k u_k)^+ < (u_i \ldots g_k u_k)^+$.

Step 2. For each $i \in \{1, \ldots, k\}$, we put

$$g_i' = g_i(u_i \ldots g_k u_k)^+.$$

Certainly

$$g_k u_k = g_k u_k^+ u_k = g_k' u_k.$$

Suppose for induction we have

$$g_i u_i \ldots g_k u_k = g_i' u_i \ldots g_k' u_k.$$

Then

$$\begin{aligned} g_{i-1}u_{i-1}g_i u_i \ldots g_k u_k &= g_{i-1}(u_{i-1}g_i u_i \ldots g_k u_k)^+ u_{i-1}g_i u_i \ldots g_k u_k \\ &= g_{i-1}' u_{i-1} g_i u_i \ldots g_k u_k \\ &= g_{i-1}' u_{i-1} g_i' u_i \ldots g_k' u_k. \end{aligned}$$

Hence

$$x = s_0 f_1 s_1 \ldots f_m s_m = u_0 g_1 u_1 \ldots g_k u_k = u_0 g_1' u_1 \ldots g_k' u_k.$$

Notice that for each $i \in \{1, \ldots, k\}$ we have that

$$g_i' < (u_i g_{i+1} \ldots g_k u_k)^+ = (u_i g_{i+1}' \ldots g_k' u_k)^+$$

and observe that no g_i' can be equal to 1.
Step 3. Finally we delete any interior u_i's that are 1. Suppose that $u_i = 1$ for some $i \in \{1, \ldots, k-1\}$ and let $\ell \in \{1, \ldots, k-1\}$ be greatest such that $u_\ell = u_{\ell-1} = \ldots = u_h = 1$, but $u_{h-1} \neq 1$ (or $h = 1$). Then

$$g_h' u_h \ldots g_k' u_k = g_h' g_{h+1}' \ldots g_\ell' g_{\ell+1}' u_{\ell+1} g_{\ell+2}' \ldots g_k' u_k$$

and

$$g_h' g_{h+1}' \ldots g_\ell' g_{\ell+1}' \leq g_{\ell+1}' < (u_{\ell+1} g_{\ell+2}' \ldots g_k' u_k)^+.$$

We have

$$x = u_0 g_1' u_1 \ldots u_{h-1}(g_h' g_{h+1}' \ldots g_\ell' g_{\ell+1}') u_{\ell+1} g_{\ell+2}' \ldots g_k' u_k$$

where

$$u_{h-1}(g_h' g_{h+1}' \ldots g_\ell g_{\ell+1}') u_{\ell+1} g_{\ell+2}' \ldots g_k' u_k = u_{h-1} g_h' u_h \ldots g_k' u_k.$$

Delete the next right-most block of u_i's that are equal to 1 and continue until there are no interior 1's and x is now reduced to the required form.

It is worth noticing that in the last reductions, a product $g_h' g_{h+1}' \ldots g_\ell' g_{\ell+1}'$ is in fact equal to g_h'. Since, if $u_h = 1$, then

$$g_h' < (u_h g_{h+1}' \ldots g_k' u_k)^+ = (g_{h+1}' \ldots g_k' u_k)^+ = g_{h+1}'(u_{h+1} \ldots g_k' u_k)^+ \leq g_{h+1}'.$$

□

We will say that an element $x = t_0 e_1 t_1 \ldots e_n t_n$ expressed as in the statement of Lemma 3.1 is in *T-normal form*; if $T = M$ then we say 'normal form'. Notice that $x^+ = (t_0 e_1)^+$. If every element of M has a unique expression in T-normal form, then we say that M has *uniqueness of T-normal forms*.

Since the above reduction process and the concept of uniqueness of T-normal forms will be crucial to some later results, in particular in Proposition 3.9, we pause to make a number of comments. Let M and T be as in Lemma 3.1. If

$$x = t_0e_1t_1 \ldots e_nt_n,$$

where $n \geq 0$, $e_1, \ldots, e_n \in E$, and $t_0, t_1, \ldots, t_n \in T$, then the first two steps of the reduction process eliminate any e_i's with $e_i \geq (t_ie_{i+1} \ldots e_nt_n)^+$. We are then left with

$$x = u_0f_1u_1 \ldots f_mu_m$$

where $u_i \in T, i \in \{0, \ldots, m\}$ and $f_j \in E$ with $f_j < (u_jf_{j+1} \ldots f_mu_m)^+$, $1 \leq j \leq m$. As a temporary definition, let us say that x is in *weak T-normal form.* The remaining step is to eliminate an of $u_1, \ldots, u_{m-1}$ ('interior u_i's') that are 1. Notice that at this stage none of the f_j's will dispappear, but they may be amalgamated with other idempotents by virtue of eliminating u_i's.

Suppose now that M has uniqueness of T-normal forms and

$$u_0f_1u_1 \ldots f_{k-1}u_{k-1}f_ku_k \ldots f_mu_m = v_0g_1v_1 \ldots g_\ell v_\ell f_ku_k \ldots f_mu_m \quad (*)$$

where both sides are in weak T-normal form and $u_j \neq 1, k \leq j \leq m-1$. Put $a = u_0f_1 \ldots u_{k-1}$ and $b = v_0g_1 \ldots v_\ell$. The reduction of both sides of (*) to T-normal form involves eliminating from a any of $u_0, \ldots, u_{k-1}$ that are equal to 1 to obtain an expression a', and eliminating from b any of $v_0, \ldots, v_\ell$ that are equal to 1 to obtain b', where

$$a'f_ku_k \ldots f_mu_m = b'f_ku_k \ldots f_mu_m$$

and both sides are in T-normal form. Consequently, $a'f_k = b'f_k$ and (as $a = a'$ and $b = b'$), we have that

$$u_0f_1u_1 \ldots u_{k-1}f_k = v_0g_1v_1 \ldots v_\ell f_k.$$

Observing the algorithm in Lemma 3.1, we notice that we only delete (interior) elements of T that are equal to 1, and this does not affect the value of any (non-empty) product.

Remark 3.2. *Let $x = u_0f_1u_1 \ldots f_mu_m$ be a product of elements of T and E, as in the proof of Lemma 3.1, and suppose that the reduction process described in that lemma yields $x = t_0e_1t_1 \ldots e_nt_n$ in T-normal form. Then*

$$t_0t_1 \ldots t_n = u_0u_1 \ldots u_m.$$

We introduce a class of left Ehresmann monoids that play the role for the class of *all* left Ehresmann monoids that groups play for inverse monoids, cancellative monoids play for ample monoids, etc.

Definition 3.3. Let M be a left Ehresmann monoid. We say that M is *reduced* if $m^+ = 1$ for all $m \in M$, i.e. $E = \{1\}$.

Note that any monoid is a reduced left Ehresmann monoid - we have simply augmented the monoid signature.

Proposition 3.4. *Let M be a left Ehresmann monoid where $M = \langle E \cup T \rangle_{(2)}$ for some submonoid T of M. Suppose that M has uniqueness of T-normal forms. Then the map $c_T : M \to T$ given by $c_T(a) = t_0 t_1 \dots t_n$ where $t_0, \dots, t_n \in T, e_1, \dots, e_n \in E$ and $a = t_0 e_1 t_1 \dots e_n t_n$, is a well-defined monoid morphism with* $\ker c_T = \sigma_E$ *so that $M/\sigma_E \cong T$ as monoids. Moreover, if T is regarded as being a reduced left Ehresmann monoid, then c_T is a $(2,1,0)$-morphism and $M/\sigma_E \cong T$ as left Ehresmann monoids.*

Proof. Let $a = t_0 e_1 t_1 \dots e_n t_n = t_0' e_1' t_1' \dots e_m' t_m'$ where $t_0, \dots, t_n, t_0' \dots, t_m' \in T$ and $e_1, \dots, e_n, e_1', \dots, e_m' \in E$. Then reducing both decompositions of a to T-normal forms as in Lemma 3.1, we arrive at the same T-normal form $a = s_0 f_1 s_1 \dots f_k s_k$ for some $s_0, \dots, s_k \in T$ and $f_1, \dots, f_k \in E$. As commented in Remark 3.2,

$$t_0 t_1 \dots t_n = s_0 \dots s_k = t_0' t_1' \dots t_m'.$$

It follows that $c_T : M \to T$ is well defined.

It is clear that c_T is a monoid morphism.

Observe that for any $e \in E$, $c_T(e) = 1$. Regarding T as reduced left Ehresmann, we have that for any $a \in M$ with $c_T(a) = t$ say,

$$c_T(a^+) = 1 = t^+ = c_T(a)^+,$$

so that c_T is a $(2,1,0)$-morphism.

It is clear that c_T is onto. If $a = t_0 e_1 t_1 \dots e_n t_n$ and $b = s_0 f_1 s_1 \dots f_m s_m \in M$ are in T-normal form, then if $c_T(a) = c_T(b)$ we have that

$$a = t_0 e_1 t_1 \dots e_n t_n \,\sigma_E\, t_0 1 t_1 1 \dots 1 t_n = t_0 t_1 \dots t_n =$$

$$s_0 s_1 \dots s_m = s_0 1 s_1 1 \dots 1 s_m \,\sigma_E\, s_0 f_1 s_1 \dots f_m s_m = b$$

so that $\ker c_T \subseteq \sigma_E$. On the other hand, if $a \,\sigma_E\, b$, then $a = b$ (so that certainly $c_T(a) = c_T(b)$), or there exist $c_1, d_1, \dots, c_k, d_k \in M$, $g_1, h_1, \dots, g_k, h_k \in E$ such that

$$a = c_1 g_1 d_1,\ c_1 h_1 d_1 = c_2 g_2 f_2, \dots, c_k h_k d_k = b.$$

Now $c_T(e) = 1$ for any $e \in E$. Hence

$$c_T(a) = c_T(c_1 g_1 d_1) = c_T(c_1) c_T(g_1) c_T(d_1) = c_T(c_1) c_T(d_1)$$

$$= c_T(c_1) c_T(h_1) c_T(d_1) = c_T(c_1 h_1 d_1) = c_T(c_2 g_2 d_2) = \dots = c_T(b).$$

so that $\sigma_E \subseteq \ker c_T$. Hence $\sigma_E = \ker c_T$ as claimed. $\square$

Let M, T and a as in Proposition 3.4, the *T-content* of a is defined to be $c_T(a) = t_0 t_1 \dots t_n$. We warn the reader that we are *not* saying that T is a $(2,1,0)$-subalgebra of M.

With the above consideration of behaviour under generators, we feel that new notions of 'proper' are needed for left Ehresmann monoids. We propose the following.

Definition 3.5. Let M be a left Ehresmann monoid and let T be a submonoid of M such that $M = \langle E \cup T \rangle_{(2)}$. Then M is *T-proper* if whenever

$$a = t_0e_1t_1 \dots e_nt_n \text{ and } b = u_0e_1u_1 \dots e_nu_n$$

are in T-normal form, and we have for all $i \in \{0, \dots, n\}$:
(s) $t_i \, \sigma_E \, u_i$ and
(r) $(t_ie_{i+1} \dots e_nt_n)^+ = (u_ie_{i+1} \dots e_nu_n)^+$,
then $a = b$.

If M is T-proper, and S is a monoid isomorphic to T, then we may also say that M is *S-proper*; this convention will be useful in the case where S is a monoid of well known type.

Note (a) We could equally well rephrase (r) by saying that

$$t_ie_{i+1} \dots e_nt_n \, \widetilde{\mathcal{R}}_E \, u_ie_{i+1} \dots e_nu_n$$

and in view of (s), we could replace $\widetilde{\mathcal{R}}_E$ by $\widetilde{\mathcal{R}}_E \cap \sigma_E$.

Note (b) The reader might also wonder why we assume that the idempotents appearing in a and b are the same, and why our condition (r) applies only to right factors of a and b that begin with elements of T. Observe however that if

$$a = t_0e_1t_1 \dots e_nt_n \text{ and } b = u_0f_1u_1 \dots f_nu_n$$

are in T-normal form, and for $i \in \{1, \dots, n\}$

$$e_it_i \dots e_nt_n \, \widetilde{\mathcal{R}}_E \, f_iu_i \dots f_nu_n,$$

then as $e_i < (t_i \dots e_nt_n)^+$ and $f_i < (u_i \dots f_nu_n)^+$, we have that

$$e_i = e_i(t_i \dots e_nt_n)^+ \, \widetilde{\mathcal{R}}_E \, e_it_i \dots e_nt_n \, \widetilde{\mathcal{R}}_E \, f_iu_i \dots f_nu_n \, \widetilde{\mathcal{R}}_E \, f_i(u_i \dots f_nu_n)^+ = f_i,$$

whence $e_i = f_i$.

Lemma 3.6. *Let M be a left Ehresmann monoid and let T be a submonoid of M such that $M = \langle E \cup T \rangle_{(2)}$.*
(i) If $T \subseteq S \subseteq M$ and M is S-proper, then M is T-proper.
(ii) If M is left restriction, then M is M-proper if and only if it is proper.

Proof. (i) This is clear.
(ii) Clearly if M is proper, then in view of the fact that (r) applies to the case where $i = 0$, and Note (a) following Definition 3.5, it is M-proper.

Conversely, suppose that M is M-proper and $a, b \in M$ with $a \, (\widetilde{\mathcal{R}}_E \cap \sigma_E) \, b$, then $a = a_0$, $b = b_0$ are normal forms satisfying (s) and (r), so that $a = b$ and M is proper. □

Notice that in the next lemma, we do not need condition (r); uniqueness of T-normal forms gives, effectively, a stronger notion of T-proper.

Lemma 3.7. *Let M be a left Ehresmann monoid and let T be a submonoid of M such that $M = \langle E \cup T \rangle_{(2)}$. If M has uniqueness of T-normal forms, then M is T-proper.*

Proof. This follows immediately from Proposition 3.4. □

In the case where M/σ_E is right cancellative we can simplify the notion of T-proper to obtain one more reminiscent of that in the left ample case.

Lemma 3.8. *Let M be a left Ehresmann monoid and let T be a submonoid of M such that $M = \langle E \cup T\rangle_{(2)}$. Suppose that M/σ_E is right cancellative. Then M is T-proper if and only if for any $x, y \in T$ and $m \in M$ we have*

$$xm\,(\widetilde{\mathcal{R}}_E \cap \sigma_E)\,ym \text{ implies that } xm = ym.$$

Proof. Suppose first that for any $x, y \in T$ and $m \in M$ we have

$$xm\,(\widetilde{\mathcal{R}}_E \cap \sigma_E)\,ym \text{ implies that } xm = ym.$$

Let $a, b \in M$ be in T-normal form

$$a = t_0e_1t_1 \dots e_nt_n \text{ and } b = u_0e_1u_1 \dots e_nu_n$$

such that (s) and (r) hold.

If $n = 0$, then $a = t_0, b = u_0$ and

$$a = t_0\,1\,(\widetilde{\mathcal{R}}_E \cap \sigma_E)\,s_0\,1 = b,$$

so by assumption, $a = b$. Suppose now the result is true for elements in T-normal form of length $n-1$. Then certainly $t_1e_2 \dots e_nt_n = u_1e_2 \dots e_nu_n$, so that with $m = e_1t_1e_2 \dots e_nt_n = e_1u_1e_2 \dots e_nu_n$, we have that $a = t_0m, b = u_0m$ and $a\,(\widetilde{\mathcal{R}}_E \cap \sigma_E)\,b$. Hence $a = b$ and M is T-proper.

Conversely, suppose that M is T-proper and $xm\,(\widetilde{\mathcal{R}}_E \cap \sigma_E)\,ym$ where $x, y \in T$ and $m \in M$. Write m in T-normal form as $m = t_0e_1 \dots e_nt_n$, so that $xt_0t_1 \dots t_n\,\sigma_E\,yt_0t_1 \dots t_n$. Since M/σ_E is right cancellative, we obtain that $xt_0\,\sigma_E\,yt_0$. Now, $xm = (xt_0)e_1 \dots e_nt_n$ and $ym = (yt_0)e_1 \dots e_nt_n$ are elements in T-normal form satisfying (s) and (r), so that as M is T-proper, $xm = ym$ as required. □

We end this section with a technical result, which is crucial for the sequel [10]. In that article, we construct a T-proper left Ehresmann monoid $\mathcal{P}(T, Y)$ from a monoid (that is, a reduced left Ehresmann monoid) T acting by order-preserving maps on a semilattice Y. We show that $\mathcal{P}(T, Y)$ has uniqueness of T-normal forms and then call upon the result below for specialisations to other quasi-varieties.

Proposition 3.9. *Let M be a left Ehresmann monoid and let T be a submonoid of M such that $M = \langle E \cup T\rangle_{(2)}$. Suppose that M has uniqueness of T-normal forms. Let $a = t_0e_1 \dots e_nt_n \in M$ be in T-normal form.*

(i) The element a is idempotent if and only if

$$t_0 = t_0t_1 \dots t_nt_0$$

and for all $i \in \{1, \dots, n\}$ we have that

$$(t_it_{i+1} \dots t_nt_0e_1)^+ \leq e_i.$$

(ii) If T is unipotent, then $a \in E(M)$ if and only if $t_0t_1 \dots t_n = 1$ and for all $i \in \{1, \dots, n\}$ we have that

$$(t_i t_{i+1} \dots t_n t_0 e_1)^+ \leq e_i.$$

(iii) If T is right cancellative, then M is left E-adequate.

(iv) If T is right cancellative and has no invertible elements other than 1, then M is left adequate.

Proof. (i) Put $u = t_1e_2 \dots e_nt_n$ so that $a = t_0e_1u$.

Suppose the given conditions hold. Then

$$\begin{aligned}
a^2 &= (t_0e_1t_1 \dots t_{n-1}e_nt_n)(t_0e_1t_1 \dots t_{n-1}e_nt_n) && \\
&= (t_0e_1t_1 \dots t_{n-1})e_n(t_nt_0e_1)^+(t_nt_0e_1)u && \\
&= (t_0e_1t_1 \dots t_{n-1})(t_nt_0e_1)^+(t_nt_0e_1)u && \text{as } (t_nt_0e_1)^+ \leq e_n \\
&= t_0e_1t_1 \dots e_{n-1}t_{n-1}t_nt_0e_1u && \\
&= (t_0e_1t_1 \dots t_{n-2})e_{n-1}(t_{n-1}t_nt_0e_1)^+(t_{n-1}t_nt_0e_1)u && \\
&= (t_0e_1t_1 \dots t_{n-2})(t_{n-1}t_nt_0e_1)^+(t_{n-1}t_nt_0e_1)u && \text{as } (t_{n-1}t_nt_0e_1)^+ \leq e_{n-1} \\
&= (t_0e_1 \dots e_{n-2})(t_{n-2}t_{n-1}t_nt_0e_1)u && \\
&= \dots && \\
&= t_0t_1 \dots t_nt_0e_1u && \\
&= t_0e_1u && \text{as } t_0t_1 \dots t_nt_0 = t_0 \\
&= a &&
\end{aligned}$$

so that a is idempotent.

Conversely, suppose that a is idempotent. We are given that a is in T-normal form, and a^2 must be reducible to the same form. We have that

$$a^2 = t_0e_1t_1 \dots e_n(t_nt_0)e_1u.$$

Notice that the occurrence of e_1 preceding u can never be erased in the reduction algorithm given in Lemma 3.1 when applied to a^2, so the factor of u will remain untouched. In view of said algorithm, we must have that

$$e_n \geq (t_nt_0e_1u)^+ = (t_nt_0e_1u^+)^+ = (t_nt_0e_1)^+$$

so that

$$a^2 = t_0e_1t_1 \dots e_{n-1}t_{n-1}t_nt_0e_1u.$$

Continuing in this way we obtain that for all $i \in \{1, \dots, n\}$,

$$e_i \geq (t_it_{i+1} \dots t_nt_0e_1)^+,$$

and

$$a^2 = t_0t_1 \dots t_nt_0e_1t_1 \dots e_nt_n.$$

But this expression is in T-normal form, so by uniqueness of such we must have that $t_0t_1 \dots t_nt_0 = t_0$ as required.

(ii) Observe that if $t_0t_1 \dots t_nt_0 = t_0$ then $t_0t_1 \dots t_n \in E(T)$, so that if T is unipotent, $t_0t_1 \dots t_n = 1$. Clearly, if $t_0t_1 \dots t_n = 1$, then $t_0t_1 \dots t_nt_0 = t_0$.

(iii) Suppose now that T is right cancellative. We know that $a\,\widetilde{\mathcal{R}}_E\,a^+$, so that $a^+a = a$, it remains to show that for any $x, y \in M$, if $xa = ya$, then $xa^+ = ya^+$.

To this end, let

$$x = u_0 f_1 u_1 \dots f_m u_m \text{ and } y = v_0 g_1 v_1 \dots g_\ell v_\ell$$

be elements in T-normal form.

Consider the process of reducing xa to T-normal form. We have that

$$xa = u_0 f_1 u_1 \dots f_m (u_m t_0) e_1 t_1 \dots e_n t_n;$$

the first two steps of the reduction tell us to delete any f_i with

$$f_i (u_i f_{i+1} \dots f_m u_m a)^+ = (u_i f_{i+1} \dots f_m u_m a)^+$$

and for any f_i with

$$f_i (u_i f_{i+1} \dots f_m u_m a)^+ < (u_i f_{i+1} \dots f_m u_m a)^+,$$

replace f_i with $f_i' = f_i (u_i f_{i+1} \dots f_m u_m a)^+$. We thus obtain

$$\begin{aligned} xa &= c_0 h_1 c_1 \dots c_{r-1} h_r (c_r' u_m t_0) e_1 t_1 \dots e_n t_n \\ &= c_0 h_1 c_1 \dots c_{r-1} h_r (c_r' u_m) a \end{aligned} \tag{1}$$

where $c_r', c_i \in T, i \in \{0, \dots, r-1\}$ and $h_j \in E \setminus \{1\}$ with

$$h_j < (c_j h_{j+1} \dots c_{r-1} h_r c_r' u_m a)^+,\ 1 \le j \le r.$$

The T-normal form of xa is then obtained by deleting any (interior) c_i that are 1, for $i \in \{1, \dots, r\}$, where $c_r = c_r' u_m t_0$. We therefore assume, without loss of generality, that $c_1, \dots, c_{r-1} \in T \setminus \{1\}$.

Similarly,

$$\begin{aligned} ya &= d_0 k_1 d_1 \dots d_{s-1} k_s (d_s' v_\ell t_0) e_1 t_1 \dots e_n t_n \\ &= d_0 k_1 d_1 \dots d_{s-1} k_s (d_s' v_\ell) a \end{aligned} \tag{2}$$

where $d_s', d_i \in T, i \in \{0, \dots, s-1\}$ and $k_j \in E \setminus \{1\}$ with

$$k_j < (d_j k_{j+1} \dots d_{s-1} k_s d_s' v_\ell a)^+,\ 1 \le j \le s.$$

The T-normal form of ya is then obtained by deleting any (interior) d_i that are 1, for $i \in \{1, \dots, s\}$, where $d_s = d_s' v_\ell t_0$. We therefore assume, without loss of generality, that $d_1, \dots, d_{s-1} \in T \setminus \{1\}$. Thus (1) and (2) are the T-normal forms of xa and ya, up to $c_r' u_m t_0$ and $d_s' v_\ell t_0$, respectively, being 1.

Observe that as $(u_i f_{i+1} \dots f_m u_m a)^+ = (u_i f_{i+1} \dots f_m u_m a^+)^+$, and $(v_i g_{i+1} \dots g_\ell v_\ell a)^+ = (v_i g_{i+1} \dots g_\ell v_\ell a^+)^+$, the first two steps in the same reduction processes yield that

$$xa^+ = c_0 h_1 c_1 \dots c_{r-1} h_r c_r' u_m a^+$$

and

$$ya^+ = d_0 k_1 d_1 \dots d_{s-1} k_s d_s' v_\ell a^+.$$

Suppose now that $xa = ya$; recall that every element of M has a unique T-normal form.

First observe that $c'_r u_m t_0 = 1$ if and only if $d'_s v_\ell t_0 = 1$. If $n = 0$ this is clear. Suppose that $n > 0$ and assume, without loss of generality, that $c'_r u_m t_0 \neq 1$ and $d'_s v_\ell t_0 = 1$. Then we must have $s > 0$. Looking at the T-normal forms of xa and of ya, from right to left, in view of their uniqueness we obtain $e_1 = k_s e_1$, i.e. $e_1 \leq k_s$. But $k_s < (e_1 t_1 \dots e_n t_n)^+ = e_1(t_1 \dots e_n t_n)^+ < e_1$ and so $k_s < e_1$, a contradiction.

If $c'_r u_m t_0 = d'_s v_\ell t_0 = 1$, then $c'_r u_m = d'_s v_\ell$, since T is right cancellative, and equalities (1) and (2) give $c_0 h_1 c_1 \dots h_{r-1} c_{r-1} = d_0 k_1 d_1 \dots k_{s-1} d_{s-1}$ and $h_r e_1 = k_s e_1$. But $h_r e_1 = h_r$ and $k_s e_1 = k_s$. Hence $xa^+ = ya^+$.

If $c'_r u_m t_0$ and $d'_s v_\ell t_0$ are not 1, then, by the uniqueness of the T-normal forms (1) and (2), $c'_r u_m t_0 = d'_s v_\ell t_0$ and $c_0 h_1 c_1 \dots h_{r-1} c_{r-1} h_r = d_0 k_1 d_1 \dots k_{s-1} d_{s-1} k_s$. Again $c'_r u_m = d'_s v_\ell$ and $xa^+ = ya^+$ follows.

(iv) Suppose now that T is right cancellative and has no units other than 1. In view of (iii) we need only show that $E(M) = E$. Let $b = s_0 f_1 s_1 \dots f_k s_k$ be in T-normal form and assume that b is idempotent. In view of (ii) we have that $s_0 s_1 \dots s_k = 1$, whence $s_i = 1$ for all $i \in \{0, \dots, k\}$. It follows that $k = 0$ and $b = 1$, or $k = 1$ and $b = f_1$; in either case, $b \in E$ as required. □

4. A covering theorem

The aim of this section is to give a covering result for monoids in classes in our list ($\mathscr{L}$). A refinement of this result in the one-sided case will be given in [10].

Theorem 4.1. *Let M be a left Ehresmann monoid with $M = \langle X \rangle_{(2,1,0)}$ for some $X \subseteq M$. Then M has an X^*-proper cover $\widehat{M}$. Moreover, if M lies in any of the classes of left Ehresmann monoids in the list ($\mathscr{L}$), then so does $\widehat{M}$.*

Proof. For convenience we denote the inclusion map of X in M by ι, so that $M = \langle X\iota \rangle_{(2,1,0)}$. We remark that ι lifts to a (unique) morphism $X^* \to M$, also denoted by ι.

Certainly X^* is reduced left Ehresmann, so we may consider X^* and $M \times X^*$ as $(2,1,0)$-algebras; as such, $M \times X^*$ is left Ehresmann with distinguished semilattice

$$E' = E \times \{1\},$$

and for any $(a, w) \in M \times X^*$ we have that

$$(a, w)^+ = (a^+, w^+) = (a^+, 1) \in E'.$$

Suppose that $a = (m, u), b = (n, v) \in M \times X^*$ are such that $a \, \sigma_{E'} \, b$. Then $a = b$ or there exist

$$(c_1, c'_1), (d_1, d'_1), \dots, (c_\ell, c'_\ell), (d_\ell, d'_\ell) \in M \times X^*$$

and

$$(e_1, 1), (f_1, 1), \dots, (e_\ell, 1), (f_\ell, 1) \in E'$$

such that

$$\begin{array}{rcl}(m,u)=a & = & (c_1,c_1')(e_1,1)(d_1,d_1'),\\ (c_1,c_1')(f_1,1)(d_1,d_1') & = & (c_2,c_2')(e_2,1)(d_2,d_2'),\\ & \vdots & \\ (c_\ell,c_\ell')(f_\ell,1)(d_\ell,d_\ell') & = & b=(n,v).\end{array}$$

In either case we obtain that $u=v$.

We put

$$X'=\{(x\iota,x):x\in X\}\subseteq M\times X^*$$

and let $\widehat{M}=\langle X'\rangle_{(2,1,0)}$, so that $\widehat{M}$ is left Ehresmann. For $\overline{x}=(x_1,\dots,x_k)$ with $x_i\in X$, we put $\overline{x}\iota=(x_1\iota,\dots,x_k\iota)$. Notice that for a $(2,1,0)$-term $t(y_1,\dots,y_k)$, certainly $t\overline{x}$ and $t\overline{x}\iota$ are elements of X^* and M respectively, but we may not have that $(t\overline{x})\iota=t\overline{x}\iota$. Clearly $\widehat{M}$ is the set of elements of $M\times X^*$ of the form $(t\overline{x}\iota,t\overline{x})$, where t is a $(2,1,0)$-term and $\overline{x}=(x_1,\dots,x_k)$ for $x_1,\dots,x_k\in X$ and $k\geq 0$.

For any $e\in E$ with $e=t\overline{x}\iota$ where $t=t(y_1,\dots,y_k)$, we have that $e=s\overline{x}\iota$ where $s=s(y_1,\dots,y_k)=t(y_1,\dots,y_k)^+$. It follows that $(e,1)=(s\overline{x}\iota,s\overline{x})$ and so $\widehat{M}$ has distinguished semilattice E'.

Let $S=\langle X'\rangle_{(2,0)}=\{(w\iota,w):w\in X^*\}$. Clearly the projection maps $p_1:\widehat{M}\to M$ and $p_2:\widehat{M}\to X^*$ are both onto, $p_1|_{E'}$ is an isomorphism from E' onto E, and $p_2|_S$ is an isomorphism from S onto X^*.

We claim that $\widehat{M}$ is S-proper. From Corollary 1.12, $\widehat{M}=\langle E'\cup S\rangle_{(2)}$. Suppose that

$$a=(u_0\iota,u_0)(e_1,1)(u_1\iota,u_1)\dots(e_n,1)(u_n\iota,u_n)$$

and

$$b=(v_0\iota,v_0)(e_1,1)(v_1\iota,v_1)\dots(e_n,1)(v_n\iota,v_n)$$

are elements of $\widehat{M}$, where $u_0,v_0,\dots,u_n,v_n\in X^*$ and $e_1,\dots,e_n\in E$, and suppose that

$$(u_i\iota,u_i)\,\sigma_{E'}\,(v_i\iota,v_i)$$

in $\widehat{M}$, for $0\leq i\leq n$. Certainly then

$$(u_i\iota,u_i)\,\sigma_{E'}\,(v_i\iota,v_i)$$

in $M\times X^*$ so that by a comment above, $u_i=v_i$ whence $u_i\iota=v_i\iota$ for $0\leq i\leq n$. Consequently, $a=b$ and thus $\widehat{M}$ is S-proper.

Since X^* is left ample, it lies in all of the one-sided classes in $(\mathscr{L})$; since all of these classes are quasi-varieties (if not varieties), if M lies in any of these classes, then so does $\widehat{M}$, as $\widehat{M}$ is a subalgebra of a direct product of two members of the class. $\square$

The same approach gives the following; here an Ehresmann monoid is *proper* if it is proper both as a left and as a right Ehresmann monoid.

Theorem 4.2. *Let M be an Ehresmann monoid with $M = \langle X\rangle_{(2,1,1,0)}$ for some $X \subseteq M$. Then M has an X^*-proper cover $\widehat{M}$. Moreover, if M lies in any of the classes of Ehresmann monoids in the list ($\mathscr{L}$), then so does $\widehat{M}$.*

5. General results on free objects

Without, at this stage, determining their structure completely, we can make a number of statements concerning free algebras in the classes ($\mathscr{L}$) listed in Section 1.

Theorem 5.1. *Let X be a non-empty set and let F_X be the free $\mathcal{C}$-object on X, where $\mathcal{C}$ is any of the quasi-varieties in our list ($\mathscr{L}$). Denote the inclusion of X into F_X by ι, so that*

$$F_X = \langle X\iota\rangle_{(2,1,0)}$$

or in the two-sided cases,

$$F_X = \langle X\iota\rangle_{(2,1,1,0)};$$

put $T = \langle X\iota\rangle_{(2,0)}$. For any $a \in F_X$ with

$$a = t_0e_1t_1 \dots e_nt_n$$

where $t_i \in T$ and $e_i \in E$, we define the content $c(a)$ *of a by*

$$c(a) = t_0t_1 \dots t_n.$$

Then

(i) $T \cong X^*$;

(ii) content is a well defined map; regarding T as a reduced left Ehresmann monoid, $c : F_X \to T$ is a $(2,1,0)$-morphism; moreover,

$$c(a) = c(b) \text{ if and only if } a\,\sigma_E\,b;$$

(iii) $F_X/\sigma_E \cong X^*$.

Proof. We give the argument for the one-sided classes; a similar proof holds in the two-sided cases.

Let $j : X \to X^*$ be the canonical embedding. Then there exists a $(2,1,0)$-morphism $\theta : F_X \to X^*$ such that

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "j"'] & F_X \arrow[d, "\theta"] \\ & X^* \end{tikzcd}$$

commutes, since X^* belongs to any such class.

On the other hand, there exists a $(2,0)$-morphism $\psi : X^* \to F_X$ such that

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\iota"'] & X^* \arrow[d, "\psi"] \\ & F_X \end{tikzcd}$$

commutes, since F_X is a monoid.

Now $xj\psi\theta = x\iota\theta = xj$ and $xjI_{X^*} = xj$, so both the diagrams

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "j"'] & X^* \arrow[d, "\psi\theta"] \\ & X^* \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "j"'] & X^* \arrow[d, "I_{X^*}"] \\ & X^* \end{tikzcd}$$

commute. Certainly $\psi\theta$ is a $(2,0)$-morphism, so that $\psi\theta = I_{X^*}$. Consequently, ψ is one-one and θ is onto. Further,

$$\text{im } \psi = \langle Xj\rangle_{(2,0)}\psi = \langle Xj\psi\rangle_{(2,0)} = \langle X\iota\rangle_{(2,0)} = T$$

and so $T \cong X^*$.

Notice now that for any $x \in X$, so that $x\iota \in T$, we have that

$$x\iota\theta\psi = xj\psi = x\iota,$$

and as $\theta\psi$ is again a $(2,0)$-morphism, we have that $\theta\psi|_T = I_T$.

Suppose now that $a \in F_X$; by Corollary 1.12, a can be written as $a = t_0e_1t_1 \dots e_nt_n$ where $e_i \in E$ and $t_i \in T$. Since $e_i\theta\psi = 1\psi = 1$ for each i, we conclude that

$$a\theta\psi = t_0\theta\psi t_1\theta\psi \dots t_n\theta\psi = t_0 \dots t_n = c(a).$$

It follows that content is a well-defined map, indeed $c = \theta\psi$. Now, regarding T as reduced, for any $m \in F_X$,

$$c(m^+) = (m^+)\theta\psi = 1 = c(m)^+,$$

so that c is a $(2,1,0)$-morphism.

Since ψ is one-one,

$$\ker\, c = \ker\, \theta\psi = \ker\, \theta.$$

For any $e \in E$ we have that $e\theta = 1$, so that $E \times E \subseteq \ker\, \theta$, whence $\sigma_E \subseteq \ker\, \theta$.

Still with a as above, suppose that $b = u_0f_1 \dots f_mu_m \in F_X$ where $u_i \in T$ and $f_i \in E$. Then if $a \ker\, \theta\, b$, we have that

$$a\, \sigma_E\, c(a) = c(b)\, \sigma_E\, b.$$

Hence $\sigma_E = \ker\, \theta$ so that

$$F_X/\sigma_E = F_X/\ker\, \theta \cong \text{ im } \theta = X^*.$$

$\square$

Theorem 5.2. *Let F_X be the free object on X in any class $\mathcal{C}$ in our list ($\mathscr{L}$). Then F_X is X^*-proper (regarded as a left Ehresmann monoid).*

Proof. We take $\mathcal{C}$ to be the class of left Ehresmann monoids; proofs for the other classes are similar.

Following the proof of Theorem 4.1, taking $M = F_X$, we know that M has an $(X')^*$-proper cover $\widehat{M}$, where $|X'| = |X|$. Let $j : X \to X'$ be the bijection from X to X' that takes x to $(x\iota, x)$. Then the covering morphism $p_1 : \widehat{M} \to F_X$ has the property that $xjp_1 = x\iota$. Since F_X is free on X, there is a $(2,1,0)$-morphism $\psi : F_X \to \widehat{M}$ such that $\iota\psi = j$. It follows that for any $x \in X$ we have that

$$(x\iota)\psi p_1 = x\iota \text{ and } (xj)p_1\psi = xj,$$

so that as F_X and $\widehat{M}$ are generated by $X\iota$ and Xj respectively,

$$\psi p_1 = I_{F_X} \text{ and } p_1\psi = I_{\widehat{M}},$$

that is, ψ and p_1 are mutually inverse isomorphisms. Observe that

$$\langle X' \rangle_{(2,0)} p_1 = \langle X' p_1 \rangle_{(2,0)} = \langle Xjp_1 \rangle_{(2,0)} = \langle X\iota \rangle_{(2,0)} \cong X^*,$$

the last equality following from Theorem 5.1, so that F_X is X^*-proper. □

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