Left idempotent bounded C–semigroups

John Fountain and Victoria Gould

Abstract

A semigroup S with zero is *categorical at zero* or a C-*semigroup* if given any $a, b, c \in S$ where $ab, bc \in S \setminus \{0\}$, then it follows that $abc \in S \setminus \{0\}$. A semigroup with zero is *left (right) idempotent bounded* if it is the 0-direct union of idempotent generated principal left (right) ideals and *idempotent bounded* if it is both left and right idempotent bounded. We abbreviate these concepts by LIB, RIB and IB respectively. The aim of this paper is to give a structure theorem for the class of LIB C-semigroups in terms of what we call *left blocked Rees matrix semigroups.* In a sequel, we specialise our work to describe IB C-semigroups in terms of *double blocked Rees matrix semigroups.*

The classes of LIB C-semigroups and IB C-semigroups are extremely large, including the classes of primitive abundant semigroups and completely 0-simple semigroups. Consideration of semigroups that are idempotent bounded *only* on the left allow us to move away from structure theorems for C-semigroups satisfying conditions that are inherently one sided, such as regularity.

1 Introduction

A celebrated theorem of Rees [R] states that a semigroup is completely 0–simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group. Such a semigroup $\mathcal{M}^0$ has the property that for all $a, b, c \in \mathcal{M}^0$, if ab and bc are non–zero, then abc is non–zero, that is, $\mathcal{M}^0$ is *categorical at zero*, or is a *C–semigroup*. An immediate question is : to what extent can the Rees theorem be extended to larger classes of C–semigroups?

One approach is to investigate Rees matrix semigroups over monoids other than groups with generalisations of the regularity conditions on the sandwich matrix. Steinfeld [S1], Lallement and Petrich [LP] and Lawson [L2] have each adopted this approach and characterised the classes of semigroups which arise from their constructions. In the theory of regular semigroups, the regular part of Rees matrix semigroups over inverse semigroups play an important role [McA].

Another direction is to generalise the Rees construction itself, for example, in [F], blocked Rees matrix semigroups are introduced to describe primitive abundant semigroups. Loosely speaking, in a blocked Rees matrix semigroup a non–zero entry in the (i, λ)'th place lies in a set which depends on the position of the coordinates i and λ. We adopt here this second approach.

It is well known that a semigroup is completely 0–simple if and only if it is regular, primitive and 0 is a prime ideal. Of course, a semigroup is regular if and only if every $\mathcal{L}$–class contains an idempotent and this is true if and only if every $\mathcal{R}$–class contains an idempotent. The class of completely 0–simple semigroups therefore cannot be extended by considering those that are 'right regular'. So we turn our attention to abundant or right abundant semigroups.

On a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $(a, b) \in \mathcal{L}^*$ if and only if the elements a, b of S are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually. A semigroup is *right abundant* if every $\mathcal{L}^*$–class contains an idempotent. *Left abundant* semigroups are defined dually and a semigroup is *abundant* if it is both right and left abundant. It is shown in [F] that the class of primitive abundant semigroups is (up to isomorphism) precisely the class of PA blocked Rees matrix semigroups.

Unlike the situation for regular semigroups, a semigroup can be right abundant, indeed primitive right abundant, without being left abundant. However, it is not difficult to see that all primitive right abundant semigroups are categorical at zero. Moreover a primitive right abundant semigroup S is the

0–direct union of idempotent generated principal left ideals. This inspires the following definition.

A semigroup with zero is *left (right) idempotent bounded* if it is the 0–direct union of idempotent generated principal left (right) ideals and *idempotent bounded* if it is both left and right idempotent bounded. We abbreviate these concepts by LIB, RIB and IB, respectively. The aim of this paper is to give a structure theorem for the class of LIB C–semigroups in terms of what we call *left blocked Rees matrix semigroups.*

After some preliminaries in Section 2 we give a number of supporting examples in Section 3. In the next section we introduce left blocked Rees matrix semigroups. These cannot quite be regarded as a generalisation of blocked Rees matrix semigroups, for the non–zero entries in the 'sandwich matrix' come not from the underlying partial semigroup M, but are functions on M. We then give conditions on the constituents of a left blocked Rees matrix semigroup that ensure it is a LIB semigroup ; by construction, it is a C–semigroup. We call left blocked Rees matrix semigroups satisfying these conditions LIB–LB Rees matrix semigroups. Following this we determine the conditions under which two LIB–LB Rees matrix semigroups are isomorphic. In Section 5 we show that every LIB C–semigroup is isomorphic to a left blocked Rees matrix semigroup. The final section is concerned with a number of special cases ; in particular, we give a structure theorem for primitive right abundant semigroups.

In a sequel [FG] we specialise our work to describe IB C–semigroups in terms of *double blocked Rees matrix semigroups.* These are perhaps a truer generalisation of blocked Rees matrix semigroups than are the left blocked Rees matrix semigroups of this paper. The characterisation of IB C–semigroups given in [FG] itself specialises naturally to the class of primitive abundant semigroups (first considered in [F]) and therefore to completely 0–simple semigroups.

We remark that the non–zero elements of a Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ can be regarded as elements of $I \times G \times \Lambda$ or as matrices over $G \cup \{0\}$ with precisely one non–zero entry. Similar comments apply to related constructions : we will keep both viewpoints in mind.

2 Preliminaries

Where possible we use the notation and terminology of [H]. In particular, for any semigroup S we denote by $E(S)$ the set of idempotents of S. Unless stated otherwise, we assume throughout that any semigroup under consideration has a zero element 0. For any subset U of a semigroup S, the set of non–zero elements of U is denoted by U^*.

Central to our approach are the relations $\sim_r$ and $\sim_\ell$. We define these relations for both semigroups with zero and partial semigroups. If S is a semigroup with zero, then for $a, b \in S$

$$a \sim_r b \text{ if and only if } a = b = 0 \text{ or } aS^1 \cap bS^1 \neq 0$$

and

$$a \sim_\ell b \text{ if and only if } a = b = 0 \text{ or } S^1a \cap S^1b \neq 0.$$

The relations $\sim_r$ and $\sim_\ell$ are often denoted by ρ and λ respectively. However, for the purposes of this paper and its sequel it is convenient to reserve the symbols ρ and λ for other uses. Clearly the relations $\sim_r$ and $\sim_\ell$ are reflexive and symmetric, but in general they are not transitive. We denote the transitive closures of $\sim_r$ and $\sim_\ell$ by $\mathcal{R}'$ and $\mathcal{L}'$ respectively. In many of the examples of C–semigroups we consider, one or both of the equivalence relations $\mathcal{R}'$ and $\mathcal{L}'$ will have a neater description.

Recall that a right (left) ideal I of a semigroup S is *indecomposable* if $I \neq A \cup B$ for any right (left) ideals of S with $A \cap B = 0$. Any semigroup S may be decomposed uniquely as a 0–direct union of indecomposable right ideals and as a 0–direct union of indecomposable left ideals . The components of these decompositions are the maximal indecomposable one–sided ideals. This approach allows us to view the relations $\mathcal{R}'$ and $\mathcal{L}'$ in a different light, for it is clear that the non–zero elements of maximal indecomposable right (left) ideals form the non–zero $\mathcal{R}'$ ($\mathcal{L}'$)–classes.

An equivalence relation θ on a semigroup S is a *left* 0*–congruence* if θ is a left congruence such that 0 is a θ–class and $xy = 0$ or $xy\theta x$ for any $x, y \in S$. A *right* 0*–congruence* is defined dually . A congruence θ on S is a 0*–matrix congruence* if 0 is a θ–class and S/θ is a rectangular 0–band.

Lemma 2.1 *[LP] The intersection of a left* 0*–congruence and a right* 0*–congruence is a* 0*–matrix congruence. Conversely, every* 0*–matrix congruence on a semigroup can be written uniquely as the intersection of a left* 0*–congruence and a right* 0*–congruence.*

Lemma 2.2 *[LP] On a C–semigroup S, the least left 0–congruence, the least right 0–congruence and the least 0–matrix congruence are $\mathcal{R}'$,$\mathcal{L}'$ and $\mathcal{R}' \cap \mathcal{L}'$ respectively.*

In later sections it is necessary to consider partial semigroups. A non–empty set M is a *partial semigroup* if there is a partial binary operation on M such that for all $a, b, c \in M$, $(ab)c$ is defined if and only if $a(bc)$ is defined, and if $(ab)c$ is defined, then $(ab)c = a(bc)$. On a partial semigroup M the relation $\sim_r$ is defined by

$$a \sim_r b \text{ if and only if } aM^1 \cap bM^1 \neq \emptyset.$$

One can also define *right ideals* of M ; then a right ideal I of M is *indecomposable* if I is not the disjoint union of two right ideals of M. Further, M is the disjoint union of its maximal indecomposable right ideals, which are its $\mathcal{R}'$–classes, where $\mathcal{R}'$ is the transitive closure of $\sim_r$ as before. The relations $\sim_\ell$ and $\mathcal{L}'$ are defined dually to $\sim_r$ and $\mathcal{R}'$ and dual comments apply.

If a is an element of a semigroup S, then $\ell(a) = \{s \in S : sa = 0\}$ is the *left annihilator* of a. Clearly, $\ell(a)$ is a left ideal of S. The *right annihilator* $r(a)$ of a is defined dually. Consideration of left and right annihilators gives rise to relations ann_ℓ and ann_r on S, defined by

$$a \, ann_\ell \, b \text{ if and only if } \ell(a) = \ell(b)$$

and

$$a \, ann_r \, b \text{ if and only if } r(a) = r(b)$$

for all $a, b \in S$. It is easy to see that ann_ℓ is a left congruence and ann_r is a right congruence on S.

Lemma 2.3 *If θ is a left (right) 0–congruence on S, then $\theta \subseteq ann_\ell$ ($\theta \subseteq ann_r$). In particular, if S is a C–semigroup, then $\mathcal{R}' \subseteq ann_\ell$ and $\mathcal{L}' \subseteq ann_r$.*

Proof Let θ be a left 0–congruence on a semigroup S and suppose that $a\theta b$ where $a, b \in S$. As θ is a left congruence, $ca\theta cb$ for all $c \in S$, so that as 0 is a θ–class, $ca = 0$ if and only if $cb = 0$ for all $c \in S$. Thus $\ell(a) = \ell(b)$ and $a \, ann_\ell \, b$.

We remark that even if S is a C–semigroup, one cannot say in general that ann_ℓ is a left 0–congruence, for it is possible that 0 is not an ann_ℓ–class.

Suppose now that S is a C–semigroup and let I, Λ index the non–zero $\mathcal{R}'(\mathcal{L}')$–classes of S respectively, where the $\mathcal{R}'(\mathcal{L}')$–class indexed by $i \in I(\lambda \in \Lambda)$ is denoted by $R'_i(L'_\lambda)$. Note that if $ab = 0$ for some $a \in L'_\lambda, b \in R'_i$, then as $\mathcal{R}' \subseteq ann_\ell$ we have that $aR'_i = 0$ and then as $\mathcal{L}' \subseteq ann_r, L'_\lambda R'_i = 0$. Our aim is to represent S as a semigroup of matrices, where this representation is 'as fine as possible'. By this we mean that S is represented by $I \times \Lambda$ matrices, our justification in this being Lemma 2.2. The elements of the matrices should come from an algebra whose structure is less complicated than that of S. For example, in the Rees theorem, $\mathcal{R}' = \mathcal{R}, \mathcal{L}' = \mathcal{L}$ and the non–zero entries of the matrices lie in a group.

For an arbitrary C–semigroup S it seems we can make only the trivial move, which for the sake of completeness we now describe.

Let $\mathcal{N}^0 = (I \times S^* \times \Lambda) \cup \{0\}$, where $I(\Lambda)$ indexes the $\mathcal{R}'(\mathcal{L}')$–classes of S and 0 is a symbol we identify with the zero of S. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix over $\{1, 0\}$ where 1 is an identity adjoined to S and where $p_{\lambda i} = 0$ if and only if $L'_\lambda R'_i = 0$, where $\lambda \in \Lambda$ and $i \in I$. Define a multiplication on $\mathcal{N}^0$ by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) = (i, ab, \mu) & \text{if } p_{\lambda j} = 1 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

$$x0 = 0 = 0x$$

for all $(i, a, \lambda), (j, b, \mu) \in I \times S^* \times \Lambda$ and for all $x \in \mathcal{N}^0$. We thus have

Lemma 2.4 *Let S be a C–semigroup. Then S is isomorphic to $\mathcal{N}^0$, constructed as above.*

Of course, this tells us nothing of the more detailed structure of S. Regarding in the usual way the elements of $\mathcal{N}^0$ as matrices, the non–zero elements lie in S itself, which is clearly unsatisfactory. To make proper progress we must restrict the class of C–semigroups under consideration. Thus we turn our attention to those C–semigroups that are 0–direct unions of idempotent generated left ideals. As stated in Introduction, such semigroups are called *left idempotent bounded* or LIB. We remark that if S is a left idempotent bounded C–semigroup and $S = \bigcup^0_{\lambda \in \Lambda} Sf_\lambda$ where $f_\lambda \in E(S)^*, \lambda \in \Lambda$ and the notation $\bigcup^0$ indicates the union is 0–direct, then the non–zero $\mathcal{L}'$–classes are the sets $Sf_\lambda \setminus \{0\}, \lambda \in \Lambda$. The idempotents in the $\mathcal{L}$–classes of the f_λ's are called *$\mathcal{L}'$–generators*. Further, for $a, b \in S$,

$$a \sim_r b \text{ if and only if } a = b = 0 \text{ or } aS \cap bS \neq 0.$$

A regular or abundant semigroup S is *primitive* if all non–zero idempotents are minimal under the natural partial order on $E(S)$. To extend the notion of primitivity to arbitrary semigroups, we use the partial order $\leq$ introduced in [M]. The relation $\leq$ is defined on a semigroup S by the rule that if $a, b \in S$ then $a \leq b$ if and only if $a = xb = by, xa = a$ for some $x, y \in S^1$. Then S is *primitive* if all non–zero elements of S are minimal under $\leq$. If S is abundant (and so of course if S is regular) this coincides with the definition given above [L].

Finally in this section we remark that to consider only semigroups with zero, here and in the sequel [FG], is no real restriction. For defining a semigroup T without zero to be left idempotent bounded if T is the disjoint union of idempotent generated principal left ideals, it is clear that T^0 is a LIB C–semigroup. But the structure of T is immediately deducible from that of T^0, since T^0 has no zero–divisors.

3 Examples

In this section we present several examples of LIB-semigroups to illustrate both the diversity of contexts in which they arise and the different properies they may have. Although some of the examples are IB-semigroups our main concern in this paper is with LIB-semigroups which are not RIB. In the sequel [FG] we give more examples of IB-semigroups.

For our first class of examples we use subsemigroups of the monoids of S-endomorphisms of certain left S-systems.

Let S be a monoid (with identity 1), let I be a non-empty set and for each $i \in I$ let A_i be an S-system of cardinality at least two. Let 0 be a symbol not in A_i: put $B_i = \{0\} \cup A_i$ and make B_i into a centred S-system by defining $s0 = 0$ for all $s \in S$. In the category of centred S-systems we form the coproduct $B = \bigcup^0 \{B_i : i \in I\}$ and let $\mathrm{End}_S B$ denote the monoid of S-endomorphisms of B. We now define subsemigroups T and V of $\mathrm{End}_S B$ as follows:

$$T = \{\alpha \in \mathrm{End}_S B : B\alpha \subseteq B_i \text{ for some } i\},$$

$$V = \{\alpha \in T : \text{ for each } j \in I, \text{ either } A_j\alpha = 0 \text{ or } A_j\alpha \cap 0 = \emptyset\}.$$

It is easy to see that T and V actually are subsemigroups; we shall show that they are both LIB-semigroups but not, in general, RIB-semigroups. Further, although V is always a C-semigroup this is not the case for T.

For each $i \in I$, let $\epsilon_i : B \to B$ be defined by $x\epsilon_i = x$ if x is in B_i and $x\epsilon_i = 0$ otherwise. Then ϵ_i is in V and for any α in T with $B\alpha \subseteq B_i$ it is clear that $\alpha = \alpha\epsilon_i$. It is easy to see that $T\epsilon_i \cap T\epsilon_j = 0$ when $i \neq j$. Thus T (and consequently also V) is a 0-direct union of idempotent-generated left ideals, that is, T and V are indeed LIB.

Now suppose that for some i, j in I with $i \neq j$ there are S-homomorphisms $\alpha_j : A_j \to A_i$ and $\alpha_i : A_i \to A_i$ such that $\mathrm{Im}\alpha_i \cap \mathrm{Im}\alpha_j$ is properly contained in both $\mathrm{Im}\alpha_i$ and $\mathrm{Im}\alpha_j$. Now define $\alpha : B \to B$ as follows:-

$$x\alpha = \begin{cases} x\alpha_i & \text{if } x \in A_i \\ x\alpha_j & \text{if } x \in A_j \\ 0 & \text{otherwise.} \end{cases}$$

Then α is a member of V. Let ϵ be an idempotent of T and suppose that $\epsilon\alpha = \alpha$. Let $\mathrm{Im}\epsilon$ be contained in B_k. If k is neither i nor j, then we get $\epsilon\alpha = 0$, a contradiction. If $k = i$, then for all x in A_j, since we cannot have that $x\epsilon = 0$, it follows that

$$x\alpha_j = x\alpha = x(\epsilon\alpha) = (x\epsilon)\alpha_i,$$

which is a contradiction since $\mathrm{Im}\alpha_j$ is not contained in $\mathrm{Im}\alpha_i$. Similarly, for $k = j$ we obtain the contradiction that $\mathrm{Im}\alpha_i$ is contained in $\mathrm{Im}\alpha_j$. Thus $\alpha \neq \epsilon\alpha$ for any idempotent ϵ of T so that neither T nor V is RIB. The conditions used to obtain this conclusion are often satisfied . For example, if we take A_j to be a free S-system and A_i to be any S-system containing two disjoint S-systems one of which is a homomorphic image of A_i, then appropriate homomorphisms α_i and α_j certainly exist.

Turning to categoricity at zero, we start with V. Let $\alpha, \beta, \gamma \in V$ be such that $\alpha\beta \neq 0$ and $\beta\gamma \neq 0$. Let $\mathrm{Im}\beta$ be contained in B_i and let $x \in B$ be such that $x\alpha\beta \neq 0$. Now $\beta\gamma \neq 0$ so that $A_i\gamma \neq 0$ and hence $A_i\gamma$ is contained in A_j for some j. This gives $x\alpha\beta\gamma \in A_j$ so that $\alpha\beta\gamma \neq 0$ and V is a C-semigroup.

By way of contrast, we consider the semigroup T in the following situation: for some i the S-system A_i can be written as a disjoint union of subsystems C_i, D_i and there is a S-homomorphism $\theta : A_j \to C_i$ for some j. Then we define endomorphisms α, β of B by putting $x\alpha = x\theta$ if $x \in A_j$, $x\alpha = 0$ if $x \notin A_j$ and $x\beta = x$ if $x \in D_i$, $x\beta = 0$ if $x \notin D_i$. Certainly, α, β are members of T and $\alpha\epsilon_i\beta = \alpha\beta = 0$ but $\alpha\epsilon_i = \alpha$ and $\epsilon_i\beta = \beta$ whence T is not a C-semigroup.

Next we consider two examples where V is not E-primitive, one being right abundant and the other not. First, suppose that $S = \{1\}$ so that each A_i

is simply a set and every mapping from B to B which leaves 0 fixed is an endomorphism. Let α be a non-zero element of V and suppose that $\mathrm{Im}\alpha$ is contained in B_i. Let $a \in \mathrm{Im}\alpha, a \neq 0$. Define $\epsilon : B \to B$ by putting $x\epsilon = 0$ if $x \notin A_i, x\epsilon = a$ if $x \in A_i \backslash \mathrm{Im}\alpha$ and $x\epsilon = x$ if $x \in \mathrm{Im}\alpha$. Clearly, ϵ is in V, $\epsilon^2 = \epsilon$ and $\alpha\epsilon = \alpha$. If $\alpha\beta = \alpha\gamma$ where $\beta, \gamma \in V^1$, then $\beta|_{\mathrm{Im}\alpha} = \gamma|_{\mathrm{Im}\alpha}$ and since $\mathrm{Im}\alpha = \mathrm{Im}\epsilon$, it follows that $\epsilon\beta = \epsilon\gamma$. Thus $\alpha\mathcal{L}^*\epsilon$ so that every element of V is $\mathcal{L}^*$-related to an idempotent and V is right abundant. Since we are assuming that each A_i has at least two elements, it is easy to see that V is not E-primitive.

To obtain an example where V is not right abundant we put $I = \{1, 2, 3\}$ and let A_1, A_2 be totally ordered sets where $\theta : A_1 \to A_2$ is an order-isomorphism onto a proper dense subset of A_2. Of course, we may make A_1, A_2 into semigroups by taking the product of two elements to be the minimum of the two. Then θ is also a semigroup homomorphism and putting $S = A_1^1$ we make A_1, A_2 into S-systems in the obvious way. We now put $A_3 = \{a, b\}$ and let $t \in A_2 \setminus A_1\theta$. For $x \in A_1$ we define $xa = a$ and $xb = a$ if $x\theta < t$, $xb = b$ if $x\theta > t$. Then A_3 is an S-system and we define α, β, γ in V as follows:

$$x\alpha = \begin{cases} x\theta & \text{if } x \in A_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} a & \text{if } x \in A_2, x < t \\ b & \text{if } x \in A_2, x \geq t \\ 0 & \text{otherwise} \end{cases}$$

$$x\gamma = \begin{cases} a & \text{if } x \in A_2, x \leq t \\ b & \text{if } x \in A_2, x > t \\ 0 & \text{otherwise} \end{cases}$$

Then $\alpha\beta = \alpha\gamma$. Let ϵ be an idempotent in V such that $\alpha\epsilon = \alpha$. Then $(x\theta)\epsilon = x\theta$ for all x in A_1 and, of course, the image of $\epsilon|_{A_2}$ is contained in A_2. Let $x, y \in A_2$ with $x < y$; then $x < z\theta < y$ for some z in A_1. Hence

$x\epsilon = ((z\theta)x)\epsilon = (z.x)\epsilon = z.(x\epsilon) = (z\theta)(x\epsilon) \leq z\theta = (z\theta)\epsilon = ((z\theta)y)\epsilon = (z.y)\epsilon = z.(y\epsilon) = (z\theta)(y\epsilon) \leq y\epsilon$

so that $\epsilon|_{A_2}$ is order-preserving. Since $A_1\theta$ is dense in A_2 it follows that $\epsilon|_{A_2}$ is the identity map on A_2. Hence $\epsilon\beta \neq \epsilon\gamma$ and so α is not $\mathcal{L}^*$-related to any idempotent. Again, it is easy to see that V is E-primitive.

To produce an E-primitive LIB C-semigroup V which is neither RIB nor right abundant, we let S be the free monoid on two generators s, t. Let $A = \{z, p, y_1, y_2, y_3, ..., x_1, x_{21}, x_{22}, x_3, x_4, ...\}$ be a set of distinct symbols. For an element v of S let $|v|$ denote the length of v. Now define a monoid action of S on A inductively as follows:-

$$sz = x_1, sp = x_{21}, sy_k = y_{k+1}, sx_1 = x_{21},$$

$$sx_{21} = sx_{22} = x_3,$$

$$tz = y_1, tp = y_2, ty_k = y_{k+1}, tx_1 = x_{22},$$

$$tx_{21} = tx_{22} = x_3$$

and for $k \geq 3$

$$sx_k = x_{k+1}, tx_k = x_{k+1}$$

and for $v \in S$ with $2 \leq |v|$, and $a \in A$, va is obtained by using the above rules $|v|$ times.

Clearly we do have an action and so given a set I with at least two elements we can let A_i be an S-system isomorphic to A for each $i \in I$; then we obtain an S-system B and an LIB C-semigroup V.

We are interested in S-endomorphisms of A and we note first that if $\theta \in \mathrm{End}_S A$, then $y_k\theta, x_h\theta, x_{21}\theta$ and $x_{22}\theta$ are determined by $z\theta$. Thus θ is specified by choosing $z\theta$ and $p\theta$ appropriately. We now define three endomorphisms α, β, γ of A as follows:-

$$z\alpha = y_1, p\alpha = y_2; z\beta = x_1, p\beta = x_{21}; z\gamma = x_1, p\gamma = x_{22}.$$

It is easy to see that these prescriptions do indeed define endomorphisms. Notice that $\alpha\beta = \alpha\gamma$. Next let ϵ be an idempotent endomorphism of A. If $z\epsilon = vz$ for some $v \in S$, then $vz = (vz)\epsilon = v(z\epsilon) = v^z$, then $|v| = |v^2|$, from which it follows that $v = 1$. The only other possibility is $z\epsilon = p$ but this leads to a contradiction : $y_2 = tp = t(z\epsilon) = (tz)\epsilon = y_1\epsilon$ so that $y_2 = y_2\epsilon = (t^2z)\epsilon = t^2(z\epsilon) = t^2p = y_3$. Hence $z\epsilon = z$ and it follows that $a\epsilon = a$ for $a \in A \setminus \{p\}$. But $x_{21} = x_{21}\epsilon = (sp)\epsilon = s(p\epsilon)$ and $y_2 = y_2\epsilon = (tp)\epsilon = t(p\epsilon)$ and it follows that $p\epsilon = p$. Thus the identity map is the only idempotent in $\mathrm{End}_S A$.

From the fact that $\mathrm{Im}\alpha$ and $\mathrm{Im}\beta$ are disjoint it follows that if i, j are distinct members of I, then we can find S-homomorphisms $\alpha_j : A_j \to A_i$ and $\alpha_i : A_i \to A_i$ such that neither of $\mathrm{Im}\alpha_j$, $\mathrm{Im}\alpha_i$ is contained in the other. Hence V is not RIB.

Now if η, ϵ are non-zero idempotents of V such that $\epsilon\eta = \eta = \eta\epsilon$, then we must have both $\mathrm{Im}\eta$ and $\mathrm{Im}\epsilon$ contained in B_i for some i. But ϵ, η are in V, are non-zero and idempotent and consequently $A_i\epsilon \subseteq A_i$ and $A_i\eta \subseteq A_i$. It follows from the previous paragraph that the restriction of either ϵ or η to A_i is the identity map. Hence $\epsilon = \epsilon\eta$ and so $\epsilon = \eta$. Thus V is E-primitive.

Let $i \in I$ and identify A_i with A. Then we can extend the endomorphisms α, β, γ of A to endomorphisms of B by putting $x\alpha = x\beta = x\gamma = 0$ for $x \in B\backslash A$. Then α, β, γ are in V and we still have $\alpha\beta = \alpha\gamma$. If ϵ is an idempotent of V such that $\alpha\epsilon = \alpha$, then Im$\epsilon \subseteq B_i$ and ϵ is non-zero so that ϵ is the identity on A. Hence $p\epsilon\beta = x_{21}$ but $p\epsilon\gamma = x_{22}$ so that $\epsilon\beta \neq \epsilon\gamma$. Thus α is not $\mathcal{L}^*$-related to any idempotent and V is not right abundant.

Next we use endomorphisms of groups to give some examples of primitive right abundant semigroups which are not left abundant. For these examples we change to writing functions on the left of their arguments. For any group G we can define a subsemigroup S of End(G) as follows:-

$$S = \{\alpha \in \text{End}(G) : \text{Im}\alpha \text{ is cyclic } \}.$$

We now restrict our attention to groups G of the following type. Let I be an index set with at least two elements and for each $i \in I$ let A_i be a torsion-free group which is (relatively) free in some variety on (relatively) free generators X_i. Take G to be the free product $*A_i$ and let $Y = \bigcup\{X_i : i \in I\}$. Given any element g of G and any function $\alpha : Y \to < g >$, there is a unique element α of S whose restriction to Y is the given α. We denote the trivial endomorphism by 0.

let η_1, η_2 be non-zero idempotents of S such that $\eta_1\eta_2 = \eta_2 = \eta_2\eta_1$. Then Im$\eta_2$ =Im$\eta_1\eta_2$ $\subseteq$Imη_1. Hence, if Im$\eta_i =< w_i >$ for $i = 1, 2$, then $w_2 = w_1^k$ for some non-zero integer k. Since G is torsion-free and $\eta_2(w_2) = w_2$, then if we put $\eta_2(w_1) = w_2^j$ it follows that $jk = 1$ and Imη_1 =Imη_2. Then $\eta_2\eta_1 = \eta_1$ so that $\eta_1 = \eta_2$. Thus all non-zero idempotents of S are primitive.

Let $\gamma \in S$ and let Im$\gamma =< w >$. Let $v \in G$ be such that $\gamma(v) = w$ and for each y in Y, let $\gamma(y) = w^{m(y)}$. Now let $\epsilon \in S$ be determined by putting $\epsilon(y) = v^{m(y)}$ for each $y \in Y$. Since Y generates G, it is easy to see that for $u \in G$, if $\gamma(u) = w^k$, then $\epsilon(u) = v^k$. In particular, $\epsilon(v) = v$ and so Im$\epsilon =< v >$ and $\epsilon = \epsilon^2$. Furthermore, for each y in Y,

$$\gamma\epsilon(y) = \gamma(v^{m(y)}) = w^{m(y)} = \gamma(y)$$

and it follows that $\gamma\epsilon = \gamma$. If $\theta, \phi \in S$ and $\gamma\theta = \gamma\phi$, then for all $y \in Y$, $\gamma\theta(y) = \gamma\phi(y) = w^{t(y)}$ for some integer $t(y)$. Then$\epsilon\theta(y) = v^{t(y)} = \epsilon\phi(y)$ and consequently, $\epsilon\theta = \epsilon\phi$. Similarly, if $\gamma\theta = \gamma$, then $\epsilon\theta = \epsilon$. Thus $\gamma\mathcal{L}^*\epsilon$ and so S is right abundant.

However, S is not left abundant. Since $2 \leq |I|$, there are elements a, b in G which do not commute. Let $c = aba^{-1}b^{-1}$. Let α be the element of S determined by putting $\alpha(y) = c$ for all $y \in Y$. Then $\alpha \neq 0$ but if $\beta \in S$, then $\beta(a), \beta(b)$ must commute so that $\beta(c) = 1$ and $\beta\alpha = 0$. Hence α does not belong to any idempotent generated right ideal of S.

In the case when each A_i is the infinite cyclic group so that G is a free group, the semigroup S has been studied in [FP].

Now we give examples of indecomposable PRA semigroups which are not left abundant. Let D be an integral domain with identity 1. Let E_{ij} denote the $n \times n$ matrix with 1 in the i, j position and 0 elsewhere. Let S be the left ideal of $M_n(D)$ defined by $S = \sum_{i=1}^{n} DE_{i1}$, that is, S consists of those matrices whose only non-zero entries are in the first column. It is routine to verify that $S = SE_{11}$, that every element of S is $\mathcal{L}^*$-related to E_{11} and that the non-zero idempotents of S are primitive. However, S is not left abundant because the right annihilator of S is non-zero.

Finally, we construct examples to show that even with just one idempotent, LIB C-semigroups and PRA semigroups can have a complicated structure. Let Y be a semilattice and for each β in Y, let F_β be a free semigroup. Let $S = \bigcup\{F_\beta : \beta \in Y\}$ be the semilattice of the F_β where for $a \in F_\beta, b \in F_\gamma$, with $\beta, \gamma \in Y$ and $\gamma < \beta$ we define $ab = b = ba$. Now let I be a non-empty set and $\{S_i : i \in I\}$ be a disjoint set of semigroups each isomorphic to S. For $i \in I$, let $\theta_i : S_i \to S$ be an isomorphism and let 0 be a fixed element of I. Let e be a symbol not in $T = \bigcup\{S_i : i \in I\}$. Define a multiplication $\cdot$ on $V = \{e\} \cup T$ as follows: $e^2 = e$ and for $s_i \in S_i, s_j \in S_j$,

$$s_i \cdot s_j = ((s_i\theta_i)(s_j\theta_j)\theta_0^{-1}, e \cdot s_j = (s_j\theta_j)\theta_0^{-1}, s_j \cdot e = s_j.$$

A straightforward verification shows that V is a semigroup; clearly e is the only idempotent of V and $Ve = V$. Thus V is LIB but V is not RIB since $eV = S_0 \cup \{e\}$. When $1 < |Y|$ and $1 < |I|$, V is not a PRA semigroup for, if $\beta, \gamma \in Y$ with $\beta < \gamma$ and $a \in F_\beta, b, c \in F_\gamma$, with $b \neq c$, then $ab = a = ac$ so that if $i \in I, i \neq 0$, then $(a\theta_i^{-1}) \cdot (b\theta_i^{-1}) = (a\theta_i^{-1}) \cdot (c\theta_i^{-1})$ but $e \cdot (b\theta_i^{-1}) = b\theta_0^{-1} \neq c\theta_0^{-1} = e \cdot (c\theta_i^{-1})$.

In the case when $|Y| = 1$, however, it is easy to show that every element of V is $\mathcal{L}^*$-related to e so that V is a PRA semigroup.

4 Left blocked Rees matrix semigroups

In this section we show how to construct a class of C–semigroups from specific ingredients. We call semigroups obtained in this manner left blocked Rees matrix semigroups or LB Rees matrix semigroups. We then give sufficient conditions on a LB Rees matrix semigroup for it to be left idempotent bounded. An additional condition is introduced which enables us to locate the $\mathcal{L}'$–generators.

Consider the following :

I, Λ, Γ	non–empty sets
$P(\Lambda) = \{\Lambda_\gamma : \gamma \in \Gamma\}$	partition of Λ indexed by Γ
$\{M_{i\alpha} : i \in I, \alpha \in \Gamma\}$	set of pairwise–disjoint sets

and a function $s : \Gamma \to I$. Let $M = \bigcup\{M_{i\alpha} : i \in I, \alpha \in \Gamma\}$ and for $i \in I$ put $U_i = \bigcup\{M_{i\alpha} : \alpha \in \Gamma\}$. We suppose there is a partial binary operation on M such that xy is defined if and only if $x \in M_{i\alpha}$ and $y \in M_{s(\alpha)\beta}$ for some i, α, β and then, if defined, $xy \in M_{i\beta}$. Further, we assume that under this multiplication M is a partial semigroup.

Let 0 be a symbol not in M and let P be a $\Lambda \times I$ matrix such that for any $(\lambda, i) \in \Lambda \times I, p_{\lambda i} = 0$ or $p_{\lambda i}$ is a function $p_{\lambda i} : U_i \to U_{s(\alpha)}$, where $\lambda \in \Lambda_\alpha$. In the latter case, $p_{\lambda i} \mid M_{i\beta} : M_{i\beta} \to M_{s(\alpha)\beta}$ for all $\beta \in \Gamma$ where $M_{i\beta} \neq \emptyset$; also, for any $\beta, \gamma \in \Gamma, x \in M_{i\beta}$ and $y \in M_{s(\beta)\gamma}$, we have $p_{\lambda i}(x)y = p_{\lambda i}(xy)$.

Let $\mathcal{LM}^0 = \mathcal{LM}^0(M; I; \Lambda, \Gamma; P) =$

$$= \{(i, u, \lambda) : i \in I, \alpha \in \Gamma, \lambda \in \Lambda_\alpha, u \in M_{i\alpha}\} \cup \{0\}.$$

Define a product on $\mathcal{LM}^0$ by

$$(i, u, \lambda)(j, v, \mu) = \begin{cases} (i, up_{\lambda j}(v), \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

$$0x = x0 = 0 \text{ for all } x \in \mathcal{LM}^0.$$

Proposition 4.1 *With the given multiplication, $\mathcal{LM}^0$ is a C–semigroup.*

Proof Note first that if $(i, u, \lambda)(j, v, \mu) \neq 0$ then since $u \in M_{i\alpha}, v \in M_{j\beta}$ where $\lambda \in \Lambda_\alpha$ and $\mu \in \Lambda_\beta$, we have that $p_{\lambda j}(v) \in M_{s(\alpha)\beta}$ so that $up_{\lambda j}(v)$ is defined and lies in $M_{i\beta}$, giving that $(i, up_{\lambda j}(v), \mu) \in \mathcal{LM}^0$.

To see that the product is associative, consider non–zero elements $(i, u, \lambda), (j, v, \mu), (k, w, \nu)$ of $\mathcal{LM}^0$, where $\lambda \in \Lambda_\alpha, \mu \in \Lambda_\beta$ and $\nu \in \Lambda_\gamma$. Put

$$X = (u, u, \lambda)((j, v, \mu)(k, w, \nu))$$

and

$$Y = ((i,u,\lambda)(j,v,\mu))(k,w,\nu).$$

Clearly, $X = Y = 0$ if and only if at least one of $p_{\lambda j}$ and $p_{\mu k}$ is zero. Suppose therefore that $p_{\lambda j}$ and $p_{\mu k}$ are non–zero. Then

$$X = (i,u,\lambda)(j, vp_{\mu k}(w),\nu) = (i, up_{\lambda j}(vp_{\mu k}(w)),\nu).$$

Using a property of the function $p_{\lambda j}$ we have that

$$X = (i, u(p_{\lambda j}(v)p_{\mu k}(w)),\nu) = (i,(up_{\lambda j}(v))p_{\mu k}(w),\nu) =$$

$$= (i, up_{\lambda j}(v),\mu)(k,w,\nu) = ((i,u,\lambda)(j,v,\mu))(k,w,\nu) = Y.$$

Thus $\mathcal{LM}^0$ is a semigroup that is clearly categorical at zero.

A semigroup $\mathcal{LM}^0$ constructed as above is called a *left blocked Rees matrix semigroup* or a *LB Rees matrix semigroup.* The next step is to impoose conditions on $\mathcal{LM}^0$ which ensure it is left idempotent bounded. These we list below. Only LIB(i) and (ii) are strictly necessary, however, LIB(iii) ensures that I indexes the non–zero $\mathcal{R}'$–classes and LIB(iv) allows one to handle $\mathcal{L}'$–generators more easily.

LIB(i) For $\alpha \in \Gamma, M_{s(\alpha)\alpha}$ contains an idempotent g_α such that $xg_\alpha = x$ for all $i \in I$ and $x \in M_{i\alpha}$.

LIB(ii) For each $\alpha \in \Gamma$ and $\lambda \in \Lambda_\alpha$ there exists $i \in I$ with $p_{\lambda i} \neq 0$ and $u \in M_{i\alpha}$ with $p_{\lambda i}(u) = g_\alpha$.

LIB(iii) The $\mathcal{R}'$–classes of M are $\{U_i : i \in I\}$.

LIB(iv) Given $\alpha, \beta \in \Gamma$ and $a \in M_{s(\alpha)\beta}, b \in M_{s(\beta)\alpha}$ where $aba = a, g_\alpha ab = g_\alpha$ and $g_\beta ba = g_\beta$, then $\alpha = \beta$.

Proposition 4.2 *Let $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$ be a LB Rees matrix semigroup satisfying LIB(i),(ii),(iii) and (iv). Then $\mathcal{LM}^0$ is a LIB semigroup. Further :*
(i) an element (i,a,λ) is idempotent if and only if $p_{\lambda i} \neq 0$ and $ap_{\lambda i}(a) = a$;
(ii) the elements $(i,a,\lambda),(j,b,\mu)$ are $\mathcal{R}'(\mathcal{L}')$–related if and only if $i = j(\lambda = \mu)$;
(iii) an element (i,u,λ) where $\lambda \in \Lambda_\alpha$ is an $\mathcal{L}'$–generator if and only if $g_\alpha p_{\lambda i}(u) = g_\alpha$;
(iv) $\mathcal{L}'$–generators $(i,a,\lambda),(j,b,\mu)$ where $\lambda \in \Lambda_\alpha, \mu \in \Lambda_\beta$ are $\mathcal{D}$–related if and only if $\alpha = \beta$.

Proof We remark that (i) is straightforward. Let $\alpha \in \Gamma$ and $\lambda \in \Lambda_\alpha$. By LIB(ii) there exists $i \in I$ and $u \in M_{i\alpha}$ with $p_{\lambda i}(u) = g_\alpha$. Then $(i,u,\lambda) \in \mathcal{LM}^0$ and given any $(j,a,\lambda) \in \mathcal{LM}^0$, $(j,a,\lambda)(i,u,\lambda) = (j, ap_{\lambda i}(u), \lambda) = (j, ag_\alpha, \lambda) = (j,a,\lambda)$, using LIB(i). Thus if $\overline{L}'_\lambda = \{(j,a,\lambda) : j \in I, a \in M_{j\alpha}\}$ then $\overline{L}'_\lambda \cup \{0\}$ is a left ideal generated by the idempotent (i,u,λ). It follows that $\mathcal{LM}^0$ is left idempotent bounded and the non–zero $\mathcal{L}'$–classes are the sets $\overline{L}'_\lambda = L'_\lambda, \lambda \in \Lambda$.

Consider now an element (k,v,λ) where $\lambda \in \Lambda_\alpha$. If (k,v,λ) is an $\mathcal{L}'$–generator then $(s(\alpha), g_\alpha, \lambda)(k,v,\lambda) = (s(\alpha), g_\alpha, \lambda)$ and so $g_\alpha p_{\lambda k}(v) = g_\alpha$. Conversely, if $g_\alpha p_{\lambda k}(v) = g_\alpha$ then given any $(j,a,\lambda) \in L'_\lambda, (j,a,\lambda)(k,v,\lambda) = (j, ap_{\lambda k}(v), \lambda) = (j, (ag_\alpha)p_{\lambda k}(v), \lambda) = (j, a(g_\alpha p_{\lambda k}(v)), \lambda) = (j, ag_\alpha, \lambda) = (j,a,\lambda)$ and so (k,v,λ) is an $\mathcal{L}'$–generator. This shows that (iii) holds.

Now let $(i,a,\lambda),(j,b,\mu) \in \mathcal{LM}^0$ where $\lambda \in \Lambda_\alpha$ and $\nu \in \Lambda_\beta$. It is clear that if (i,a,λ) and (j,b,μ) are $\mathcal{R}'$–related, then $i = j$. Conversely, suppose that $i = j$. Then $a, b \in U_i$ and so by LIB(iii),$a\mathcal{R}'b$ in M. Thus there are elements $a = a_1, a_2, ..., a_n = b$ in U_i with $a_xM^1 \cap a_{x+1}M^1 \neq \emptyset, 1 \leq x \leq n-1$. In view of LIB(i),we may choose $s_x, t_x \in M$ with $a_xs_x = a_{x+1}t_x, 1 \leq x \leq n-1$. Let $\alpha_1, ..., \alpha_n$ be the elements of Γ such that $a_x \in M_{i\alpha_x}, 1 \leq x \leq n$, so that in particular, $\alpha_1 = \alpha$ and $\alpha_n = \beta$. Note that if $x \in \{1, ..., n-1\}$, then $s_x \in M_{s(\alpha_x)\gamma_x}, t_x \in M_{s(\alpha_{x+1})\gamma_x}$ for some $\gamma_x \in \Gamma$. Let $\mu_1 = \lambda, \mu_n = \mu$ and choose and fix $\mu_x \in \Lambda_{\alpha_x}, x \in \{2, ..., n-1\}$; also, choose and fix $\nu_x \in \Lambda_{\gamma_x}, x \in \{1, ..., n-1\}$. By LIB(ii), for each $x \in \{1, ..., n\}$ there exists $i_x \in I$ with $p_{\mu_x i_x} \neq 0$ and $p_{\mu_x i_x}(u_x) = g_{\alpha_x}$ for some $u_x \in M_{i_x\alpha_x}$. Then $(i_x, u_xs_x, \nu_x), (i_{x+1}, u_{x+1}t_x, \nu_x) \in \mathcal{LM}^0$ for $x \in \{1, ..., n-1\}$ and $(i, a_x, \mu_x) \in \mathcal{LM}^0$ for $x \in \{2, ..., n-1\}$. Computing gives that

$$(i,a,\lambda)(i_1, u_1s_1, \nu_1) = (i, a_2, \mu_2)(i_2, u_2t_1, \nu_1),$$

$$(i, a_x, \mu_x)(i_x, u_xs_x, \nu_x) = (i, a_{x+1}, \mu_{x+1})(i_{x+1}, u_{x+1}t_x, \nu_x)$$

for $x \in \{2, ..., n-1\}$ and

$$(i, a_{n-1}, \mu_{n-1})(i_{n-1}, u_{n-1}s_{n-1}, \nu_{n-1}) = (i, b, \mu)(i_n, u_nt_{n-1}, \nu_{n-1}).$$

Thus $(i,a,\lambda)\mathcal{R}'(j,b,\mu)$ in $\mathcal{LM}^0$ as required.

To show that (iv) holds, let $e = (i,a,\lambda), f = (j,b,\mu)$ be $\mathcal{L}'$–generators with $\lambda \in \Lambda_\alpha, \mu \in \Lambda_\beta$. Suppose that $e\mathcal{D}f$. In view of Proposition II 3.6 of [H], there is an element h in $\mathcal{LM}^0$ with inverse h' such that $e = hh'$ and $h'h = f$. It is clear that $h = (i,c,\mu)$ and $h' = (j,d,\lambda)$ for some $c \in M_{i\beta}, d \in M_{j\alpha}$. From $h = hh'h$

we obtain $c = cp_{\mu j}(dp_{\lambda i}(c)) = cp_{\mu j}(d)p_{\lambda i}(c)$, giving $p_{\lambda i}(c) = p_{\lambda i}(c)p_{\mu j}(d)p_{\lambda i}(c)$. Now $p_{\lambda i}(c) \in M_{s(\alpha)\beta}$ and $p_{\mu j}(d) \in M_{s(\beta)\alpha}$. Further, $e = ehh'$ gives that

$$(i, a, \lambda) = (i, a, \lambda)(i, cp_{\mu j}(d), \lambda) = (i, ap_{\lambda i}(c)p_{\mu j}(d), \lambda)$$

and so using (iii),

$$g_\alpha = g_\alpha p_{\lambda i}(a) = g_\alpha p_{\lambda i}(a)p_{\lambda i}(c)p_{\mu j}(d) = g_\alpha p_{\lambda i}(c)p_{\mu j}(d).$$

Similarly, $g_\beta = g_\beta p_{\mu j}(d)p_{\lambda i}(c)$, so that by LIB(iv), $\alpha = \beta$.

Conversely, consider $\mathcal{L}'$–generators e, f as above and assume that $\alpha = \beta$. Then $(i, a, \mu), (j, b, \lambda) \in \mathcal{LM}^0$ and a straightforward computation gives that $(i, a, \lambda)(i, a, \mu) = (i, a, \mu)$ and $(i, a, \mu)(j, b, \lambda) = (i, a, \lambda)$ so that $(i, a, \lambda)\mathcal{R}'(i, a, \mu)$. Similarly, $(i, a, \mu)(j, b, \mu) = (i, a, \mu)$ and $(j, b, \lambda)(i, a, \mu) = (j, b, \mu)$ so that $(i, a, \mu)\mathcal{L}(j, b, \mu)$. Thus $e\mathcal{D}f$.

We draw attention to the fact that if $\mathcal{LM}^0$ is as in Proposition 4.2 and $e = (i, a, \lambda)$ is an $\mathcal{L}'$–generator, then $L'_\lambda = (\mathcal{LM}^0(i, a, \lambda))^*$. A LB Rees matrix semigroup $\mathcal{LM}^0$ satisfying LIB(i),(ii),(iii) and (iv) is called a LIB–LB Rees matrix semigroup. The justification for this terminology comes from Proposition 4.2, where we show such semigroups are certainly left idempotent bounded, and Theorem 5.2, which states that conversely, every left idempotent bounded C–semigroup is isomorphic to a LB Rees matrix semigroup satisfying LIB(i),(ii),(iii) and (iv).

Having shown how to construct LIB–LB Rees matrix semigroups, the question naturally arises of determining when two such semigroups are isomorphic.

Proposition 4.3 *Let $\mathcal{LM}^0_i = \mathcal{LM}^0_i(M^i; I^i, \Lambda^i, \Gamma^i; P^i)$, $i = 1, 2$ be LIB–LB Rees matrix semigroups. Then $\mathcal{LM}^0_1$ is isomorphic to $\mathcal{LM}^0_2$ if and only if there are bijections $\phi : I^1 \to I^2, \psi : \Lambda^1 \to \Lambda^2, \rho : \Gamma^1 \to \Gamma^2$ such that*
(i) if $\alpha \in \Gamma^1$ and $\lambda \in \Lambda^1_\alpha$, then $\psi(\lambda) \in \Lambda^2_{\rho(\alpha)}$;
(ii) for any $i \in I^1, \alpha \in \Lambda^1, M^1_{i\alpha} \neq \emptyset$ if and only if $M^2_{\phi(i)\rho(\alpha)} \neq \emptyset$ and if $M^1_{i\alpha} \neq \emptyset$ then for each $\lambda \in \Lambda_\alpha$ there is a bijection $\nu_{i\lambda} : M^1_{i\alpha} \to M^2_{\phi(i)\rho(\alpha)}$;
(iii) for all $i \in I^1, \alpha \in \Gamma^1, p^1_{\lambda i} = 0$ if and only if $p^2_{\psi(\lambda)\phi(i)} = 0$;
(iv) for all $j \in I^1, \alpha \in \Gamma^1, \lambda \in \Lambda^1_\alpha$, if $p^1_{\lambda j} \neq 0$ then

$$\nu_{i\mu}(ap^1_{\lambda j}(b)) = \nu_{i\lambda}(a)p^2_{\psi(\lambda)\phi(j)}(\nu_{j\mu}(b))$$

for all $a \in M^1_{i\alpha}, b \in M^1_{j\beta}, \mu \in \Lambda^1_\beta$.

Proof Suppose that $\theta : \mathcal{LM}_1^0 \to \mathcal{LM}_2^0$ is an isomorphism. Clearly θ induces a bijection between the $\mathcal{R}'$–classes of $\mathcal{LM}_1^0$ and $\mathcal{LM}_2^0$ and so $\phi : I^1 \to I^2$ is a bijection, where for $i \in I$ and for some (any) $(i, a, \lambda) \in \mathcal{LM}_1^0, \theta(i, a, \lambda) = (\phi(i), a', \lambda')$. Similarly, θ induces a bijection between the $\mathcal{L}'$–classes of $\mathcal{LM}_1^0$ and $\mathcal{LM}_2^0$, so that $\psi : \Lambda^1 \to \Lambda^2$ is a bijection where for some (any) $(i, a, \lambda) \in \mathcal{LM}_1^0, \theta(i, a, \lambda) = (\phi(i), a', \psi(\lambda))$.

For $\alpha \in \Gamma^1$ let $\rho(\alpha) \in \Gamma^2$ be such that $\psi(\lambda) \in \Lambda^2_{\rho(\alpha)}$ for some $\lambda \in \Lambda^1_\alpha$. To see that ρ is well defined, suppose that $\lambda, \mu \in \Lambda^1_\alpha$. In view of Proposition 4.2 there are $\mathcal{L}'$–generators $f_\lambda = (i, a, \lambda)$ and $f_\mu = (j, b, \mu)$ in $\mathcal{LM}_1^0$; (iv) of that propostion yields $f_\lambda \mathcal{D} f_\mu$. Certainly then $\theta(f_\lambda) = (\phi(i), a', \psi(\lambda))$ and $\theta(f_\mu) = (\phi(j), b', \psi(\mu))$ are $\mathcal{D}$–related $\mathcal{L}'$–generators in $\mathcal{LM}_2^0$. Thus $\psi(\lambda), \psi(\mu) \in \Lambda^2_\nu$ for some $\nu \in \Gamma^2$ and so ρ is well defined. A similar argument shows that ρ is an injection. Then ρ is a bijection, for given any $\xi \in \Gamma^2, \Lambda^2_\xi$ is non–empty so that $\psi(\lambda) \in \Lambda^2_\xi$ for some $\lambda \in \Lambda^1$, say $\lambda \in \Lambda^1_\alpha$, and then $\rho(\alpha) = \xi$. Note that (i) holds in view of the definition of ρ.

Suppose now that $i \in I^1, \alpha \in \Gamma^1$ and $M^1_{i\alpha} \neq \emptyset$. Then $(i, a, \lambda) \in \mathcal{LM}_1^0$ for any $\lambda \in \Lambda^1_\alpha$ and for any $a \in M^1_{i\alpha}$. Writing $\theta(i, a, \lambda) = (\phi(i), \nu_{i\lambda}(a), \psi(\lambda)) \in \mathcal{LM}_2^0$, clearly $M^2_{\phi(i)\rho(\alpha)} \neq \emptyset$ and $\nu_{i\lambda} : M^1_{i\alpha} \to M^2_{\phi(i)\rho(\alpha)}$ is a bijection. Conversely, if $M^2_{\phi(i)\rho(\alpha)} \neq \emptyset$ then it is easy to see that $M^1_{i\alpha} \neq \emptyset$.

Let $i \in I^1, \lambda \in \Lambda^1$ and pick $(j, a, \lambda), (i, b, \mu) \in \mathcal{LM}_1^0$. Then $p^1_{\lambda i} = 0$ if and only if $(j, a, \lambda)(i, b, \mu) = 0$. But this is equivalent to $\theta((j, a, \lambda)(i, b, \mu)) = \theta(0) = 0$ and hence to $\theta(j, a, \lambda)\theta(i, b, \mu) = 0$. The latter statement is true if and only if $p^2_{\psi(\lambda)\phi(i)} = 0$.

To see that (iv) holds, let $j \in I^1, \alpha \in \Gamma^1, \lambda \in \Lambda^1_\alpha$ and suppose that $p^1_{\lambda j} \neq 0$. From the above, $p^2_{\psi(\lambda)\phi(j)} \neq 0$. Consider $a \in M^1_{i\alpha}, b \in M^1_{j\beta}$ and $\mu \in \Lambda^1_\beta$. Then $(i, a, \lambda), (j, b, \mu) \in \mathcal{LM}_1^0$ and since θ is a homomorphism we obtain

$$\nu_{i\lambda}(a) p^2_{\psi(\lambda)\phi(j)}(\nu_{j\mu}(b)) = \nu_{i\mu}(a p^1_{\lambda j}(b)).$$

Conversely, suppose that bijections $\phi : I^1 \to I^2, \psi : \Lambda^1 \to \Lambda^2$ and $\rho : \Gamma^1 \to \Gamma^2$ are given such that conditions (i),(ii),(iii) and (iv) hold. Define $\theta : \mathcal{LM}_1^0 \to \mathcal{LM}_2^0$ by $\theta(0) = 0$ and $\theta(i, a, \lambda) = (\phi(i), \nu_{i\lambda}(a), \psi(\lambda))$ for any non–zero element (i, a, λ) of $\mathcal{LM}_1^0$. A straightforward computation gives that θ is an isomorphism.

5 Left idempotent bounded C–semigroups

Throughout this section S denotes a given fixed LIB C–semigroup. Our aim, achieved in Theorem 5.2, is to show that S is isomorphic to a LIB Rees matrix semigroup satisfying conditions LIB(i),(ii),(iii) and (iv), that is, a LIB–LB Rees matrix semigroup.

Let $I(\Lambda)$ index the non–zero $\mathcal{R}'(\mathcal{L}')$–classes of S. Since S is LIB, for each $\lambda \in \Lambda$ we can choose and fix an $\mathcal{L}'$–generator f_λ, so that $L'_\lambda = (Sf_\lambda)^*$. Let Γ index the $\mathcal{D}$–classes of S that contain at least one $\mathcal{L}'$–generator.

For each $\gamma \in \Gamma$ put $\Lambda_\gamma = \{\lambda \in \Lambda : f_\lambda \in D_\gamma\}$, where D_γ is the $\mathcal{D}$–class indexed by γ, and choose and fix an idempotent $g_\gamma = f_{t(\gamma)} \in D_\gamma$. Note that $P(\Lambda) = \{\Lambda_\gamma : \gamma \in \Gamma\}$ is a partition of Λ and is independent of the choice of f_λ's.

Define $s : \Gamma \to I$ by $s(\gamma) = i$ where $g_\gamma \in R'_i$. For each $i \in I$ and $\gamma \in \Gamma$ we put $M_{i\gamma} = (R'_i \cap Sg_\gamma)^*$.

Lemma 5.1 *Given non–empty sets $M_{i\alpha}, M_{s(\alpha)\beta}$, then $M_{i\alpha}M_{s(\alpha)\beta} \subseteq M_{i\beta}$.*

Proof One has $M_{i\alpha} \subseteq L'_{t(\alpha)}$ and $M_{s(\alpha)\beta} \subseteq R'_{s(\alpha)}$. Now $g_\alpha \in L'_{t(\alpha)} \cap R'_{s(\alpha)}$ so that $L'_{t(\alpha)}R'_{s(\alpha)} \neq 0$. As noted in Section 2, it follows that if $a \in L'_{t(\alpha)}, b \in R'_{s(\alpha)}$, then $ab \neq 0$. Thus $M_{i\alpha}M_{s(\alpha)\beta} \subseteq (R'_i \cap Sg_\beta)^* = M_{i\beta}$.

Let $M = \bigcup\{M_{i\alpha} : i \in I, \alpha \in \Gamma\}$. In view of Lemma 5.1, M becomes a partial semigroup where the product of two elements x, y of M is defined if and only if $x \in M_{i\alpha}, y \in M_{s(\alpha)\beta}$ for some $i \in I$ and $\alpha, \beta \in \Gamma$ and where the product of elements is defined it is simply the product in S.

For each $\gamma \in \Gamma$ and $\lambda \in \Lambda_\gamma$ we have that $f_\lambda \mathcal{D} g_\gamma$ so that by Proposition II 3.6 of [H], we may pick $d_\lambda \in D_\gamma$ with inverse $d'_\lambda \in D_\gamma$ such that $f_\lambda = d_\lambda d'_\lambda$ and $g_\gamma = d'_\lambda d_\lambda$. Hence $f_\lambda \mathcal{R} d_\lambda \mathcal{L} g_\gamma$ and $f_\lambda \mathcal{L} d'_\lambda \mathcal{R} g_\gamma$. Define a $\Lambda \times I$ matrix P by

$$p_{\lambda i} = \begin{cases} \overline{\lambda} & \text{if } f_\lambda R'_i \neq 0 \\ 0 & \text{if } f_\lambda R'_i = 0 \end{cases}$$

where $\overline{\lambda} : U_i \to S$ is given by $\overline{\lambda}(x) = d'_\lambda x$. If $p_{\lambda i} \neq 0$ where $\lambda \in \Lambda_\gamma$ and $a \in M_{i\alpha}$ we have $f_\lambda a \neq 0$ and so $d'_\lambda a \neq 0$. Then $p_{\lambda i}(a) = \overline{\lambda}(a) = d'_\lambda a = g_\gamma d'_\lambda a \in M_{s(\gamma)\alpha}$. Clearly, if in addition $b \in M_{s(\alpha)\beta}$ for some $\beta \in \Gamma$ then $p_{\lambda i}(a)b = p_{\lambda i}(ab)$.

We now have all the ingredients necessary to form the semigroup $\mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$.

Theorem 5.2 *Any LB Rees matrix semigroup satisfying conditions LIB(i),(ii),(iii) and (iv) is a LIB C–semigroup.*

Conversely, given a LIB C–semigroup S then the LB Rees matrix semigroup $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$ constructed as above is isomorphic to S. Moreover, $\mathcal{LM}^0$ satisfies conditions LIB(i),(ii),(iii) and (iv).

Proof The first part of the statement is proved in Proposition 4.2. For the converse, suppose that S is a LIB C–semigroup and $\mathcal{LM}^0$ is constructed as above. We show first that $\mathcal{LM}^0$ satisfies conditions LIB(i),(ii),(iii) and (iv).

Given $\alpha \in \Gamma$ certainly $g_\alpha \in M_{s(\alpha)\alpha}$ and if $x \in M_{i\alpha}$ for any $i \in I$ then $x \in Sg_\alpha$ so that $xg_\alpha = x$. Now let $\alpha \in \Gamma$ and $\lambda \in \Lambda_\alpha$. Then $g_\alpha = d'_\lambda d_\lambda$ and so $f_\lambda R'_i \neq 0$, where $d_\lambda \in (R'_i \cap Sg_\alpha)^* = M_{i\alpha}$. Thus $p_{\lambda i} \neq 0$ and $p_{\lambda i}(d_\lambda) = d'_\lambda d_\lambda = g_\alpha$. This gives that LIB(i) and (ii) hold.

To see that LIB(iii) holds, first note that if $a, b \in M$ and $a\mathcal{R}'b$, then clearly $a, b \in U_i$ for some $i \in I$. Conversely, let $i \in I$. Pick $a \in R'_i$ so that $a \in R'_i \cap Sf_\lambda$ for some $\lambda \in \Lambda$, say $\lambda \in \Lambda_\alpha$. Thus $ad_\lambda \neq 0$ and $ad_\lambda \in (R'_i \cap Sg_\alpha)^* = M_{i\alpha}$. Hence $U_i \neq \emptyset$.

To see that U_i is an $\mathcal{R}'$–class of M, let $a, b \in U_i$. Then $a \in M_{i\alpha}, b \in M_{i\beta}$ for some $\alpha, \beta \in \Gamma$. Thus $a, b \in S^*$ are $\mathcal{R}'$–related in S and it follows there are elements $a = a_1, a_2, ..., a_n = b$ in S with $a_jS \cap a_{j+1}S \neq 0, 1 \leq j \leq n-1$. For $j \in \{1, ..., n\}$, define $\phi(j) \in \Lambda$ by $a_j \in (R'_i \cap Sf_{\phi(j)})^*$, and let $\theta(j) \in \Gamma$ be such that $\phi(j) \in \Lambda_{\theta(j)}$. Note that $\theta(1) = \alpha$ and $\theta(n) = \beta$. For $j \in \{1, ..., n-1\}$ we have that $a_jh_j = a_{j+1}k_j \neq 0$ for some $h_j, k_j \in S$, say $h_j, k_j \in Sf_{\mu(j)}$ where $\mu(j) \in \Lambda_{\tau(j)}$.

Consider $j \in \{2, ..., n-2\}$. In this case,

$$a_jf_{\phi(j)}h_jd_{\mu(j)} = a_{j+1}f_{\phi(j+1)}k_jd_{\mu(j)} \neq 0$$

so that

$$(a_jd_{\phi(j)})(d'_{\phi(j)}h_jd_{\mu(j)}) = (a_{j+1}d_{\phi(j+1)})(d'_{\phi(j+1)}k_jd_{\mu(j)}) \neq 0$$

and $a_jd_{\phi(j)} \in M_{i\theta(j)}, d'_{\phi(j)}h_jd_{\mu(j)} \in M_{s(\theta(j))\tau(j)}, a_{j+1}d_{\phi(j+1)} \in M_{i\theta(j+1)}$ and $d'_{\phi(j+1)}k_jd_{\mu(j)} \in M_{s(\theta(j+1))\tau(j)}$.

If $j = 1$ and $n > 2$ then from $ah_1 = a_2k_1$ we have $a(g_\alpha h_1d_{\mu(1)}) = (a_2d_{\phi(2)})(d'_{\phi(2)}k_1d_{\mu(1)})$ where $g_\alpha h_1d_{\mu(1)} \in M_{s(\alpha)\nu(1)}, a_2d_2 \in M_{i\theta(2)}$ and $d'_{\phi(2)}k_1d_{\mu(1)} \in M_{s(\theta(2))\tau(1)}$. Similar treatment may be applied to the equality $a_{n-1}h_{n-1} = bk_{n-1}$ and in the case $n = 2$ to the equality $ah_1 = bk_1$. Thus we have shown that $a\mathcal{R}'b$ in M and LIB(iii) is true for $\mathcal{LM}^0$.

Let $\alpha, \beta \in \Gamma, a \in M_{s(\alpha)\beta}$ and $b \in M_{s(\beta)\alpha}$. Suppose that $aba = a, g_\alpha ab = g_\alpha$ and $g_\beta ba = g_\beta$. Then as $ab \in M_{s(\alpha)\alpha}$, we have that $abg_\alpha = ab$ and as $ba \in M_{s(\beta)\beta}$, we have also that $bag_\beta = ba$. Thus

$$g_\alpha \mathcal{L} ab \mathcal{R} a \mathcal{L} ba \mathcal{L} g_\beta$$

and so $g_\alpha \mathcal{D} g_\beta$. Hence $\alpha = \beta$ and LIB(iv) holds.

It remains to show that S is isomorphic to $\mathcal{LM}^0$. To see this, start by taking any $a \in S^*$. Then $a \in R'_i \cap Sf_\lambda$ for some $i \in I$ and $\lambda \in \Lambda$, say $\lambda \in \Lambda_\alpha$, so that $a = af_\lambda = (ad_\lambda)d'_\lambda$ where $ad_\lambda \in (R'_i \cap Sg_\alpha)^* = M_{i\alpha}$. Further, if $(ad_\lambda)d'_\lambda = ud'_\lambda$ where $u \in M_{i\alpha}$, then $ad_\lambda = ud'_\lambda d_\lambda = ug_\alpha = u$. Thus a has a unique expression as $a = \overline{a}d'_\lambda$, where $\overline{a} \in M_{i\alpha}$.

Define $\psi : S \to \mathcal{LM}^0$ by $\psi(0) = 0$ and $\psi(a) = (i, \overline{a}, \lambda)$ where $a \in S^*, a \in R'_i \cap Sf_\lambda, \lambda \in \Lambda_\alpha$ and $a = \overline{a}d'_\lambda$ where $\overline{a} \in M_{i\alpha}$. Clearly ψ is injective, for if $a, b \in S^*$ and $\psi(a) = \psi(b) = (i, \overline{a}, \lambda)$, then $a = \overline{a}d'_\lambda = b$. To see that ψ is onto, suppose that $(i, c, \lambda) \in \mathcal{LM}^0$. Then $\lambda \in \Lambda_\alpha$ for some $\alpha \in \Gamma$ and $c \in M_{i\alpha} = (R'_i \cap Sg_\alpha)^*$. Now $cd'_\lambda \in (R'_i \cap Sf_\lambda)^*$ so that $\psi(cd'_\lambda) = (i, c, \lambda)$.

Finally we must show that ψ is a homomorphism. Consider $a, b \in S^*$ where $a \in R'_i \cap Sf_\lambda, \lambda \in \Lambda_\alpha, a = \overline{a}d'_\lambda, \overline{a} \in M_{i\alpha}$ and $b \in R'_j \cap Sf_\mu, \mu \in \Lambda_\beta, b = \overline{b}d'_\mu, \overline{b} \in M_{j\beta}$. Then $\psi(a) = (i, \overline{a}, \lambda)$ and $\psi(b) = (j, \overline{b}, \mu)$. If $ab = 0$ then $\psi(ab) = 0$ and $f_\lambda R'_j = 0$ so that $p_{\lambda j} = 0$ and $\psi(a)\psi(b) = 0$. Suppose therefore that $ab \neq 0$ and so $p_{\lambda j} = \overline{\lambda}$. Also, $ab \in R'_i \cap Sf_\mu$ and $ab = a\overline{b}d'_\mu$ where $a\overline{b} \in M_{i\beta}$. Then $\psi(ab) = (i, a\overline{b}, \mu) = (i, \overline{a}(d'_\lambda\overline{b}), \mu) = (i, \overline{a}, \lambda)(j, \overline{b}, \mu) = \psi(a)\psi(b)$. It follows that ψ is a homomorphism.

6 Some special cases

In this final section we show how Theorem 5.2 can be specialised to describe LIB C–semigroups satisfying various additional properties. In particular we give a characterization of primitive right abundant semigroups.

Lemma 6.1 *Let S be a LIB C–semigroup. Then S is primitive if and only if S is isomorphic to a LIB–LB Rees matrix semigroup $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$ satisfying (P).*

(P) For all $i \in I$, for all $\alpha \in \Gamma$, if $a, b \in M_{i\alpha}$ and

$$a = bu = xp_{\tau i}(b) = au,$$

where $\tau \in \Lambda_\beta, x \in M_{i\beta}$ and $u \in M_{s(\alpha)\alpha}$ for some $\beta \in \Gamma$, then $a = b$.

Proof By Theorem 5.2, S is isomorphic to a LIB–LB Rees matrix semigroup $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$. Suppose first that S and hence $\mathcal{LM}^0$ are primitive. Let $i, \alpha, a, b, \tau, \beta, x$ and u be given as in (P). Pick $\lambda \in \Lambda_\alpha$; by LIB(ii) there is some $j \in I$ and $v \in M_{j\alpha}$ with $p_{\lambda j}(v) = g_\alpha$. Then $(i, a, \lambda), (i, b, \lambda), (j, vu, \lambda)$ and (i, x, τ) belong to S and an easy computation gives that

$$(i, a, \lambda) = (i, b, \lambda)(j, vu, \lambda) =$$

$$(i, x, \tau)(i, b, \lambda) = (i, a, \lambda)(j, vu, \lambda).$$

By the Corollary to Theorem 4 of [M], $(i, a, \lambda) \leq (i, b, \lambda)$. Thus $a = b$ and (P) holds.

Conversely, suppose that (P) holds and $(i, a, \lambda) \leq (j, b, \mu)$ in $\mathcal{LM}^0$. Again using the Corollary to Theorem 4 of [M], either $(i, a, \lambda) = (j, b, \mu)$ or there are elements $(i, x, \tau), (k, y, \lambda) \in \mathcal{LM}^0$ with $(i, a, \lambda) = (i, x, \tau)(j, b, \mu) = (j, b, \mu)(k, y, \lambda) = (i, a, \lambda)(k, y, \lambda)$. Thus $i = j, \lambda = \mu$ and $a = xp_{\tau j}(b) = bu = au$ where $u = p_{\lambda k}(y) \in M_{s(\alpha)\alpha}$ where $\lambda \in \Lambda_\alpha$. By (P), $a = b$ and so $(i, a, \lambda) = (j, b, \mu)$ and $\mathcal{LM}^0$ and S are primitive.

If S is a semigroup we will say that S is *E–primitive* if the non–zero idempotents of $E(S)$ are primitive under the natural partial order on $E(S)$. That is, if $e, f \in E(S)^*$ and $e = ef = fe$, then $e = f$. It is known that if S is abundant then S is primitive if and only if it is E–primitive [L1]. One consequence of the following series of lemmas is that this is also true of right abundant semigroups. The first lemma is straightforward.

Lemma 6.2 *[F] If $e, f \in E(S)^*$ where S is an E–primitive semigroup and $Se \subseteq Sf$, then $Se = Sf$.*

Lemma 6.3 *Let S be an E–primitive right abundant semigroup and $e \in E(S)^*$. Then $a\mathcal{L}^*e$ if and only if $a \neq 0$ and $a \in Se$.*

Proof If $a\mathcal{L}^*e$ then $a \neq 0$ and $ae = a$ so that $a \in Se$. Conversely, suppose that $a \neq 0$ and $a \in Se$. Then $ae = a$ so that $fe = f$ where $a\mathcal{L}^*f$ and $f \in E(S)^*$. Thus $Sf \subseteq Se$ and so by Lemma 6.1, $f\mathcal{L}e$. Certainly then $a\mathcal{L}^*e$.

Lemma 6.4 *Let S be an E–primitive right abundant semigroup. Then S is left idempotent bounded.*

Proof Certainly S is the union of its idempotent generated principal left ideals. Suppose that $e, f \in E(S)^*$ and $Se \cap Sf \neq 0$. Then if a is a non–zero element of this intersection, the previous lemma shows that $e\mathcal{L}^* a\mathcal{L}^* f$. Thus $e\mathcal{L} f$ and $Se = Sf$.

Lemma 6.5 *Let S be an E–primitive right abundant semigroup. Then S is a LIB C–semigroup. Further, $\mathcal{L}' = \mathcal{L}^*$ on S.*

Proof Let $a, b, c \in S$ and suppose that $ab, bc \in S^*$. By Lemma 6.3, $ab\mathcal{L}^* b$ and as $\mathcal{L}^*$ is a right congruence, $abc\mathcal{L}^* bc$. Thus $abc \neq 0$, since 0 is an $\mathcal{L}^*$–class.

We have shown that S is a LIB C–semigroup. That $\mathcal{L}^* = \mathcal{L}'$ follows from Lemmas 6.3 and 6.4.

We are now in a position to show

Corollary 6.6 *Let S be a right abundant semigroup . Then S is E–primitive if and only if it is primitive.*

Proof It is easy to show that if S is primitive then it is E–primitive. Conversely, suppose that S is E–primitive and $a, b \in S^*$ with $a \leq b$. By the Corollary to Theorem 4 of [M], either $a = b$ or there are elements $w, z \in S$ with $a = wb = bz = az$. If follows that $a\mathcal{L}^* b\mathcal{L}^* f$ for some $f \in E(S)^*$. Now $a = az$ gives $f = fz$ and then $a = bz = bfz = bf = b$. Thus S is primitive.

The class of primitive abundant semigroups was investigated by the first author in [F]. The structure of such semigroups can be described in terms of blocked Rees matrix semigroups. Here we show that the structure of primitive right abundant semigroups can be described by LIB–LB Rees matrix semigroups.

Theorem 6.7 *A semigroup S is primitive right abundant if and only if S is isomorphic to a LIB–LB Rees matrix semigroup $\mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$ satifying condition (RA).*

(RA) Given $i \in I, \alpha, \beta \in \Gamma, a \in M_{i\alpha}, x, y \in M^1_{s(\alpha)\beta}$, if $ax = ay$ then $g_\alpha x = g_\alpha y$.

In this case,

(i) if $\alpha, \beta \in \Gamma, a \in M_{s(\alpha)\beta}, b \in M_{s(\beta)\alpha}$ and $aba = a$, then $\alpha = \beta$;

(ii) $(i, a, \lambda) \in \mathcal{LM}^0$ is idempotent if and only if $g_\alpha p_{\lambda i}(a) = g_\alpha$, where $\lambda \in \Lambda_\alpha$;

(iii) the elements $(i,a,\lambda),(j,b,\mu)$ are $\mathcal{L}^*$–related if and only if $\lambda=\mu$;
(iv) the idempotents $e=(i,a,\lambda), f=(j,b,\mu)$ where $\lambda\in\Lambda_\alpha$ and $\mu\in\Lambda_\beta$ are $\mathcal{D}$–related if and only if $\alpha=\beta$.

Proof Suppose first that S is isomorphic to a LIB–LB Rees matrix semigroup $\mathcal{LM}^0=\mathcal{LM}^0(M;I,\Lambda,\Gamma;P)$ satisfying condition (RA). Let $(i,a,\lambda)\in\mathcal{LM}^0$ where $\lambda\in\Lambda_\alpha$ so that $a\in M_{i\alpha}$, and let (j,b,λ) be an $\mathcal{L}'$–generator. Suppose that $\overline{x},\overline{y}\in(\mathcal{LM}^0)^1$ and $(i,a,\lambda)\overline{x}=(i,a,\lambda)\overline{y}$. Clearly $(i,a,\lambda)\overline{x}=(i,a,\lambda)\overline{y}=0$ if and only if $(j,b,\lambda)\overline{x}=(j,b,\lambda)\overline{y}=0$. Thus we suppose that $(i,a,\lambda)\overline{x}\neq 0$. Assume first that $\overline{x},\overline{y}\in\mathcal{LM}^0$ so that $\overline{x}=(k,x,\mu),\overline{y}=(\ell,y,\mu)$ for some $k,\ell\in I,\mu\in\Lambda_\beta,x\in M_{k\beta}$ and $y\in M_{\ell\beta}$. Then $ap_{\lambda k}(x)=ap_{\lambda\ell}(y)$ so (RA) gives $g_\alpha p_{\lambda k}(x)=g_\alpha p_{\lambda\ell}(y)$. Then $(j,b,\lambda)\overline{x}=(j,b,\lambda)(k,x,\mu)=(j,bp_{\lambda k}(x),\mu)=(j,bg_\alpha p_{\lambda k}(x),\mu)=(j,bg_\alpha p_{\lambda\ell}(y),\mu)=(j,bp_{\lambda\ell}(y),\mu)=(j,b,\lambda)(\ell,y,\mu)=(j,b,\lambda)\overline{y}$. A similar argument works if $\overline{x}$ or $\overline{y}$ is 1. Thus $(i,a,\lambda)\mathcal{L}^*(j,b,\lambda)$ and $\mathcal{LM}^0$ is right abundant.

We have shown in addition that $\mathcal{L}^*=\mathcal{L}'$. Thus every non–zero idempotent is an $\mathcal{L}'$–generator and so by (iii) of Proposition 4.2, (ii) holds.

To show that $\mathcal{LM}^0$ is E–primitive, let $(i,a,\lambda),(j,b,\mu)$ be non–zero idempotents of $\mathcal{LM}^0$ with $(i,a,\lambda)(j,b,\mu)=(j,b,\mu)(i,a,\lambda)=(i,a,\lambda)$. Thus $i=j$ and $\lambda=\mu$ and $a=ap_{\lambda i}(b)=bp_{\lambda i}(a)$. By (ii), $g_\alpha p_{\lambda i}(a)=g_\alpha$, where $\lambda\in\Lambda_\alpha$. Now $a=bg_\alpha p_{\lambda i}(a)=bg_\alpha=b$, as required. By Corollary 6.6, S is primitive.

Since $\mathcal{L}^*=\mathcal{L}'$, (iii) holds, and (iv) holds by (iv) of Proposition 4.2. Suppose now that $\alpha,\beta\in\Gamma,a\in M_{s(\alpha)\beta},b\in M_{s(\beta)\alpha}$ and $aba=a$. Then $ba\in M_{s(\beta)\beta}$ so that by (RA), $g_\beta ba=g_\beta$. Using LIB(i), $abg_\alpha=ab$ so that $abg_\alpha ab=abab=ab$. Again from (RA), $g_\alpha g_\alpha ab=g_\alpha$ so that $g_\alpha ab=g_\alpha$ and LIB(iv) yields $\alpha=\beta$. Thus (i) holds.

For the converse, suppose that S is a primitive right abundant semigroup. By Lemma 6.5, S is a LIB C–semigroup and so by Theorem 5.2, S is isomorphic to a LIB–LB Rees matrix semigroup $\mathcal{LM}^0=\mathcal{LM}^0(M;I,\Lambda,\Gamma;P)$. We must show that $\mathcal{LM}^0$ satisfies (RA). Let $i\in I,\alpha,\beta\in\Gamma,a\in M_{i\alpha}$ and $x,y\in M^1_{s(\alpha)\beta}$ be given with $ax=ay$. Suppose first that $x\neq 1$ and $y\neq 1$. Choose $\lambda\in\Lambda_\alpha$ and $j\in I,u\in M_{j\alpha}$ with $p_{\lambda j}(u)=g_\alpha$. Choose also $\mu\in\Lambda_\beta$. Since $\mathcal{L}'=\mathcal{L}^*$ on S, $(i,a,\lambda)\mathcal{L}^*(s(\alpha),g_\alpha,\lambda)$. Now $(j,ux,\mu),(j,uy,\mu)\in\mathcal{LM}^0$ and $(i,a,\lambda)(j,ux,\mu)=(i,ap_{\lambda j}(u)x,\mu)=(i,ag_\alpha x,\mu)=(i,ax,\mu)=...=(i,a,\lambda)(j,uy,\mu)$, giving that $(s(\alpha),g_\alpha,\lambda)(j,ux,\mu)=(s(\alpha),g_\alpha,\lambda)(j,uy,\mu)$. Thus $g_\alpha x=g_\alpha g_\alpha x=g_\alpha p_{\lambda j}(u)x=g_\alpha p_{\lambda j}(u)y=g_\alpha g_\alpha y=g_\alpha y$ as required. A similar argument works if $x=1$ or $y=1$.

Finally we turn out attention to LIB–LB Rees matrix semigroups $\mathcal{LM}^0$ for which $\mid \Gamma \mid = 1$. Adapting the terminology of Steinfeld [S2], we say that left ideals L_1, L_2 of a semigroup S are *left similar* if there is a bijection $\phi : L_1 \to L_2$ such that $\phi(sx) = s\phi(x)$ for all $x \in L_1$ and for all $s \in S$; that is, L_1 and L_2 are isomorphic as left S^1–systems. Then a LIB C–semigroup S is *left similarly decomposable* if $S = \bigcup^0 \{Sf_\lambda : \lambda \in \Lambda, f_\lambda \in E(S)^*\}$ such that for all $\lambda, \mu \in \Lambda, Sf_\lambda$ and Sf_μ are left similar.

The next lemma is Proposition 4.10 of [S3].

Lemma 6.8 *Let S be a semigroup and $e, f \in E(S)$. Then $e\mathcal{D}f$ if and only if Se and Sf are left similar.*

Corollary 6.9 *Let S be a LIB C–semigroup with $S \simeq \mathcal{LM}^0 = \mathcal{LM}^0(M; I, \Lambda, \Gamma; P)$. Then S is left similarly decomposable if and only if $\mid \Gamma \mid = 1$.*

Proof If $\mathcal{LM}^0$ is left similarly decomposable, then given any $\alpha, \beta \in \Gamma$, choosing $\lambda \in \Lambda_\alpha$ and $\mu \in \Lambda_\beta$ there are $\mathcal{L}'$–generators $f_\lambda \in L'_\lambda$ and $f_\mu \in L'_\mu$ with Sf_λ and Sf_μ left similar. By Lemma 6.8, $f_\lambda \mathcal{D} f_\mu$ and so by Proposition 4.2, $\alpha = \beta$. Thus $\mid \Gamma \mid = 1$. The converse is proved in a similar manner.

References

[F] J.B. FOUNTAIN, 'Abundant semigroups', *Proc. London Math. Soc.* 44 (1982), 103–129.

[FG] J.B. FOUNTAIN and V.A.R. GOULD, 'Idempotent bounded C–semigroups', preprint.

[FP] J.B.FOUNTAIN and M. PETRICH, 'Completely 0-simple semigroups of quotients III', *Math. Proc. Cambridge Phil. Soc.* 105 (1989), 263-275.

[H] J.M. HOWIE, *An introduction to semigroup theory*, Academic Press (1976).

[M] H. MITSCH, 'A natural partial order for semigroups', *Proc. American Math. Soc.* 97 (1986),384–388.

[LP] G. LALLEMENT and M. PETRICH, 'Decompositions I–matricielles d'un demi–groupe', *J. Math. Pures et Appl.* 45 (1966), 67–117.

[L1] M.V. LAWSON, 'The natural partial order on an abundant semigroup', *Proc. Edinburgh Math. Soc.* 30 (1987), 169–186.

[L2] M.V. LAWSON, 'Rees matrix semigroups',*Proc. Edinburgh Math. Soc.*, to appear.

[McA] D.B. McALISTER,'Rees matrix covers for regular semigroups', *J. Algebra* 89 (1984), 264–279.

[S1] O. STEINFELD, 'On a generalisation of completely 0–simple semigroups', *Acta. Sci. Math. (Szeged)*, 28 (1967), 135–145.

[S2] O. STEINFELD, 'On a faithful representation of similarly decomposable semigroups', *Estudos de Matematica, Lisboa* (1974), 113–118.

[S3] O. STEINFELD, 'Quasi–ideals in rings and semigroups', *Akademiai Kiado, Budapest* (1978).

[R] D. REES, 'On semi–groups', *Proc. Cambridge Phil. Soc.* 36 (1940), 387–400.