SEMIGROUPS OF LEFT QUOTIENTS - THE LAYERED APPROACH

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ABSTRACT. A subsemigroup S of a semigroup Q is a *left order* in Q and Q is a *semigroup of left quotients* of S if every element of Q can be expressed as $a^\sharp b$ where $a, b \in S$ and if, in addition, every element of S that is *square cancellable* lies in a subgroup of Q. Here $a^\sharp$ denotes the inverse of a in a subgroup of Q. We say that a left order S is *straight* in Q if in the above definition we can insist that $a\,\mathcal{R}\,b$ in Q. A complete characterisation of straight left orders in terms of *embeddable $*$-pairs* is available. In this paper we adopt a different approach, based on *partial order decompositions* of semigroups. Such decompositions include semilattice decompositions and decompositions of a semigroup into principal factors or principal $*$-factors. We determine when a semigroup that can be decomposed into straight left orders is itself a straight left order. This technique gives a unified approach to obtaining many of the early results on characterisations of straight left orders.

1. INTRODUCTION

We assume the reader has a familiarity with Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ and the preorders $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$, as may be found in [1]. This paper makes frequent use of the 'starred' generalisations $\mathcal{L}^*, \leq_{\mathcal{L}^*}$ etc. of the relations $\mathcal{L}, \leq_{\mathcal{L}}$ etc. Recall that for elements a, b of a semigroup S, $a \leq_{\mathcal{L}^*} b$ if and only if for all $x, y \in S^1$,

$$bx = by \text{ implies that } ax = ay.$$

Clearly $\leq_{\mathcal{L}^*}$ is a right compatible preorder containing $\leq_{\mathcal{L}}$; its associated equivalence $\mathcal{L}^*$ is thus a right congruence containing $\mathcal{L}$. The relations $\leq_{\mathcal{R}^*}$ and $\mathcal{R}^*$ are defined dually; the analogy with notation for Green's relations is continued by putting

$$\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^* \text{ and } \mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*.$$

It is worth remarking that it is not necessarily true that $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$. The best reference for these relations is [2]. For further detail concerning left orders we refer the reader to [3].

The notion of a left order in a semigroup was introduced by Fountain and Petrich in [4] and has been widely used since. Fountain and Petrich describe those semigroups that are left orders in completely 0-simple semigroups. We remark here that if S is a left order in Q, $\mathcal{H}$ is a congruence on Q and Q

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is regular, then S is perforce straight [5]. Subsequently, a number of papers have appeared characterising semigroups that are (straight) left orders in semigroups in various classes, for example [6, 7]. More recently, Theorem 4.4 of [3] gives a characterisation of straight left orders in arbitrary semigroups; we recap briefly below the approach of [3].

An ordered pair $(\leq_\ell, \leq_r)$ of preorders on a semigroup S is a $*$-*pair* if $\leq_l$ is right compatible with multiplication, $\leq_r$ is left compatible, $\leq_l \subseteq \leq_{\mathcal{L}^*}$ and $\leq_r \subseteq \leq_{\mathcal{R}^*}$. Clearly $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ is a $*$-pair for any semigroup S.

Lemma 1.1. *Let S be a subsemigroup of Q. Then*

$$(\leq_{\mathcal{L}^Q} \cap (S \times S), \leq_{\mathcal{R}^Q} \cap (S \times S))$$

is a $*$-pair for S, the $*$-pair for S induced by Q.

In particular the above applies when S is a left order in Q.

For a $*$-pair $(\leq_\ell, \leq_r)$, we denote by $\mathcal{L}'$ and $\mathcal{R}'$ the equivalence relations associated with $\leq_l$ and $\leq_r$ respectively. Notice that $\mathcal{L}'$ is a right congruence contained in $\mathcal{L}^*$ and $\mathcal{R}'$ is a left congruence contained in $\mathcal{R}^*$; $\mathcal{H}'$ denotes the intersection of $\mathcal{L}'$ and $\mathcal{R}'$ and we put

$$\mathcal{G}(S) = \{a \in S : a\, \mathcal{H}'\, a^2\}.$$

Notice that $\mathcal{G}(S) \subseteq \mathcal{S}(S)$, where

$$\mathcal{S}(S) = \{a \in S : a\, \mathcal{H}^*\, a^2\}$$

is the set of *square cancellable* elements of S. We put $\mathcal{D}' = \mathcal{L}' \vee \mathcal{R}'$ and define the relation $\leq_j$ by the rule that for any $a, b \in S$,

$$a \leq_j b \text{ if and only if } a\, \mathcal{D}'\, ubv \text{ for some } u, v \in S.$$

Finally we say that for any $a, b \in S$

$$a\, \mathcal{J}'\, b \text{ if and only if } a \leq_j b \leq_j a,$$

so that if $\leq_j$ is a preorder, then $\mathcal{J}'$ is the associated equivalence relation.

In the following definition, the slightly eccentric notation is that inherited from [3].

Definition 1.2. [3] *Let $(\leq_\ell, \leq_r)$ be a $*$-pair for S. Then $(\leq_\ell, \leq_r)$ is an* embeddable $*$-pair *if the following conditions and the duals (Eii)(r), (Ev)(r), (Evi)(r) and (Evii)(r) of (Eii)(l), (Ev)(l), (Evi)(l) and (Evii)(l) respectively hold.*

(Ei) $\mathcal{L}' \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{L}'$.

(Eii)(l) For all $b, c \in S$, $b \leq_l c$ if and only if $b\, \mathcal{L}'\, dc$ for some $d \in S$.

(Eiii) Every $\mathcal{L}'$-class and every $\mathcal{R}'$-class contains an element from $\mathcal{G}(S)$.

(Ev)(l) For all $a \in \mathcal{G}(S)$ and $b \in S$, if $b \leq_l a$, then $ba\, \mathcal{R}'\, b$.

(Evi)(l) For all $a \in \mathcal{G}(S)$ and $b, c \in S$, if $b, c \leq_l a$ and $ba = ca$, then $b = c$.

(Evii)(l) For all $a \in \mathcal{G}(S)$ and $b, c \in S$, if $b, c \leq_l a$ and $ba\, \mathcal{L}'\, ca$, then $b\, \mathcal{L}'\, c$.

Embeddable $*$-pairs are crucial in determining the existence of semigroups of straight left quotients, as the next result shows. We recall that a subset T of a semigroup S is *right reversible*, or *satisfies the left Ore condition*, if for any $a, b \in T$ there exist $c, d \in T$ such that $ca = db$.

Theorem 1.3. [3] *Let S be a semigroup having an embeddable $*$-pair $(\leq_\ell, \leq_r)$. Then S is a straight left order in a semigroup Q inducing $(\leq_\ell, \leq_r)$ if and only if $(\leq_\ell, \leq_r)$ satisfies the following conditions:*
(Gi) $\mathcal{S}(S) = \mathcal{G}(S)$;
and
(Gii) If $a \in \mathcal{S}(S)$ then H'_a is right reversible.

In some sense Theorem 1.3 is the final answer in the characterisation of straight left orders. However, earlier descriptions of straight left orders in semigroups from certain classes, for example in Clifford semigroups, concentrated on decomposing a semigroup S into subsemigroups that are straight left orders, and then glueing together the semigroups of left quotients of the subsemigroups of S to form a semigroup of left quotients of the entire semigroup S. For example, Theorem 3.1 of [6] states that S is a left order in a semilattice Y of groups $G_\alpha, \alpha \in Y$, if and only if S is a semilattice Y of right reversible, cancellative semigroups $S_\alpha, \alpha \in Y$. The latter are known by the theorem of Ore and Dubreil to be left orders in groups [8]. A similar approach is taken to straight left orders in completely simple semigroups in [7]. The approach of [6, 7] and other early papers was thus to determine the global structure of a semigroup from the local structure. Other results, for example Theorem 5.12 of [9], do not use this approach explicitly, but nevertheless, the semigroups in question have a relative ideal series for which the Rees quotients are left orders in completely 0-simple semigroups.

The aim of this paper is to investigate the relation between a semigroup being a straight left order, and its decompositions into subsemigroups or Rees quotients that are straight left orders. To this end we introduce the idea of a *partial order decomposition* of a semigroup, that allows us to consider at the same time semilattice decompositions and decompositions into Rees quotients. This technique enables us to extend the main theorem of [7] to arbitrary semilattices of orders, find an alternative proof of Theorem 5.12 of [9], and a new characterisation of straight left orders in completely semisimple semigroups.

The structure of the paper is as follows. In Section 2 we prove a number of minor technical results needed for the remainder of the paper. In Section 3 we introduce the notion of a decomposition of a semigroup S as a partial order P of subsets $S_\alpha, \alpha \in P$. Associated with each $\alpha \in P$ is a Rees quotient $\overline{S_\alpha}$.

Section 4 looks at the situation where S is a straight left order in Q and Q is a partial order P of subsets $Q_\alpha, \alpha \in P$. From the definition of partial order of subsets we have that S is a partial order P of subsets $S_\alpha = Q_\alpha \cap S$. We consider under what circumstances the associated Rees quotients $\overline{S_\alpha}$ are straight left orders. We investigate the relation between P being associated with the $\leq_{\mathcal{J}}$ order on Q and P being associated with the $\leq_{\mathcal{J}^*}$ order on S. Section 5 concentrates on the converse to the scenario in Section 4, namely, given a semigroup S that is a partial order P of subsets $S_\alpha, \alpha \in P$, such that each $\overline{S_\alpha}$ is a straight left order, when is S a straight left order? Theorem 5.4 gives necessary and sufficient conditions for this to be the case.

The remaining sections of the paper are specialisations of Theorem 5.4. In Section 6 we consider semilattices of straight left orders. In Section 7 we look at straight left orders that are *fully stratified*, that is, the $*$-pair induced by the semigroup of left quotients is $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$. In the case where S is *idempotent connected abundant*, we obtain an appropriately clean version of the results of Sections 4 and 5. This enables us to show that the semigroup of endomorphisms of finite rank of a stable basis algebra is a straight left order. The final section gives a new characterisation of straight left orders in completely semisimple semigroups.

2. Preliminaries

We make frequent use of the fact that if S is a straight (weak) left order in Q, then S has non-empty intersection with every $\mathcal{H}$-class of Q. For if $q \in Q$, then $q = a^\sharp b$ for some $a, b \in S$ with $a \,\mathcal{R}\, b$ in Q; it is easy to see that $q \,\mathcal{H}\, b$ follows.

We begin with a technical result on zeroes.

Lemma 2.1. *Let S be a straight left order in Q. Then*

(A) S has a zero if and only if Q has a zero;

(B) if S has a zero, then S has no zero divisors if and only if Q has no zero divisors.

Proof. (A) If Q has a zero 0, then as $0 \,\mathcal{H}\, x$ for some $x \in S$, we must have that $0 = x \in S$. The converse follows from Proposition 2.8 of [10].

(B) Suppose that S and Q have a zero 0 and that S has no zero divisors. Let $p, q \in Q \setminus \{0\}$ with $p \,\mathcal{H}\, a, q \,\mathcal{H}\, b$ and $a, b \in S \setminus \{0\}$. Then

$$pq \,\mathcal{R}\, pb \,\mathcal{L}\, ab$$

so that as $ab \neq 0$ we have $pq \neq 0$. Hence Q has no zero divisors. The converse is clear. $\square$

We recall that for a semigroup S, S^0 denotes S with a zero adjoined. The following observation will be useful.

Lemma 2.2. *Let S be a semigroup. Then S is a straight left order in Q if and only if S^0 is a straight left order in Q^0.*

Next we recall from [2] that a semigroup S is *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contains an idempotent. In a non-regular abundant semigroup S, the relationship between the idempotents in R_a^* and those in L_a^* for a given $a \in S$ need not be as strong as it is in a regular semigroup. In view of this, El-Qallali and Fountain [11] introduced the notion of an *idempotent connected*, or IC, abundant semigroup. An abundant semigroup is IC if for each $a \in S$ and each idempotent $e \leq_{\mathcal{L}^*} a$ ($e \leq_{\mathcal{R}^*} a$), there is an element $b \in S$ with $ae = ba$ ($ea = ab$) [12].

Lemma 2.3. *Let S be an IC abundant semigroup. Then for any $a, b \in S$*

$$a \leq_{\mathcal{L}^*} b \text{ if and only if } a \,\mathcal{L}^*\, cb \text{ some } c \in S,$$

and dually for $\leq_{\mathcal{R}^*}$.

Proof. If $a \,\mathcal{L}^*\, cb$, then it follows from the definitions that $a \leq_{\mathcal{L}^*} b$.

Conversely, suppose that $a \leq_{\mathcal{L}^*} b$. Now $a \,\mathcal{L}^*\, e$ and $b \,\mathcal{L}^*\, f$ for some $e, f \in E(S)$. Hence $e \leq_{\mathcal{L}^*} f$, whence $e = ef$ and

$$a \,\mathcal{L}^*\, e \,\mathcal{L}\, fe \,\mathcal{L}^*\, be.$$

But S is IC abundant, so that $be = cb$ for some $c \in S$. □

Finally in this section we consider the restrictions of $\mathcal{D}, \mathcal{J}$ and $\leq_{\mathcal{J}}$ on a semigroup of left quotients to a left order.

Lemma 2.4. *Let S be a straight left order in Q inducing $(\leq_\ell, \leq_r)$. Then*

$$\mathcal{D}' = \mathcal{L}' \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{L}',$$

consequently,

$$\mathcal{D}' = \mathcal{D} \cap (S \times S).$$

Further,

$$\leq_j \,=\, \leq_{\mathcal{J}} \cap\, (S \times S) \text{ and } \mathcal{J}' = \mathcal{J} \cap (S \times S).$$

Proof. If $a, b \in S$ and $a \,\mathcal{L}'\, c \,\mathcal{R}'\, b$ for some $c \in S$, then as Q induces $(\leq_\ell, \leq_r)$ we have that $a \,\mathcal{D}\, b$ in Q. Now S intersects every $\mathcal{H}$-class of Q, whence $a \,\mathcal{R}'\, d \,\mathcal{L}'\, b$ for some $d \in S$. Together with the dual argument we have that $\mathcal{L}'$ and $\mathcal{R}'$ commute, so that

$$\mathcal{D}' = \mathcal{L}' \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{L}'$$

as required. Clearly then $\mathcal{D}' = \mathcal{D} \cap (S \times S)$.

Suppose now that $a, b \in S$ and $a \leq_j b$. By definition, we have $a \,\mathcal{D}'\, ubv$ for some $u, v \in S$. From the previous paragraph and the fact that $\mathcal{D} \subseteq \mathcal{J}$,

$$a \,\mathcal{J}\, ubv \leq_{\mathcal{J}} b$$

in Q.

Conversely, let $a, b \in S$ and suppose that $a \leq_{\mathcal{J}} b$ in Q. Then $a = pbq$ for some $p, q \in Q$. Choose $u, v \in S$ with $p \,\mathcal{H}\, u$ and $q \,\mathcal{H}\, v$ so that

$$a = pbq \,\mathcal{R}\, pbv \,\mathcal{L}\, ubv$$

and so $a \,\mathcal{D}'\, ubv$ in S, giving that $a \leq_j b$. □

It follows from Lemma 2.4 that if S is a straight left order in Q inducing $(\leq_\ell, \leq_r)$, then $\mathcal{D}'$ is an equivalence relation and $\leq_j$ is a preorder with associated equivalence relation $\mathcal{J}'$.

3. Partial orders of semigroups

Let P be a partially ordered set and let S be a semigroup. We say that S *is a partial order P of subsets S_α* if S is the disjoint union of non-empty subsets S_α, $\alpha \in P$ such that for any $\alpha, \beta \in P$,

$$S_\alpha S_\beta \subseteq \bigcup \{S_\gamma : \gamma \leq \alpha \text{ and } \gamma \leq \beta\}.$$

Equivalently, there is a preorder $\preceq$ on S satisfying the condition that for all $a, b \in S$, $ab \preceq a$ and $ab \preceq b$, such that the classes of the associated equivalence relation on S are indexed by P.

Our definition is, of course, motivated by that of a semilattice of semigroups. If P is a semilattice and S is a semilattice P of subsemigroups $S_\alpha, \alpha \in P$, then certainly S is a partial order P of the subsets $S_\alpha, \alpha \in P$. However, there are certainly natural decompositions of semigroups as partial orders of subsets that are not necesarily associated with a semilattice decomposition.

Example 3.1. *The $\mathcal{J}$-class decomposition.*

Proof. For any elements a, b of a semigroup S we have that $ab \leq_{\mathcal{J}} a$ and $ab \leq_{\mathcal{J}} b$. $\square$

Example 3.2. *The $\mathcal{J}^*$-class decomposition.*

Proof. We recall from [2] that for an element a of a semigroup S, there is a smallest $*$-ideal $J^*(a)$ containing a, where an ideal I of S is a $*$-ideal if it is a union of $\mathcal{R}^*$-classes and of $\mathcal{L}^*$-classes.

The relation $\leq_{\mathcal{J}^*}$ is defined on S by the rule that $a \leq_{\mathcal{J}^*} b$ if and only if $J^*(a) \subseteq J^*(b)$. It is easy to see that $\leq_{\mathcal{J}^*}$ is a preorder, with associated equivalence $\mathcal{J}^*$. It follows from (3) of Lemma 1.7 of [2], that for any $a, b \in S$ we have $ab \leq_{\mathcal{J}^*} a$ and $ab \leq_{\mathcal{J}^*} b$. $\square$

We remark that in a regular semigroup, every ideal is a $*$-ideal. Consequently, $\leq_{\mathcal{J}} = \leq_{\mathcal{J}^*}$ and $\mathcal{J} = \mathcal{J}^*$. Example 3.1 is thus a special case of Example 3.2.

Suppose now that S is a semigroup and a partial order P of subsets $S_\alpha, \alpha \in P$. For each $\alpha \in P$ we associate two ideals of S,

$$I_\alpha^S = \cup\{S_\beta : \beta < \alpha\} \text{ and } J_\alpha^S = \cup\{S_\beta : \beta \leq \alpha\}$$

(we allow the possibility that $I_\alpha^S = \emptyset$). We can thus form the Rees quotient

$$\overline{S_\alpha} = J_\alpha^S / I_\alpha^S = S_\alpha \cup \{I_\alpha^S\},$$

or, in the case where $I_\alpha^S = \emptyset$, we put $\overline{S_\alpha} = S_\alpha$. We can think of these Rees quotients as 'slices' describing the local structure of S, and of S as being built up as 'layers' of these slices. In the next sections we consider the relationship between S being a straight left order and the Rees quotients $\overline{S_\alpha}, \alpha \in P$, being straight left orders.

4. Slicing

Throughout this section, S denotes a straight left order in a semigroup Q, where Q is a partial order P of subsets $Q_\alpha, \alpha \in P$. For $\alpha \in P$ put

$$S_\alpha = Q_\alpha \cap S.$$

Notice that as $\overline{Q_\alpha}$ is a quotient of ideals of Q, the set of non-zero elements Q_α is a union of $\mathcal{J}$-classes of Q. Since S intersects every $\mathcal{H}$-class of Q, certainly each $S_\alpha \neq \emptyset$. Clearly S is then a partial order P of subsets $S_\alpha, \alpha \in P$.

The semigroup Q is regular, consequently, for non-zero elements p, q of $\overline{Q_\alpha}$,

$$q \leq_{\mathcal{L}} p \text{ in } \overline{Q_\alpha} \text{ if and only if } q \leq_{\mathcal{L}} p \text{ in } Q$$

and

$$q \leq_{\mathcal{R}} p \text{ in } \overline{Q_\alpha} \text{ if and only if } q \leq_{\mathcal{R}} p \text{ in } Q.$$

Let $(\leq_\ell, \leq_r)$ denote the $*$-pair on S induced by Q. Since each Q_α is a union of $\mathcal{J}$-classes, we must have that S_α is a union of $\mathcal{L}'$-classes and of $\mathcal{R}'$-classes.

We remark that for each $\alpha \in P$, $\overline{S_\alpha}$ may be regarded as a subsemigroup of $\overline{Q_\alpha}$. Clearly $I^Q_\alpha = \emptyset$ if and only if $I^S_\alpha = \emptyset$. In the case where $I^Q_\alpha \neq \emptyset$, we identify the zeroes $\{I^Q_\alpha\}$ and $\{I^S_\alpha\}$ of $\overline{Q_\alpha}$ and $\overline{S_\alpha}$.

Lemma 4.1. *For each $\alpha \in P$, $\overline{S_\alpha}$ is a straight weak left order in $\overline{Q_\alpha}$.*

Proof. If $q \in \overline{Q_\alpha}$, then either $q = 0 = 0^\sharp 0$, or $q \in Q_\alpha$ and $q = a^\sharp b$ where $a, b \in S$ and $a \,\mathcal{R}\, b$ in Q. Since $q \,\mathcal{H}\, b$ in Q we must have that $a, b \in S_\alpha$ and $a \,\mathcal{R}\, b$ in $\overline{Q_\alpha}$. $\square$

From Lemma 4.1 we certainly have the following.

Corollary 4.2. *For any $\alpha \in P$, $\overline{S_\alpha}$ is a straight left order in $\overline{Q_\alpha}$ if and only if for any $a \in S_\alpha$,*

$$a \,\mathcal{H}^*\, a^2 \textit{ in } \overline{S_\alpha} \textit{ implies that } a \,\mathcal{H}^*\, a^2 \textit{ in } S.$$

In general we do not have enough control over the square cancellable elements, for the condition in Corollary 4.2 to hold. However, in two cases of note we have the required result.

Proposition 4.3. *Let $\alpha \in P$. If Q is completely regular or S is abundant and J^S_α and I^S_α are $*$-ideals, then $\overline{S_\alpha}$ is a straight left order in $\overline{Q_\alpha}$.*

Proof. We need only consider (ii). In this case the condition required in Corollary 4.2 follows from Lemma 2.5 of [9]. $\square$

The canonical decomposition of a semigroup of left quotients Q as a partial order P of subsets $Q_\alpha, \alpha \in P$, arises from the preorder $\leq_{\mathcal{J}}$, as in Example 3.1. In this case the Rees quotients $\overline{Q_\alpha}$ are, of course, the principal factors of Q, and we abuse terminology by saying 'P is the $\leq_{\mathcal{J}}$-order on Q'. The canonical decomposition of a left order S as a partial order P of subsets $S_\alpha, \alpha \in P$, arises from the preorder $\leq_{\mathcal{J}^*}$, as in Example 3.2; in this case we say 'P is the $\leq_{\mathcal{J}^*}$-order on S'. If P arises from the preorder $\leq_j$ we say 'P is the $\leq_j$-order'.

The relations $\leq_{\mathcal{J}}$ on Q and $\leq_{\mathcal{J}^*}$ and $\leq_j$ on S are, of course, related, and we see examples of this in the following proposition and in Sections 7 and 8.

Proposition 4.4. *Let S be a straight left order in Q where Q induces $(\leq_\ell, \leq_r)$. Suppose that Q is a partial order P of subsets $Q_\alpha, \alpha \in P$, so that S is a partial order P of subsets $S_\alpha = S \cap Q_\alpha, \alpha \in P$. Then P is the $\leq_{\mathcal{J}}$-order on Q if and only if P is the $\leq_j$-order on S.*

Moreover, if S is stratified in Q, that is $\mathcal{L}' = \mathcal{L}^*$ and $\mathcal{R}' = \mathcal{R}^*$, then P is the $\mathcal{J}$-order on Q if and only if P is the $\mathcal{J}^*$-order on S.

Proof. The first part follows from Lemma 2.4 and the facts that S intersects every $\mathcal{H}$-class of Q and each Q_α is a union of $\mathcal{H}$-classes.

For the remainder of the proof, we assume that S is stratified in Q.

Suppose that P is the $\leq_{\mathcal{J}}$-order on Q, so that P is the $\leq_j$-order on S. Let $a, b \in S$ and suppose that $a \leq_j b$, so that $a \,\mathcal{D}'\, ubv$ for some $u, v \in S$. We have that $\mathcal{D}^* = \mathcal{D}'$, so that by Lemma 1.7 of [2], $a \leq_{\mathcal{J}^*} b$ in S. On the other hand, if $c, d \in S$ and $c \leq_{\mathcal{J}^*} d$, then again by Lemma 1.7 of [2], there are elements $c_0, c_1, \ldots, c_n \in S$, say $c_i \in S_{\alpha_i}$, and $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$ such that $d = c_0, c = c_n$ and $(c_i, x_i c_{i-1} y_i) \in \mathcal{D}^*$ for $i = 1, \ldots, n$. Since $\mathcal{D}' = \mathcal{D}^*$ and each S_α is a union of $\mathcal{D}^*$-classes, we have

$$\alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_n.$$

Thus $c \leq_{\mathcal{J}} d$ in Q so that $c \leq_j d$ in S. Consequently, P is the $\leq_{\mathcal{J}^*}$-order on S.

Finally, suppose that P is the $\leq_{\mathcal{J}^*}$-order on S. Let $p \in Q_\alpha$ and $q \in Q_\beta$, and choose $a \in S_\alpha, b \in S_\beta$ with $p \,\mathcal{H}\, a$ and $q \,\mathcal{H}\, b$.

If $\alpha \leq \beta$, then $a \leq_{\mathcal{J}^*} b$, so that by Lemma 1.7 of [2], there are elements $a_0, a_1, \ldots, a_n \in S$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$ such that $b = a_0, a = a_n$ and $(a_i, x_i a_{i-1} y_i) \in \mathcal{D}^*$ for $i = 1, \ldots, n$. Hence in Q, $a_i \,\mathcal{D}\, x_i a_{i-1} y_i$ so that $a_i \,\mathcal{D}'\, x_i a_{i-1} y_i$ and

$$a = a_n \leq_j a_{n-1} \leq_j \ldots \leq_j a_0 = b.$$

It follows that $a \leq_{\mathcal{J}} b$ in Q, whence $p \leq_{\mathcal{J}} q$ in Q.

Conversely, if $p \leq_{\mathcal{J}} q$, then $a \leq_j b$ in S so that $a \,\mathcal{D}'\, ubv$ for some $u, v \in S$, giving that $\alpha \leq \beta$ as required. □

5. Layering

Suppose that a semigroup S is a partial order P of subsets $S_\alpha, \alpha \in P$ such that each $\overline{S_\alpha}$ is a straight left order in a semigroup W_α. The aim of this section is to give necessary and sufficient conditions on S such that S is a straight left order in Q and Q is a partial order P of subsets Q_α such that $\overline{Q_\alpha} = W_\alpha$ for each $\alpha \in P$.

We vary standard notation slightly by letting $\leq_\ell^\alpha$ and $\leq_r^\alpha$ be the restriction to S_α of the preorders induced on $\overline{S_\alpha}$ by $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ in W_α (that is, we exclude from the preorders pairs of the form (I_α^S, x)). We denote by $\mathcal{L}'_\alpha$ and $\mathcal{R}'_\alpha$ the equivalences on S_α induced by $\leq_\ell^\alpha$ and $\leq_r^\alpha$ respectively; thus $\mathcal{L}'_\alpha$ and $\mathcal{R}'_\alpha$ are the restrictions of Green's relations $\mathcal{L}$ and $\mathcal{R}$ on W_α to the elements of S_α.

Definition 5.1. *The relations $\leq_\ell, \leq_r$ are defined on the elements of S by the rule that for any $a, b \in S$,*

$$a \leq_\ell b \text{ if and only if } a, cb \in S_\alpha \text{ and } a\,\mathcal{L}'_\alpha\, cb$$

for some $\alpha \in P$ and $c \in S$. The relation $\leq_r$ is defined dually.

In the above definition, if $a, cb \in S_\alpha$ and $a\,\mathcal{L}'_\alpha\, cb$, then choosing a square cancellable element x of $\overline{S_\alpha}$ in the $\mathcal{R}'_\alpha$-class of cb, we have $a\,\mathcal{L}'_\alpha\, xcb$ and $xc \in S_\alpha$ as J^S_α and I^S_α are ideals. We may then immediately deduce the following.

Lemma 5.2. *Let $a, b \in S_\alpha$. Then*

$$a \leq_\ell b \text{ if and only if } a \ \leq^\alpha_\ell\ b$$

and dually,

$$a \leq_r b \text{ if and only if } a \ \leq^\alpha_r\ b.$$

Lemma 5.3. *For any $a, b \in S$,*

$$a \leq_\ell b \leq_\ell a,$$

if and only if $a, b \in S_\alpha$ for some $\alpha \in P$ and $a\,\mathcal{L}'_\alpha\, b$. The dual result holds for $\leq_r$.

Proof. Suppose that $a, b \in S$ and

$$a \leq_\ell b \leq_\ell a.$$

We have that $a, cb \in S_\alpha$ and $b, da \in S_\beta$ for some $\alpha, \beta \in P$ and $c, d \in S$. It follows from the definition of partial order of subsets that $\alpha = \beta$ and so $a\mathcal{L}'_\alpha b$ by Lemma 5.2.

Conversely, if $a, b \in S_\alpha$ and $a\,\mathcal{L}'_\alpha\, b$, then $a \leq^\alpha_\ell b$ and $b \leq^\alpha_\ell a$. By Lemma 5.2 we have that

$$a \leq_\ell b \leq_\ell a$$

as required. ∎

Consequent upon the previous result, if $\leq_\ell$ and $\leq_r$ are transitive, then the associated equivalence relations $\mathcal{L}'$ and $\mathcal{R}'$ are respectively the union of the equivalence relations $\mathcal{L}'_\alpha$ and $\mathcal{R}'_\alpha, \alpha \in P$.

Recall that if T is a left order in semigroups Q and W, then Q is *isomorphic to W over S* if there is an isomorphism from Q to W that restricts to the identity on S.

Theorem 5.4. *The following conditions on the semigroup S are equivalent:*

(I) $(\leq_\ell, \leq_r)$ is an embeddable $*$-pair and $\mathcal{G}(S) = \mathcal{S}(S)$;

(II) $(\leq_\ell, \leq_r)$ is a $*$-pair such that for all $a, b, c \in S$ with $a \in \mathcal{S}(S)$,

$$a^2 b\,\mathcal{R}'\, a^2 c \text{ implies that } ab\,\mathcal{R}'\, ac$$

and $\mathcal{S}(S) = \mathcal{G}(S)$;

(III) S is a straight left order in a semigroup Q inducing $(\leq_l, \leq_r)$.

If any of these conditions hold, then Q is a partial order P of subsets $Q_\alpha, \alpha \in P$, such that each $\overline{S_\alpha}$ is a straight left order in $\overline{Q}_\alpha$ and $\overline{Q_\alpha}$ is isomorphic to W_α over $\overline{S_\alpha}$.

Proof. (III) implies (I) This is immediate from Theorem 4.4 of [3].

(I) implies (II) Suppose that $a, b, c \in S$ with $a \in \mathcal{S}(S)$ and $a^2 b\,\mathcal{R}'\,a^2 c$. Since $(\leq_\ell, \leq_r)$ is an embeddable $*$-pair, we have $ab, ac \leq_r a$ by condition (Eii)(r) of Definition 1.2, so that by (Evii)(r), $ab\,\mathcal{R}'\,ac$.

(II) implies (III) By the construction of the relation $\mathcal{H}'$ condition G(ii) of Theorem 1.3 holds and we are given that G(i) also holds.

We verify that $(\leq_\ell, \leq_r)$ is an embeddable $*$-pair. Since each S_α is a union of $\mathcal{L}'$-classes and $\mathcal{R}'$-classes and $\mathcal{L}'_\alpha \circ \mathcal{R}'_\alpha = \mathcal{R}'_\alpha \circ \mathcal{L}'_\alpha$ we have that (Ei) holds. Conditions (Eii)(l) and (Eii)(r) follow from Definition 3.1 and the remarks preceding the statement of the theorem. Clearly (Eiii) holds.

We proceed via a series of lemmas to verify the remaining conditions.

Lemma 5.5. *Let $a \in \mathcal{G}(S)$ and let $b \in S$ with $b \leq_\ell a$. Then $b\,\mathcal{R}'\,ba$.*

Proof. We have that $b\,\mathcal{L}'_\alpha ca$ for some $\alpha \in P$ and $b, c \in S_\alpha$. Now $\mathcal{R}'$ is a left congruence so that $ca\,\mathcal{R}'\,ca^2$ and in $\overline{S_\alpha}$,

$$b = qca\,\mathcal{R}\,qca^2 = ba$$

for some $q \in W_\alpha$. Hence $b\,\mathcal{R}'_\alpha\,ba$ and so $b\,\mathcal{R}'\,ba$ as required. □

The above lemma show that condition (Ev)(l) holds; the proof for (Ev)(r) is dual.

Lemma 5.6. *Let $a \in \mathcal{G}(S)$ and let $b, c \in S$ with $b, c \leq_r a$ and $ab\,\mathcal{R}'\,ac$. Then $b\,\mathcal{R}'\,c$.*

Proof. By condition (Eii)(r) we have that $b\,\mathcal{R}'\,ax$ and $c\,\mathcal{R}'\,ay$ for some $x, y \in S$. Since $\mathcal{R}'$ is a left congruence,

$$a^2x\,\mathcal{R}'\,ab\,\mathcal{R}'\,ac\,\mathcal{R}'\,a^2y$$

so that by the given hypotheses of (II) we have that $ax\,\mathcal{R}'\,ay$ and so $b\,\mathcal{R}'\,c$. □

Lemma 5.7. *Let $a \in \mathcal{G}(S)$ and let $b, c \in S$ with $b, c \leq_r a$ and $ab = ac$. Then $b = c$.*

Proof. By condition (Ev)(r) we have

$$b\,\mathcal{L}'\,ab = ac\,\mathcal{L}'\,c.$$

Hence $b\,\mathcal{L}'_\alpha\,ab$ for some $\alpha \in P$. Since $\overline{S_\alpha}$ is a straight left order in W_α we may apply Lemma 4.7 of [3] to deduce the existence of $h \in \mathcal{S}(S), k \in S$ with $b\,\mathcal{R}'\,h\,\mathcal{R}'\,k$ and

$$hb = kab.$$

We know that $\mathcal{R}' \subseteq \mathcal{R}^*$ so that $hc = kac$ and

$$hb = kab = kac = hc.$$

By Lemma 5.6 we have that $b\,\mathcal{R}'\,c$ and so in W_α,

$$b = h^\sharp hb = h^\sharp hc = c.$$

□

Lemma 5.8. *Let $a \in \mathcal{G}(S)$ and $b, c \in S$ with $b, c \leq_\ell a$ and $ba\,\mathcal{L}'\,ca$. Then $b\,\mathcal{L}'\,c$.*

Proof. By condition (Eii)(l) we have that

$$b\,\mathcal{L}'\,xa \text{ and } c\,\mathcal{L}'\,ya$$

for some $x, y \in S$. Since $\mathcal{L}'$ is a right congruence

$$xa^2\,\mathcal{L}'\,ba\,\mathcal{L}'\,ca\,\mathcal{L}'\,ya^2$$

and so

$$xa^2\,\mathcal{L}'_\alpha\,ya^2$$

for some $\alpha \in P$. By Lemma 4.7 of [3],

$$hxa^2 = kya^2$$

for some $h \in \mathcal{S}(S), k \in S$ and

$$h\,\mathcal{R}'\,k\,\mathcal{R}'\,xa^2\,\mathcal{R}'\,xa.$$

We know that $\mathcal{R}' \subseteq \mathcal{R}^*$ and so

$$hxa = kya,$$

giving by condition (Ev)(r)

$$xa\,\mathcal{L}'\,hxa = kya \leq_\ell ya.$$

Hence $b \leq_\ell c$; dually, $c \leq_\ell b$ so that $b\,\mathcal{L}'\,c$ as required. □

Lemma 5.9. *Let $a \in \mathcal{G}(S)$ and let $b, c \in S$ with $b, c \leq_\ell a$ and $ba = ca$. Then $b = c$.*

Proof. From Lemma 5.8 we have that $b\,\mathcal{L}'\,c$, say $b\,\mathcal{L}'_\alpha\,c$ and so by Lemma 4.7 of [3], $hb = kc$ for some $h \in \mathcal{S}(S), k \in S$ with $h\,\mathcal{R}'\,k\,\mathcal{R}'\,b$. By condition (Ev)(l)

$$b\,\mathcal{R}'\,ba = ca\,\mathcal{R}'c.$$

From

$$hba = kca = kba$$

we deduce that $hb = kb$ and $hc = kc$. Hence

$$hb = kc = hc.$$

In W_α we then have

$$b = h^\sharp hb = h^\sharp hc = c.$$

□

Lemmas 5.9,5.7,5.8 and 5.6 show that conditions (Evi)(l), (Evi)(r), (Evii)(l) and (Evii)(r) hold, respectively.

We call upon Theorem 4.4 of [3] to deduce that S is a straight left order in Q where Q induces $(\leq_\ell, \leq_r)$.

Suppose now that any of the equivalent conditions (I) to (III) holds. We show that Q is a partial order P of subsets Q_α, $\alpha \in P$.

Let $q \in Q$. Since S is a straight left order in Q, $q = h^\sharp k$ where $h \in \mathcal{S}(S), k \in S$ and $h\,\mathcal{R}'\,k$ in S. Hence $h, k \in S_\alpha$ for some $\alpha \in P$ and $q\,\mathcal{H}\,k$ in Q. Thus

$$Q = \bigcup\{Q_\alpha : \alpha \in P\}$$

where

$$Q_\alpha = \{q \in Q : q\,\mathcal{H}'\,k, k \in S_\alpha\}.$$

Since each S_α is a union of $\mathcal{H}'$-classes, $Q_\alpha \cap Q_\beta = \emptyset$ for $\alpha \neq \beta$ and $S_\alpha = S \cap Q_\alpha$.

Let $p \in Q_\alpha, q \in Q_\beta$. Then $p\,\mathcal{H}\,h$, $q\,\mathcal{H}\,k$ for some $h \in S_\alpha, k \in S_\beta$. Now

$$pq\,\mathcal{R}\,pk\,\mathcal{L}\,hk$$

and $hk \in S_\gamma$ for some $\gamma \leq \alpha, \beta$. Since each S_γ is a union of $\mathcal{R}'$-classes and of $\mathcal{L}'$-classes, it follows that $pq \in Q_\gamma$. Hence Q is a partial order of the subsets $Q_\alpha, \alpha \in P$.

For each $\alpha \in P$ we let $\overline{Q_\alpha}$ be the Rees quotient associated with Q_α. From Corollary 4.2 and the construction of the relation $\mathcal{H}'$, we have that $\overline{S_\alpha}$ is a straight left order in $\overline{Q_\alpha}$.

Let $a, b \in \overline{S_\alpha}$. If $a = \{I^S_\alpha\}$ then a is the zero of both W_α and $\overline{Q_\alpha}$, so that $a \leq_{\mathcal{R}} b$ in both W_α and $\overline{Q_\alpha}$. Otherwise, $a \in S_\alpha$ and

$$\begin{aligned} a \leq_{\mathcal{R}} b \text{ in } W_\alpha &\Leftrightarrow b \in S_\alpha \text{ and } a\,\mathcal{R}'\,bc, \text{ for some } c \in S_\alpha \\ &\Leftrightarrow b \in S_\alpha \text{ and } a\,\mathcal{R}'\,bc, \text{ for some } c \in S \\ &\Leftrightarrow b \in S_\alpha \text{ and } a \leq_r b \\ &\Leftrightarrow b \in S_\alpha \text{ and } a \leq_{\mathcal{R}} b \text{ in } Q \\ &\Leftrightarrow b \in S_\alpha \text{ and } a \leq_{\mathcal{R}} b \text{ in } \overline{Q_\alpha}. \end{aligned}$$

Corollary 4.3 of [13] gives that $\overline{Q_\alpha}$ is isomorphic to W_α over $\overline{S_\alpha}$. □

The rest of the paper is devoted to specialisations of Theorem 5.4.

6. Semilattices of straight left orders

Let S be a semilattice Y of subsemigroups S_α, $\alpha \in Y$, where each S_α is a straight left order in T_α inducing $(\leq^\alpha_\ell, \leq^\alpha_r)$ with associated equivalence relations $\mathcal{L}'_\alpha$ and $\mathcal{R}'_\alpha$. As remarked in Section 3, S is certainly a partial order Y of the subsets $S_\alpha, \alpha \in Y$. We define the relations $\leq_\ell$ and $\leq_r$ as in Definition 5.1. Let $\alpha \in P$. If $I^S_\alpha = \emptyset$, then we let $W_\alpha = T_\alpha$. If $I^S_\alpha \neq \emptyset$ put $W_\alpha = T^0_\alpha$. In view of Lemma 2.2, in either case $\overline{S_\alpha}$ is a straight left order in W_α. It is easy to see that the pair $(\leq_\ell, \leq_r)$ arising from the left quotient semigroups T_α coincides with the pair of relations arising from the left quotient semigroups W_α. We can thus apply Theorem 5.4 to obtain the first part of the following result.

Theorem 6.1. *The following conditions on the semigroup S are equivalent:*

(A) $(\leq_\ell, \leq_r)$ is a $*$-pair such that for all $a \in \mathcal{S}(S), b, c \in S$,

$$a^2 b\, \mathcal{R}'\, a^2 c \text{ implies that } ab\, \mathcal{R}'\, ac$$

and $\mathcal{S}(S) = \mathcal{G}(S)$;

(B) S is a straight left order in Q inducing $(\leq_\ell, \leq_r)$.

If any of these conditions hold, then Q is a semilattice Y of subsemigroups $Q_\alpha, \alpha \in Y$, such that each S_α is a straight left order in Q_α and Q_α is isomorphic to T_α over S_α.

Proof. By Theorem 5.4, (A) and (B) are equivalent. If (A) or (B) hold, then again by Theorem 5.4, Q is a partial order Y of subsets $\overline{Q_\alpha}$, $\alpha \in Y$, such that each $\overline{S_\alpha}$ is a straight left order in $\overline{Q_\alpha}$ and $\overline{Q_\alpha}$ is isomorphic to W_α over $\overline{S_\alpha}$.

Let $\alpha \in P$. If $I_\alpha^S = \emptyset$, then as in Section 4, $I_\alpha^Q = \emptyset$. In this case, $\overline{S_\alpha} = S_\alpha$, $\overline{Q_\alpha} = Q_\alpha$ is a subsemigroup of Q, and $W_\alpha = T_\alpha$. Hence S_α is a straight left order in Q_α and Q_α is isomorphic to T_α over $S\alpha$.

Otherwise, $\overline{S_\alpha} = S_\alpha^0$. By Lemma 2.1, the zero of $\overline{S_\alpha}$ is the zero of $\overline{Q_\alpha}$, and as $\overline{S_\alpha}$ has no zero divisors, neither does $\overline{Q_\alpha}$. Thus Q_α is a subsemigroup of Q and by Lemma 2.2, S_α is a straight left order in Q_α. Now $\overline{Q_\alpha}$ is isomorphic to T_α^0 over S_α^0; clearly then Q_α is isomorphic to T_α over S_α.

Finally, let $p \in Q_\alpha$ and $Q \in Q_\beta$. Then $p\, \mathcal{H}\, a$ and $q\, \mathcal{H}\, b$ for some $a \in S_\alpha$ and $b \in S_\beta$, whence

$$pq\, \mathcal{R}\, pb\, \mathcal{L}\, ab$$

so that as each Q_γ is a union of $\mathcal{D}$-classes, we have $pq \in Q_{\alpha\beta}$. Thus Q is a semilattice Y of subsemigroups $Q_\alpha, \alpha \in Y$. $\square$

Theorem 6.1 has a number of applications. Theorem 5.5 of [7] (see also [14]) is the specialisation of Theorem 6.1 to the case where each S_α is a straight left order in a completely simple semigroup T_α. Surprisingly, the conditions in (A) are only slightly stronger than those in Theorem 5.5 of [7]. In the latter result one can use the decomposition of each S_α as a matrix of right reversible cancellative subsemigroups to deduce that $(\leq_\ell, \leq_r)$ is a $*$-pair from the assumptions that $\leq_\ell$ and $\leq_r$ are transistive, $\mathcal{R}'$ is a left congruence, $\mathcal{L}' \subseteq \mathcal{L}^*$ and $\mathcal{R}' \subseteq \mathcal{R}^*$ [7]. Moreover for any $a \in S$, $a\, \mathcal{H}'\, a^2$ so that as $\mathcal{H}' \subseteq \mathcal{H}^*$, $a\, \mathcal{H}^*\, a^2$. thus $\mathcal{S}(S) = \mathcal{G}(S)$ holds automatically. Of course one can specialise further to the case where each T_α is a group, to obtain the early result of [6] mentioned in the Introdution.

Specialisations of Theorem 6.1 to the case where S is cancellative or commutative can be found in [15] and [16].

7. The fully stratified case

Recall that a straight left order S in Q is *fully stratified* in Q if Q induces $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$. Clearly we may specialise Theorem 5.4 to fully stratified straight left orders. The cleanest result is available when S is IC abundant.

Theorem 7.1. *Let S be an idempotent connected abundant semigroup.*

If S is a fully stratified straight left order in Q, where Q is a partial order P of subsets $Q_\alpha, \alpha \in P$, then for any $\alpha \in P$, $\overline{S_\alpha}$ is an abundant fully stratified straight left order in $\overline{Q_\alpha}$.

Conversely, suppose that S is a partial order P of subsets $S_\alpha \alpha \in P$, such that each $\overline{S_\alpha}$ is an abundant fully stratified straight left order. Then S is a fully stratified straight left order if and only if for all $a \in \mathcal{S}(S)$ and $b, c \in S$,

$$a^2 b\, \mathcal{R}^*\, a^2 c \text{ implies that } ab\, \mathcal{R}^*\, ac.$$

Proof. Suppose that S is a fully stratified straight left order in Q, where Q is a partial order P of subsets $Q_\alpha, \alpha \in P$. As in Section 4, S is a partial order P of subsets $S_\alpha = S \cap Q_\alpha$, and each S_α is a union of $\mathcal{L}^*$-classes and of $\mathcal{R}^*$-classes. Thus for any $\alpha \in P$, $\underline{J_\alpha^S}$ and I_α^S are $*$-ideals. From Proposition 4.3, each $\overline{S_\alpha}$ is a straight left order in $\overline{Q_\alpha}$. It remains to show that $\overline{S_\alpha}$ is abundant and fully stratified in $\overline{Q_\alpha}$.

Let $a, b \in \overline{S_\alpha}$ and suppose that $a \leq_{\mathcal{R}^*} b$ in $\overline{S_\alpha}$. Either $a = 0$, in which case certainly $a \leq_{\mathcal{R}} b$ in $\overline{Q_\alpha}$. Otherwise, $a, b \in S_\alpha$ and by Lemma 2.5 of [9], $a \leq_{\mathcal{R}^*} b$ in S. But S is fully stratified in Q, so that $a \leq_{\mathcal{R}} b$ in Q and hence $a \leq_{\mathcal{R}} b$ in $\overline{Q_\alpha}$. Together with the dual argument for $\leq_{\mathcal{L}^*}$, we have that $\overline{Q_\alpha}$ induces $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ on $\overline{S_\alpha}$.

Again from Lemma 2.5 of [9], the non-zero $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes of $\overline{S_\alpha}$ are $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes of S and thus contain idempotents since S is abundant. Thus $\overline{S_\alpha}$ is abundant.

Conversely, suppose that S is a partial order P of subsets $S_\alpha, \alpha \in P$, such that each $\overline{S_\alpha}$ is an abundant fully stratified straight left order in a semigroup W_α. Let $a \in S_\alpha, b \in S_\beta$ and suppose that $a\, \mathcal{R}^*\, b$ in S. Since $\overline{S_\alpha}$ and $\overline{S_\beta}$ are abundant, there are idempotents $e \in \overline{S_\alpha}$ and $f \in \overline{S_\beta}$ with $a\, \mathcal{R}^*\, e$ in $\overline{S_\alpha}$ and $b\, \mathcal{R}^*\, f$ in $\overline{S_\beta}$. Hence $ea = a$ and $fb = b$. It follows that $e \in S_\alpha$ and $f \in S_\beta$. Moreover from $ea = a$ and $a\, \mathcal{R}^*\, b$ in S we have $eb = b$, so that $\beta \leq \alpha$; dually, $\alpha \leq \beta$. Hence each S_α is a union of $\mathcal{R}^*$-classes, and dually, of $\mathcal{L}^*$-classes. Consequently, for any $\alpha \in P$, I_α^S and J_α^S are $*$-ideals.

Let $\leq_\ell$ and $\leq_r$ be as in Definition 5.1.

Let $a, b \in S$ and suppose that $a \leq_\ell b$. Then for some $\alpha \in P$ and $c \in S$, $a, cb \in S_\alpha$ and $a\, \mathcal{L}'_\alpha\, cb$. As $\overline{S_\alpha}$ is fully stratified we have that $a\, \mathcal{L}^*\, cb$ in $\overline{S_\alpha}$ so that by Lemma 2.5 of [9], $a\, \mathcal{L}^*\, cb$ in S, so that $a \leq_{\mathcal{L}^*} b$ in S.

Conversely, if $a \leq_{\mathcal{L}^*} b$ in S, then by Lemma 2.3, $a\, \mathcal{L}^*\, xb$ for some $x \in S$. Since each S_α is a union of $\mathcal{L}^*$-classes, we have $a, xb \in S_\alpha$ for some $\alpha \in P$ and by Lemma 2.5 of [9], $a\, \mathcal{L}^*\, xb$ in $\overline{S_\alpha}$. Thus $a\, \mathcal{L}'_\alpha\, xb$ so that $a \leq_\ell b$. Hence $\leq_\ell = \leq_{\mathcal{L}^*}$ and dually, $\leq_r = \leq_{\mathcal{R}^*}$. Clearly $(\leq_\ell, \leq_r)$ is a $*$-pair and $\mathcal{S}(S) = \mathcal{G}(S)$. We can now apply Theorem 5.4 to deduce that S is a straight left order in a semigroup Q inducing $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$, that is, S is a fully stratified straight left order, if and only if for all $a \in \mathcal{S}(S)$ and $b, c \in S$,

$$a^2 b\, \mathcal{R}^*\, a^2 c \text{ implies that } ab\, \mathcal{R}^*\, ac.$$

$\square$

We now present an application of this theorem. Theorem 5.12 of [9] states that the semigroup $\mathrm{End}_f A$ of endomorphisms of finite rank of a stable basis algebra is a fully stratified straight left order. The proof proceeds by verifying all the conditions necessary for $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ to be an embeddable $*$-pair, together with condition (Gii). Theorem 7.1 gives an alternative approach. For further details of basis algebras, we refer the reader to [17, 9].

It is clear that

$$\mathrm{End}_f A = \bigcup\{U_n : 0 \leq n \leq \mathrm{rank}\, A, n \in \mathbb{N}\}$$

where

$$U_n = \{\alpha \in \mathrm{End}\, A : \mathrm{rank}\, A = n\}.$$

For any $\alpha, \beta \in \mathrm{End}\, A$, Lemma 5.1 of [9] gives that

$$\mathrm{rank}\, \alpha\beta \leq \min \{\mathrm{rank}\, \alpha, \mathrm{rank}\, \beta\}$$

so that in our terminology, $\mathrm{End}_f A$ is a partial order (indeed, a chain) N of subsets $U_n, n \in N$, where N is the set of finite ranks of endomorphisms of A. Our Rees quotients $\overline{U_n}$ are, in the terminology of [9], the semigroups S_n/S_{n-1}. The latter are shown in the final section of [9] to be primitive abundant semigroups in which all the non-zero idempotents are $\mathcal{D}$-related. Moreover the non-null $\mathcal{H}^*$-classes are right reversible. It follows from [2] that S_n/S_{n-1} is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(H; I, \Lambda; P)$, where H is a right reversible cancellative semigroup and I and Λ index the $\mathcal{R}^*$-classes and $\mathcal{L}^*$-classes of S, respectively. It is easy to see that $\mathcal{M}^0(H; I, \Lambda; P)$ is a fully stratified straight left order in $\mathcal{M}^0(G; I, \Lambda; P)$, where G is the group of left quotients of H.

From Theorem 3.9 and Corollary 5.2 of [9], $\mathrm{End}_f A$ is an IC abundant semigroup. We thus have exactly the right circumstances to apply Theorem 7.1. Lemma 5.11 of [9] gives that for all $a \in \mathcal{S}(S)$ and $b, c \in S$,

$$a^2 b\, \mathcal{R}^*\, a^2 c \text{ implies that } ab\, \mathcal{R}^*\, ac.$$

Consequently, $\mathrm{End}_f A$ is a fully stratified straight left order.

8. Straight left orders in completely semisimple semigroups

We recall from [8] that a semigroup is *completely semisimple* if all its principal factors are completely (0)-simple. We end the paper with a new characterisation of straight left orders in completely semisimple semigroups.

Let S be a semigroup possessing a $*$-pair $(\leq_\ell, \leq_r)$. We say that S has M_L^* (M_R^*) if the $\mathcal{L}'$-class ($\mathcal{R}'$-classes) *within any $\mathcal{D}$-class* satisfy the descending chain condition.

Theorem 8.1. *The following are equivalent for a semigroup S:*

(i) S is a straight left order in a completely semisimple semigroup Q inducing $(\leq_\ell, \leq_r)$;

(ii) S is a straight left order in a semigroup Q inducing $(\leq_\ell, \leq_r)$ such that $\mathcal{J}' = \mathcal{D}'$ and M_L^* and M_R^* hold;

(iii) S is a partial order P of subsets $S_\alpha, \alpha \in P$, each $\overline{S_\alpha}$ is a (straight) left order in a completely 0-simple (or completely simple) semigroup such that $(\leq_\ell, \leq_r)$ is a $*$-pair, for all $a, b, c \in S$ with $a \in \mathcal{S}(S)$,

$$a^2 b \, \mathcal{R}' \, a^2 c \text{ implies that } ab \, \mathcal{R}' \, ac$$

and $\mathcal{S}(S) = \mathcal{G}(S)$.

Proof. (i) implies (ii) In Q we have that $\mathcal{J} = \mathcal{D}$ and so by Lemma 2.4, $\mathcal{J}' = \mathcal{D}'$ on S. Suppose that $a_1, a_2, \ldots$ are $\mathcal{D}'$-related elements of S such that

$$a_1 \geq_\ell a_2 \geq_\ell \ldots$$

Then in Q the elements $a_1, a_2, \ldots$ are $\mathcal{D}$-related and we have

$$a_1 \geq_{\mathcal{L}} a_2 \geq_{\mathcal{L}} \ldots$$

so that the same is true in the principal factors. But these are completely (0)-simple, and so

$$a_1 \, \mathcal{L} \, a_2 \, \mathcal{L} \ldots$$

in $\overline{Q_\alpha}$ and hence in Q. This gives

$$a_1 \, \mathcal{L}' \, a_2 \, \mathcal{L}' \ldots.$$

Certainly then M_L^* holds. Dually condition M_R^* holds.

(ii) implies (i) We have $\mathcal{J} = \mathcal{D}$ on Q since $\mathcal{J}' = \mathcal{D}'$ on S. Consider a principal factor W of Q. Let $q_1, q_2, \ldots$ be non-zero elements of W such that

$$q_1 \geq_{\mathcal{L}} q_2 \geq_{\mathcal{L}} \ldots$$

in W. For each i choose $a_i \in S$ such that $a_i \, \mathcal{H} \, q_i$ in Q and hence in W. Now in W and Q

$$a_1 \geq_{\mathcal{L}} a_2 \geq_{\mathcal{L}} \ldots$$

so that

$$a_1 \geq_\ell a_2 \geq_\ell \ldots$$

But the q_i's are all $\mathcal{J}$-related in Q, so that the a_i's are all $\mathcal{D}'$-related in S. Since S has M_L^*, we have that for some n,

$$a_n \mathcal{L}' \, a_{n+1} \, \mathcal{L}' \ldots$$

whence

$$q_n \mathcal{L} \, q_{n+1} \, \mathcal{L} \ldots$$

in Q and W. Hence W has the descending chain condition on principal left ideals, and dually for principal right ideals. By Theorem 3.3.2 of [1], W is completely (0)-simple.

(i) implies (iii) We have that Q is a partial order P of subsets $Q_\alpha, \alpha \in P$, where P is the $\leq_{\mathcal{J}}$-order on Q, and S is thus a partial order P of subsets $S_\alpha = S \cap Q_\alpha$, where P is the $\leq_j$-order on S. By Lemma 2.1, each $\overline{S_\alpha}$ is a straight weak left order in $\overline{Q_\alpha}$. But if $a \in \overline{S_\alpha}$ is square cancellable in $\overline{S_\alpha}$, then either $a = 0$, so lies in a subgroup of $\overline{Q_\alpha}$, or as $a \, \mathcal{H}^* \, a^2$ in $\overline{S_\alpha}$, $a^2 \neq 0$. In the latter case it is well known that a lies in a subgroup of $\overline{Q_\alpha}$. Thus $\overline{S_\alpha}$ is a straight left order in a completely (0)-simple semigroup.

It is easy to verify that $(\leq_\ell, \leq_r)$ coincides with the pair of preorders constructed as in Section 5 from the straight left orders $\overline{S_\alpha}$. Clearly then (iii) holds.

(iii) implies (i) This is immediate from Theorem 5.4. $\square$

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