LEFT ADEQUATE AND LEFT EHRESMANN MONOIDS II

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Abstract. This article is the second of two presenting a new approach to left adequate monoids. In the first, we introduced the notion of being T-*proper*, where T is a submonoid of a left adequate monoid M. We showed that the free left adequate monoid on a set X is X^*-proper. Further, any left adequate monoid M has an X^*-proper cover for some set X, that is, there is an X^*-proper left adequate monoid $\widehat{M}$ and an idempotent separating epimorphism $\theta : \widehat{M} \to M$ of the appropriate signature.

We now show how to construct a T-proper left adequate monoid $\mathcal{P}(T, Y)$ from a monoid T acting via order preserving maps on a semilattice Y with identity. Our construction plays the role for left adequate monoids that the semidirect product of a group and a semilattice plays for inverse monoids. A left adequate monoid M with semilattice E has an X^*-proper cover $\mathcal{P}(X^*, E)$. Hence, by choosing a suitable semilattice E_X and an action of X^* on E_X, we prove that the free left adequate monoid is of the form $\mathcal{P}(X^*, E_X)$. An alternative description of the free left adequate monoid appears in a recent preprint of Kambites. We show how to obtain the labelled trees appearing in his result from our structure theorem.

Our results apply to the wider class of left Ehresmann monoids, and we give them in full generality. Indeed this is the right setting: the class of left Ehresmann monoids is the variety generated by the quasi-variety of left adequate monoids. This paper, and the two of Kambites on free (left) adequate semigroups, demonstrate the rich but accessible structure of (left) adequate semigroups and monoids, introduced with startling insight by Fountain some 30 years ago.

Introduction

This article is the second of two concerning a variety of algebras $\mathcal{LEM}$ and its sub-quasi-varieties, where $\mathcal{LEM}$ consists of monoids equipped with an additional unary operation, denoted by $a \mapsto a^+$. Thus the signature of our variety has a binary, a unary and a nullary operation: we indicate this by writing the signature

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as $(2, 1, 0)$. Our variety $\mathcal{LEM}$ is the variety of *left Ehresmann monoids* and is defined by the identities:

$$1\,x = x, (xy)z = x(yz)$$

where 1 always denotes the image of the nullary operation, and

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+(xy)^+ = (xy)^+,\ (xy)^+ = (xy^+)^+.$$

We remark that from the defining identities, the image E of the operation $^+$ always forms a semilattice under the semigroup multiplication.

It is easy to see that if M is an inverse monoid, then $M \in \mathcal{LEM}$ where we put $a^+ = aa^{-1}$. However, $\mathcal{LEM}$ contains many other classes of monoids of interest. It is generated by the sub-quasi-variety of left adequate monoids, and contains the sub-variety of left restriction monoids, which itself is generated by the sub-quasi-variety of left ample monoids. We refer the reader to [9] for background and a selection of references demonstrating the many sources from which these monoids spring. We stress that almost all of the previous results in this area relied on the additional assumption of the ample identity $xy^+ = (xy^+)x$, which is easily seen to hold for inverse monoids. This identity, innocent enough in appearance, effectively forces monoids to have a structure bearing some resemblance to that in the inverse case. Without it, new techniques are called for.

The forerunner to this article [1] was the first to consider the behaviour of left Ehresmann monoids in full generality. Previously, structure results had relied on imposing further conditions such as the ample identity, or by looking at the two-sided case of adequate monoids [15, 7]. However, even in the two-sided case, no progress had been made in the 'McAlister' direction, that is, in finding a property P for left adequate monoids, such that all left adequate monoids with property P are described by an accessible structure theorem, and such that every left adequate monoid has a cover with property P. Our aim in [1] and here is to remedy this situation.

Left adequate monoids were introduced by Fountain in [2] as monoids M for which every principal left ideal is projective as a left M-act, and such that the set $E(M)$ of idempotents forms a semilattice. The former condition is equivalent to every $\mathcal{R}^*$-class of M containing an idempotent; the latter guarantees that this idempotent is unique. Denoting by a^+ the (unique) idempotent in the $\mathcal{R}^*$-class of $a \in M$, it is easy to see that the class of left adequate monoids forms a quasi-variety of algebras of signature $(2, 1, 0)$, but does *not* form a variety. It follows from Corollary 3.2 that the variety generated by the quasi-variety of left adequate monoids is the variety of left Ehresmann monoids. We therefore present our results in the more general setting of left Ehresmann monoids. For an introduction to such monoids, and their origins in the work of Charles Ehresmann, see [15]; [9] also contains routine background details. We also remark that we concentrate on monoids rather than semigroups. For technical reasons this makes some of our arguments more straightforward; the free left adequate monoid is the free left adequate semigroup with an identity adjoined (see [12]), so there is no significant loss in generality, at least in so far as our results concern free algebras.

If M is an inverse monoid, then $\mathcal{R}^* = \mathcal{R}$ on M and certainly M is left adequate. We make two observations concerning inverse monoids. First, it follows from the description of the free inverse monoid $\mathcal{FIM}(X)$ given by Scheiblich [19] and Munn [18] that it is E-unitary. We recall that an inverse monoid is E-unitary if and only if it is *proper*, that is, $\mathcal{R} \cap \sigma = \iota$. Here σ is the least congruence on a monoid M identifying all the idempotents, so that if M is inverse, σ is the least group congruence. Secondly, as well as having a free (and proper) pre-image, McAlister showed that any inverse monoid has a proper pre-image, *under an idempotent separating morphism*, a 'cover'. Moreover, any proper inverse monoid P can be constructed from a group G acting by order automorphisms on a partially ordered set X with subsemilattice Y (P is isomorphic to a 'P-semigroup' $\mathcal{P} = \mathcal{P}(G, X, Y)$) [16, 17]. In the case $X = Y$, the semigroup $\mathcal{P}$ becomes a semidirect product.

Naturally, one would wish for similar theory for left Ehresmann monoids. That is, can we find a property P such that the structure of any left Ehresmann monoid with P is well-determined, and is such that every left Ehresmann monoid has a cover with P. As explained in [1], the obvious generalisation of the notion of proper from the inverse case has no chance of success.

The main aim of [1] was to introduce a new notion of *T-proper* for a left Ehresmann monoid M having submonoid T. We showed that every left Ehresmann monoid has an X^*-proper cover, and, moreover, the free left Ehresmann monoid $\mathcal{FLE}M(X)$ is X^*-proper. We will see that, in fact, $\mathcal{FLE}M(X)$ and $\mathcal{FLA}d\mathcal{M}(X)$ coincide.

In this current article we develop a 'recipe' for constructing a T-proper left Ehresmann monoid $\mathcal{P}(T, Y)$ from a monoid T acting by order-preserving maps on a semilattice Y with identity, that is in some loose sense an analogue of a semidirect product. We show that if T is cancellative and has no units other than 1, then $\mathcal{P}(T, Y)$ is left adequate. Our construction is inspired by that of the free left h-adequate monoid given in [3], where it occurs in the very special case of T being free. Left h-adequate monoids need not be left ample, but neither is every left adequate monoid left h-adequate [2]. We also show that every left Ehresmann monoid M has a proper left adequate cover of the form $\mathcal{P}(X^*, E)$, so that, consequently, $\mathcal{LEM}$ is generated as a variety by left adequate monoids. We then use our recipe to determine the structure of $\mathcal{FLEM}(X)$ (and show it coincides with $\mathcal{FLA}d\mathcal{M}(X)$); an alternative description of $\mathcal{FLEM}(X)$ appears in the preprint [12] of Kambites.

After giving some preliminaries in Section 1, we concentrate in Section 2 on constructing left Ehresmann monoids of the form $\mathcal{P}(T, Y)$. Once we have established that elements of $\mathcal{P}(T, Y)$ have a unique normal form, we call upon the results of [1] to deduce that $\mathcal{P}(T, Y)$ is T-proper and has a number of additional properties, some of which depend upon those of T. In Section 3, we show that every left Ehresmann monoid M generated as a semigroup by $T \cup E$, where T is a submonoid and E its distinguished semilattice of idempotents, has a cover of the form $\mathcal{P}(T, E)$ and in addition, M has a cover of the form $\mathcal{P}(X^*, E)$. Finally in Section 4, we show that $\mathcal{FLEM}(X)$ is isomorphic to $\mathcal{P}(X^*, E_X)$ for a semilattice E_X, and connect our result to that of [12].

1. Preliminaries

To make this article self-contained we give some basic definitions and results concerning left adequate and left Ehresmann monoids. Further details may be found in the notes [9]. We also describe the notion of T-proper introduced in [1].

We approach left adequate and left Ehresmann monoids via the equivalence relations $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}_E$, repectively, since this angle will be of use in later arguments. The relation $\mathcal{R}^*$ is defined on a monoid M by the rule that for any $a, b \in M$, $a \, \mathcal{R}^* \, b$ if and only if for all $x, y \in M$,

$$xa = ya \text{ if and only if } xb = yb.$$

It is easy to see that $\mathcal{R}^*$ is a left congruence, $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{R} = \mathcal{R}^*$ if M is regular. In general, however, the inclusion can be strict.

Suppose now that $E \subseteq E(M)$ and E forms a commutative subsemigroup of M; we will say simply that E is a *semilattice* in M.

Definition 1.1. A monoid M is *left E-adequate* if E is a semilattice in M, and every $\mathcal{R}^*$-class contains an idempotent of E. If $E = E(M)$ then we say that M is *left adequate*.

If M is left E-adequate, then from the commutativity of idempotents, it is clear that any $\mathcal{R}^*$-class contains exactly one idempotent of E. We denote the unique idempotent of E in the $\mathcal{R}^*$-class of a by a^+ (where E is understood). Observe that we are forced to have $1^+ = 1$, so that $1 \in E$. We may thus regard M as an algebra of signature $(2, 1, 0)$, where $^+$ is the basic unary operation. As such, morphisms must preserve the unary operation of $^+$ (and hence the relation $\mathcal{R}^*$). We may refer to such morphisms as '$(2, 1, 0)$-morphisms' if there is danger of ambiguity. Similarly, if X is a set of generators of a left E-adequate monoid as an algebra with the augmented signature, then we say that X is a set of $(2, 1, 0)$-*generators* and write $M = \langle X \rangle_{(2,1,0)}$ for emphasis. We remark here that if M is inverse and $E = E(M)$, then $a^+ = aa^{-1}$ for all $a \in M$.

Definition 1.2. A left adequate monoid M is *left ample* if the *left ample* identity (AL) holds:

$$xy^+ = (xy^+)^+ x.$$

Left ample monoids may be determined by their representations by partial one-one maps. They are precisely the submonoids of symmetric inverse monoids closed under $^+$ (see, for example [9]). We observe that there is no need to define 'left E-ample monoid', since if a left E-adequate monoid satisfies (AL), the semilattice E is forced to be $E(M)$.

Remark 1.3. [9] The class of left E-adequate monoids forms a quasi-variety of algebras of signature $(2, 1, 0)$ with sub-quasi-varieties the classes of left adequate and left ample monoids.

We now turn our attention to left Ehresmann monoids. Again, let E be a semilattice in M. The relation $\widetilde{\mathcal{R}}_E$ on M is defined by the rule that for any $a, b \in M$, $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if for all $e \in E$,

$$ea = a \text{ if and only if } eb = b,$$

that is, a and b have the same set of left identities from E. It is easy to see that for any monoid M, we have $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$, with both inclusions equalities if M is regular and $E = E(M)$; in general, however, these inclusions can be strict. The relation $\widetilde{\mathcal{R}}_E$ is certainly an equivalence; however, unlike the case for $\mathcal{R}$ and $\mathcal{R}^*$, it need not be left compatible, not even when $E = E(M)$.

It is clear that any $\widetilde{\mathcal{R}}_E$-class contains at most one idempotent from E. If every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E, we again have a unary operation $a \mapsto a^+$, where a^+ is now the (unique) idempotent of E in the $\widetilde{\mathcal{R}}_E$-class of a. Again, we must have that $1^+ = 1 \in E$, and we may consider M as an algebra of signature $(2, 1, 0)$. In the case that $E = E(M)$, we drop the 'E' from notation and terminology, for example, we write $\widetilde{\mathcal{R}}_{E(M)}$ more simply as $\widetilde{\mathcal{R}}$.

Definition 1.4. A monoid M with semilattice E is *left Ehresmann* (with distinguished semilattice E), or *left E-Ehresmann*, if every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E and $\widetilde{\mathcal{R}}_E$ is a left congruence.

Definition 1.5. A monoid M with semilattice E is *left restriction* (with distinguished semilattice E), or *left E-restriction*, if it is left Ehresmann and satisfies (AL).

As for left ample monoids, left restriction monoids have a natural representation, this time as submonoids of partial transformation monoids closed under the operation $\alpha \mapsto \alpha^+$, where α^+ is the identity map in the domain of α.

Remark 1.6. [9] The class of all left Ehresmann monoids is a variety of algebras of signature $(2, 1, 0)$, with sub-quasi-varieties the classes of left Ehresmann monoids M having distinguished semilattice $E(M)$, left E-adequate monoids, left adequate monoids and left restriction monoids.

We stress that a left Ehresmann monoid M has augmented signature $(2, 1, 0)$; we normally denote by E the image of the unary operation $^+$, so that

$$E = \{a^+ : a \in M\}$$

is a semilattice, the *distinguished semilattice* of M. The identity of M must lie in E, for we must have that $1^+ = 1$.

We now give a technical result which will be useful in the subsequent sections. It follows immediately from the fact that in a left Ehresmann monoid, $\widetilde{\mathcal{R}}_E$ is a left congruence. The relation $\leq$ appearing in its statement is the natural partial order on E.

Lemma 1.7. *Let M be a left Ehresmann monoid. Then for any $a, b \in M$ and $e \in E$,*

$$(ab)^+ = (ab^+)^+,\ (ea)^+ = ea^+ \text{ and } (ab)^+ \leq a^+.$$

The following idea is central to our approach. Let M be a left Ehresmann monoid and let T be a submonoid of M. Then T *acts on E on the left via order preserving maps* by

$$(t, e) \mapsto t \cdot e = (te)^+.$$

This is essentially folklore; a proof may be found in [1]. A left Ehresmann monoid M is said to be *hedged* [7] if the action of M on E is by morphisms, that is, if for any $e, f \in E$ and $m \in M$, we have that $(mef)^+ = (me)^+(mf)^+$. A hedged left adequate monoid is *left h-adequate* [3].

We pause to explain why we might be interested in monoids acting on semilattices in the context of left Ehresmann monoids. In the theory of inverse monoids, groups acting on semilattices play a major role. A group may be regarded as an inverse monoid possessing exactly one idempotent. In the same way, we may regard a monoid as a left Ehresmann monoid in which $a^+ = 1$, for every $a \in M$, that is, as a *reduced* left Ehresmann monoid.

The notion of least group congruence on an inverse semigroup is central to the McAlister approach to inverse semigroups [16, 17]. The least group congruence on an inverse semigroup is precisely the least congruence identifying all the idempotents. We explore this notion in our current context.

Let M be a monoid and suppose that $E \subseteq E(M)$. We define the relation σ_E to be the semigroup (monoid) congruence on M generated by $E \times E$; that is, for any $a, b \in M$ we have that $a \,\sigma_E\, b$ if and only if $a = b$ or there exists a sequence

$$a = c_1e_1d_1, c_1f_1d_1 = c_2e_2d_2, \ldots, c_nf_nd_n = b,$$

where $c_1, d_1, \ldots, c_n, d_n \in M$ and $(e_1, f_1), \ldots, (e_n, f_n) \in E \times E$. If $E = E(M)$ then we write σ for $\sigma_{E(M)}$.

Lemma 1.8. [9, 1]. *Let M be a left Ehresmman monoid with distinguished semilattice E. Then E is contained in a σ_E-class, σ_E is a $(2, 1, 0)$-congruence and M/σ_E is reduced.*

For an inverse monoid, or a left restriction monoid M, the description of σ simplifies to $a \,\sigma\, b$ if and only if $ea = eb$ for some $e \in E(M)$ (see [11, 9]). For left Ehresmann monoids in general, we have no such useful description. Nevertheless, we showed in [1] that if M is the free left Ehresmann monoid on X and if $T = \langle X\rangle_{(2,0)}$, then $T \cong X^* \cong M/\sigma_E$.

Let T be a submonoid of a left Ehresmann monoid M; we mean here that T is a $(2,0)$-subalgebra of the $(2,1,0)$-algebra M. It is easy to see that $T \cup E$ generates M as a left Ehresmann monoid if and only if $T \cup E$ generates M as a semigroup, which we denote by $M = \langle T \cup E\rangle_{(2)}$.

Lemma 1.9. [1, Lemma 3.1] *Let M be a left Ehresmann monoid. Suppose that we have $M = \langle E \cup T\rangle_{(2)}$ for some submonoid T of M. Then any $m \in M$ can be written as*

$$m = t_0e_1t_1 \dots e_nt_n,$$

where $n \geq 0$, $e_1, \dots, e_n \in E \setminus \{1\}$, $t_1, \dots, t_{n-1} \in T \setminus \{1\}$, $t_0, t_n \in T$ and for $1 \leq i \leq n$,

$$e_i < (t_ie_{i+1} \dots e_nt_n)^+.$$

We will say that an element $m = t_0e_1t_1 \dots e_nt_n$ expressed as in the statement of the above Lemma is in *T-normal form*; if $T = M$, then we simply say *normal form.* If every element of M has a unique expression in T-normal form, then we say that M has *uniqueness of T-normal forms.* Noticing that $m^+ = (t_0e_1)^+$, and comparisons with the theory of proper inverse and proper left ample monoids, led us to introduce the following concept in [1].

Definition 1.10. Let M be a left Ehresmann monoid and let T be a submonoid of M such that $M = \langle E \cup T\rangle_{(2)}$. Then M is *T-proper* if whenever

$$a = t_0e_1t_1 \dots e_nt_n \text{ and } b = u_0e_1u_1 \dots e_nu_n$$

are in T-normal form, and we have for all $i \in \{0, \dots, n\}$:
(s) $t_i \, \sigma_E \, u_i$ and
(r) $(t_ie_{i+1} \dots e_nt_n)^+ = (u_ie_{i+1} \dots e_nu_n)^+$,
then $a = b$.

Note that we also say that M above is T-proper, and S is any monoid isomorphic to T, then we may also say that M is S-proper. We show in [1] that if M is left restriction, then it is M-proper if and only if it is proper.

A left Ehresmann monoid M with submonoid T with uniqueness of T-normal forms has many pleasant properties. We quote these from [1] as we need them but remark at this point that such a monoid is certainly T-proper and $M/\sigma_E \cong T$ as monoids. Indeed, regarding T as a reduced left Ehresmann monoid, $M/\sigma_E \cong T$ as left Ehresmann monoids. Note however, that T need not be a $(2,1,0)$-subalgebra of M.

2. A construction

In this section we give a recipe for constructing T-proper left Ehresmann monoids from monoids acting on semilattices via order-preserving maps. We were motivated by the pointers given in [3, 1], which suggest that the free left

Ehresmann monoid must be constructed in this way; this indeed proves to be the case (see Section 4, and also Kambites [13]). The construction itself follows that given in [3] by Fountain in the very special case of a free monoid acting by morphisms on a particular semilattice. In our case we must pay some attention to the fact that our monoids may not be cancellative and, more troubling in this instance, may not have trivial group of units.

Our first simple observation provides us much of our motivation.

Let M be a left Ehresmann monoid that $M = \langle E \cup T\rangle_{(2,1,0)} = \langle E \cup T\rangle_{(2)}$ for some submonoid T of M. In Lemma 1.9 we claim that any element can be written as a product of elements of T and E in T-normal form. Suppose now that

$$a = t_0e_1t_1 \dots e_nt_n \text{ and } b = u_0f_1u_1 \dots f_mt_m$$

are in T-normal form. To transform ab into normal form the first steps are to consider $(t_nu_0f_1 \dots u_m)^+$, that is, $t_nu_0 \cdot (f_1 \dots u_m)^+ = t_nu_0 \cdot f_1$, and then multiply this with e_n. This manoevre (and subsequent ones of the same kind) was key in [1] to the reduction of ab to normal form.

After explaining our motivation, we proceed with the construction.

Let T be a monoid with identity 1_T acting (as a monoid) on the left of a semilattice Y with identity 1_Y, via order preserving maps. We denote the action of $t \in T$ on $y \in Y$ by $t \cdot y$.

Let $T * Y$ be the free *semigroup* product of T and Y. Since T acts on the left of Y via order-preserving maps, there is a monoid morphism

$$\phi : T \to \mathcal{O}_Y^*, (t\phi)(y) = t \cdot y,$$

where $\mathcal{O}_Y$ is the monoid of order-preserving maps of Y and a * denotes the dual of a monoid, so that in $\mathcal{O}_Y^*$, maps are composed from right to left. Now, Y acts on the left of itself by order-preserving maps via multiplication, so that there is a monoid morphism, also denoted ϕ, given by

$$\phi : Y \to \mathcal{O}_Y^*, (z\phi)(y) = zy.$$

By the universal propery of free products, we have a semigroup morphism

$$\phi : T * Y \to \mathcal{O}_Y^*$$

defined by

$$(s_1 \dots s_n)\phi = s_1\phi \dots s_n\phi,$$

where each $s_i \in T \cup Y$. We thus have a *semigroup* action of $T * Y$ on Y, which we may without ambiguity denote by $\cdot$, so that

$$s_1 \dots s_n \cdot y = s_1 \cdot (s_2 \cdot (\dots (s_n \cdot y) \dots)).$$

We now define w^+ (for $w \in T * Y$) to be

$$w^+ = w \cdot 1_Y,$$

so that $e^+ = e$ for all $e \in Y$. We remark that for any $w \in T * Y$, if v is obtained from w via insertion or deletion of elements 1_Y or 1_T, then $w^+ = v^+$. Notice also that $1_T^+ = 1_Y$ and for $v, w \in T * Y$, we have

$$(vw)^+ = (vw) \cdot 1_Y = v \cdot (w \cdot 1_Y) = v \cdot w^+$$

so that if $v \in Y$ we have $(vw)^+ = vw^+$.

Lemma 2.1. *If T and Y are as above and $t \in T$ has a right inverse, then $t^+ = 1_Y$.*

Proof. Suppose that $tu = 1_T$. Then

$$1_Y = 1_T \cdot 1_Y = tu \cdot 1_Y = t \cdot (u \cdot 1_Y) \leq t \cdot 1_Y \leq 1_Y,$$

so that $1_Y = t \cdot 1_Y = t^+$. □

The remainder of this section is devoted to the proof of the following theorem.

Theorem 2.2. *Let T be a monoid acting on the left of a semilattice Y with identity, via order-preserving maps. Let the unary operation of $^+$ be defined on $T * Y$ as above. Let $\sim$ be the semigroup congruence on $T * Y$ generated by*

$$H = \{(\alpha^+\alpha, \alpha) : \alpha \in T * Y\} \cup \{(1_T, 1_Y)\}.$$

Let $\mathcal{P} = \mathcal{P}(T, Y) = (T * Y)/\sim$. Then $\mathcal{P}$ is a left Ehresmann monoid with $[\alpha]^+ = [\alpha^+]$, identity $[1_Y]$ and distinguished semilattice

$$Y' = \{[y] : y \in Y\}.$$

Let $T' = \{[t] : t \in T\}$. Then Y' is isomorphic to Y and T' is isomorphic to T under restrictions of the natural morphism, and T' is a submonoid of P such that $\mathcal{P}$ has uniqueness of T'-normal forms. Consequently,

(i) $\mathcal{P}$ is T'-proper;

(ii) $\mathcal{P}/\sigma_{Y'} \cong T'$;

(iii) if T is right cancellative, $\mathcal{P}$ is left Y'-adequate;

(iv) if T acts by morphisms, then $\mathcal{P}$ is hedged;

(v) if T is right cancellative and has no invertible elements other than 1_T, then $Y' = E(\mathcal{P})$.

Proof. We begin with some terminology. A tuple

$$(t_0, e_1, t_1, \ldots, e_n, t_n)$$

such that $t_i \in T$, $0 \leq i \leq n$, and $e_j \in Y \setminus \{1_Y\}, e_j < (t_j e_{j+1} \ldots e_n t_n)^+$ for $1 \leq j \leq n$ is a *weak T-normal form.* If we insist that $t_k \neq 1$ for $1 \leq k \leq n-1$, we say our tuple is in *T-normal form.* We are using the same terminology as that established in Section 1, but, of course, we do not have that $T * Y$ is left Ehresmann with distinguished semilattice Y. If $\alpha = (t_0, e_1, t_1, \ldots, e_n, t_n)$ is a (weak) T-normal form, we put $\overline{\alpha} = t_0 e_1 t_1 \ldots e_n t_n$ and where there is no danger of ambiguity we may say that $\overline{\alpha}$ is in (weak) T-normal form.

We denote the set of all T-normal forms by $\mathcal{N}$. Our first aim is to show that each element of $\mathcal{P}$ can be represented uniquely in T-normal form.

Lemma 2.3. *For any $\beta \in T * Y$, there exist $\alpha = (u_0, g_1, u_1, \ldots, g_m, u_m)$ in T-normal form such that*

$$\beta \sim \overline{\alpha} = u_0 g_1 u_1 \ldots g_m u_m.$$

Proof. From the fact that $1_T \sim 1_Y$ we have that for any $\beta \in T * Y$,

$$\beta \sim t_0 e_1 t_1 \ldots e_n t_n$$

for some $t_0, \ldots, t_n \in T$ and $e_1, \ldots, e_n \in Y$. If $n = 0$, then certainly (t_0) is a T-normal form and $\beta \sim \overline{(t_0)} = t_0$.

Suppose inductively that $n > 0$ and

$$t_1 e_2 \ldots e_n t_n \sim s_0 f_1 s_1 \ldots f_m s_m$$

where $(s_0, f_1, s_1, \ldots, f_m, s_m)$ is a T-normal form.

We have that

$$\beta \sim t_0 e_1 s_0 f_1 s_2 \ldots f_m s_m \sim t_0 e_1 (s_0 f_1 \ldots f_m s_m)^+ (s_0 f_1 \ldots f_m s_m).$$

If $e_1 (s_0 f_1 \ldots f_m s_m)^+ = (s_0 f_1 \ldots f_m s_m)^+$ then

$$\beta \sim t_0 s_0 f_1 \ldots f_m s_m$$

and $(t_0 s_0, f_1, s_1, \ldots, f_m, s_m)$ is a T-normal form.

Suppose on the other hand that $f = e_1 (s_0 f_1 \ldots f_m s_m)^+ < (s_0 f_1 \ldots f_m s_m)^+$. Then

$$\beta \sim t_0 f s_0 f_1 \ldots f_m s_m.$$

If $m = 0$ or $m > 0$ and $s_0 \neq 1_T$, then $(t_0, f, s_0, f_1, \ldots, f_m, s_m)$ is a T-normal form. Otherwise, i.e. $m > 0$ and $s_0 = 1_T$, we have that $f < (f_1 s_1 \ldots f_m s_m)^+ \leq f_1$, and

$$\beta \sim t_0 f s_1 f_2 \ldots f_m s_m$$

and $(t_0, f, s_1, f_2, \ldots, f_m, s_m)$ is a T-normal form. $\square$

We now set out to show that for any $\beta \in T * Y$, we have that $\beta \sim \overline{\alpha}$ for a *unique* T-normal form α.

First, we construct a semigroup morphism from the free product $T * Y$ to $\mathcal{T}^*(\mathcal{N})$, the (dual of the) full transformation semigroup on $\mathcal{N}$.

For $t \in T$ we define $\psi(t)$ by

$$\psi(t)(t_0, e_1, t_1, \ldots, e_n, t_n) = (t t_0, e_1, t_1, \ldots, e_n, t_n).$$

Clearly $\psi(s)\psi(t) = \psi(st)$ for all $s, t \in T$ and $\psi(1_T) = I_{\mathcal{N}}$, so that $\psi : T \to \mathcal{T}^*(\mathcal{N})$ is a monoid morphism.

For $e \in Y$ we define $\psi(e)$ by $\psi(1_Y) = I_{\mathcal{N}}$ and for $e \neq 1_Y$,

$$\psi(e)(1_T) = (1_T, e, 1_T)$$

and for $\alpha = (t_0, e_1, t_1, \ldots, e_n, t_n) \neq (1_T)$, we put

$$\psi(e)(\alpha) = \begin{cases} (t_0, ee_1, t_1, \ldots, e_n, t_n) & \text{if } t_0 = 1_T \\ (t_0, e_1, t_1, \ldots, e_n, t_n) = \alpha & \text{if } t_0 \neq 1_T \text{ and } \overline{\alpha}^+ \leq e \\ (1_T, e\overline{\alpha}^+, t_0, e_1, t_1, \ldots, e_n, t_n) & \text{if } t_0 \neq 1_T \text{ and } \overline{\alpha}^+ \not\leq e. \end{cases}$$

Lemma 2.4. *The function $\psi : Y \to \mathcal{T}^*(\mathcal{N})$ is a monoid morphism.*

Proof. Let $e, f \in Y$. If e or f is 1_Y, then clearly $\psi(ef) = \psi(e)\psi(f)$. We assume therefore that $e, f \in Y \setminus \{1_Y\}$. Then

$$\begin{aligned} \psi(e)\psi(f)(1_T) &= \psi(e)(1_T, f, 1_T) \\ &= (1_T, ef, 1_T) \\ &= \psi(ef)(1_T). \end{aligned}$$

Similarly, if $\alpha = (1_T, e_1, t_1, \ldots, e_n.t_n)$ where $n > 0$, then

$$\psi(e)\psi(f)(\alpha) = \psi(e)(1_T, fe_1, t_1, \ldots, e_n, t_n) = (1_T, efe_1, t_1, \ldots, e_n, t_n) = \psi(ef)(\alpha).$$

Suppose now that $\alpha = (t_0, e_1, t_1, \ldots, e_n, t_n)$ where $t_0 \neq 1_T$. If $\overline{\alpha}^+ \leq e$ and $\overline{\alpha}^+ \leq f$, then $\overline{\alpha}^+ \leq ef$, so that

$$\psi(e)\psi(f)(\alpha) = \alpha = \psi(ef)(\alpha).$$

If $\overline{\alpha}^+ \not\leq e$ and $\overline{\alpha}^+ \leq f$, then $\overline{\alpha}^+ \not\leq ef$ and

$$\begin{aligned} \psi(e)\psi(f)(\alpha) &= \psi(e)(t_0, e_1, t_1, \ldots, e_n, t_n) \\ &= (1_T, e\overline{\alpha}^+, t_0, e_1, \ldots, e_n, t_n) \\ &= (1_T, ef\overline{\alpha}^+, t_0, e_1, \ldots, e_n, t_n) \\ &= \psi(ef)(\alpha). \end{aligned}$$

If $\overline{\alpha}^+ \not\leq f$, then $\overline{\alpha}^+ \not\leq ef$ and

$$\begin{aligned} \psi(e)\psi(f)(\alpha) &= \psi(e)(1_T, f\overline{\alpha}^+, t_0, e_1, \ldots, e_n, t_n) \\ &= (1_T, ef\overline{\alpha}^+, t_0, e_1, \ldots, e_n, t_n) \\ &= \psi(ef)(\alpha). \end{aligned}$$

It follows that $\psi : Y \to \mathcal{T}^*(\mathcal{N})$ is a monoid morphism. $\square$

The universal property of free products ensures that ψ extends to a *semigroup* morphism $\psi : T * Y \to \mathcal{T}^*(\mathcal{N})$. Notice that if $w \in T * Y$ and v is obtained from w by insertion and deletion of elements 1_Y and 1_T, then $\psi(v) = \psi(w)$. We also remark that if $(t_0, e_1, \ldots, e_n, t_n) \in \mathcal{N}$ and $x \in T * Y$, then for ease on the eye we write $\psi(x)(t_0, e_1, \ldots, e_n, t_n)$ rather than $\psi(x)((t_0, e_1, \ldots, e_n, t_n))$.

Lemma 2.5. *The relation $\sim$ is contained in ker ψ.*

Proof. It suffices to show that $H \subseteq \ker \psi$.

Since $\psi(1_T) = I_{\mathcal{N}} = \psi(1_Y)$, we certainly have $(1_T, 1_Y) \in \ker \psi$.

We wish to show that $\psi(x^+x) = \psi(x)$ for all $x \in T*Y$, that is, $\psi(x^+)\psi(x)(\alpha) = \psi(x)(\alpha)$ for all $x \in T * Y$ and for all $\alpha \in \mathcal{N}$.

For any $\beta \in \mathcal{N}$ and $e \in Y$, we note that $\psi(e)(\beta) = \beta$ if and only if $e = 1_Y$ or $e \neq 1_Y, \beta \neq (1_T)$ and either

$$\beta = (1_T, e_1, t_1, \dots, e_n, t_n) \text{ with } ee_1 = e_1,$$

or

$$\beta = (t_0, e_1, t_1, \dots, e_n, t_n) \text{ where } t_0 \neq 1_T \text{ and } \overline{\beta}^+ \leq e.$$

Notice that if $\beta = (1_T, e_1, t_1, \dots, e_n, t_n)$, then

$$\begin{array}{rcl} \overline{\beta}^+ & = & (1_T e_1 t_1 \dots e_n t_n)^+ \\ & = & (e_1 t_1 \dots e_n t_n)^+ \\ & = & e_1 (t_1 \dots e_n t_n)^+ \text{ by comment preceding Lemma 2.1} \\ & = & e_1 \text{ by definition of } T\text{-normal form.} \end{array}$$

We deduce that $\psi(e)(\beta) = \beta$ if and only if $e = 1_Y$ or $e \neq 1_Y$ and $\beta = (t_0, e_1, t_1, \dots, e_n, t_n) \neq (1_T)$ with $\overline{\beta}^+ \leq e$. So, we need to show that for all $x \in T * Y$ we have $x^+ = 1_Y$ or $x^+ \neq 1_Y$ and, for all $\alpha \in \mathcal{N}$, $\psi(x)(\alpha) \neq (1_T)$ with $\overline{\psi(x)(\alpha)}^+ \leq x^+$.

It is clear that if $x^+ = 1_Y$, then $\psi(x^+)\psi(x)(\alpha) = \psi(x)(\alpha)$, for any $\alpha \in \mathcal{N}$.

Our next step is to argue that if $\psi(x)(\alpha) = (1_T)$, then $x^+ = 1_Y$. First note that if $\psi(x)(\alpha) = (1_T)$, then $\alpha = (t)$ for some $t \in T$. If $x \in T$, then $\psi(x)(t) = (1_T)$ gives that $xt = 1_T$ and so by Lemma 2.1, $x^+ = 1_Y$. On the other hand, if $x \in Y \setminus \{1_Y\}$, then for $\psi(x)(t) = (1_T)$ we must have that $t \neq 1_T$ (else $\psi(x)(t) = (1_T, x, 1_T)$) and also that $\psi(x)(t) = (t)$ (else again, $\psi(x)$ increases the length of the normal form (t)). The latter condition gives $t = 1_T$, a contradiction. Thus if $x \in Y$ and $\psi(x)(t) = (1_T)$, then we must have that $x = 1_Y$ and so $x^+ = 1_Y$.

Since insertion and deletion of 1_T and 1_Y in $x \in T * Y$ does not affect the value of $\psi(x)$ nor x^+, we now suppose that $x = s_0 e_1 s_1 e_2 \dots e_n s_n \in T * Y \setminus T \cup Y$ where $s_0, \dots, s_n \in T$ and $e_1, \dots, e_n \in Y \setminus \{1_Y\}$. Then $\psi(s_0 e_1 s_1 \dots e_n s_n)(t) = (1_T)$ gives that

$$(1_T) = \psi(s_0 e_1 \dots s_{n-1})\psi(e_n)\psi(s_n)(t) = \psi(s_0 e_1 \dots s_{n-1})\psi(e_n)(s_n t).$$

It follows that $s_n t \neq 1_T, (s_n t)^+ \leq e_n$ and $\psi(e_n s_n)(t) = (s_n t)$. We also have that

$$(e_n s_n t)^+ = e_n (s_n t)^+ = (s_n t)^+.$$

Suppose for induction that for $n \geq \ell > 1$ we have

$$\psi(e_\ell s_\ell \dots e_n s_n)(t) = (s_\ell s_{\ell+1} \dots s_n t)$$

and

$$(e_\ell s_\ell e_{\ell+1} \dots e_n s_n t)^+ = (s_\ell s_{\ell+1} \dots s_n t)^+.$$

Then

$$\begin{array}{rcl} \psi(e_{\ell-1} s_{\ell-1} e_\ell s_\ell e_{\ell+1} \dots e_n s_n)(t) & = & \psi(e_{\ell-1})\psi(s_{\ell-1})\psi(e_\ell s_\ell e_{\ell+1} \dots e_n s_n)(t) \\ & = & \psi(e_{\ell-1})\psi(s_{\ell-1})(s_\ell s_{\ell+1} \dots s_n t) \\ & = & \psi(e_{\ell-1})(s_{\ell-1} s_\ell s_{\ell+1} \dots s_n t) \\ & = & (s_{\ell-1} s_\ell s_{\ell+1} \dots s_n t) \end{array}$$

and for this to happen, we must have that $(s_{\ell-1}s_\ell s_{\ell+1}\dots s_nt)^+\leq e_{\ell-1}$. Hence

$$\begin{array}{rcl}(e_{\ell-1}s_{\ell-1}e_\ell s_\ell\dots e_ns_nt)^+ &=& e_{\ell-1}s_{\ell-1}\cdot(e_\ell s_\ell e_{\ell+1}\dots e_ns_nt)^+\\ &=& e_{\ell-1}s_{\ell-1}\cdot(s_\ell s_{\ell+1}\dots s_nt)^+\\ &=& e_{\ell-1}(s_{\ell-1}s_\ell s_{\ell+1}\dots s_nt)^+\\ &=& (s_{\ell-1}s_\ell s_{\ell+1}\dots s_nt)^+.\end{array}$$

It follows that $\psi(x)(t)=(1_T)=(s_0s_1\dots s_nt)$ and

$$\begin{array}{rcl}x\cdot t^+ &=& (s_0e_1s_1\dots e_ns_n)\cdot t^+\\ &=& (s_0e_1s_1\dots e_ns_nt)^+\\ &=& s_0\cdot(e_1s_1\dots e_ns_nt)^+\\ &=& s_0\cdot(s_1s_2\dots s_nt)^+\\ &=& (s_0s_1\dots s_nt)^+\\ &=& s_0s_1\dots s_n\cdot t^+.\end{array}$$

Finally,

$$1_Y=1_T\cdot 1_Y=s_0s_1\dots s_nt\cdot 1_Y=s_0s_1\dots s_n\cdot t^+=x\cdot t^+\leq x\cdot 1_Y\leq 1_Y$$

and so $x^+=x\cdot 1_Y=1_Y$ as claimed. Hence, if $\psi(x)(\alpha)=(1_T)$, we have from the above that $\psi(x^+)\psi(x)(\alpha)=\psi(x)(\alpha)$.

Suppose now that $x^+\neq 1_Y$ so that $\psi(x)(\alpha)\neq(1_T)$. It remains to show that $\overline{\psi(x)(\alpha)}^+\leq x^+$.

To this end, observe that for any $t\in T$ and $\alpha=(t_0,e_1,\dots,e_n,t_n)$, we have

$$\begin{array}{rcl}\overline{\psi(t)(\alpha)}^+ &=& \overline{\psi(t)(t_0,e_1,\dots,e_n,t_n)}^+\\ &=& \overline{(tt_0,e_1,\dots,e_n,t_n)}^+\\ &=& (tt_0e_1\dots e_nt_n)^+\\ &=& tt_0e_1\dots e_nt_n\cdot 1_Y\\ &=& t\cdot\big((t_0e_1\dots e_nt_n)\cdot 1_Y\big)\\ &=& t\cdot\overline{\alpha}^+.\end{array}$$

Clearly, for any $\alpha\in\mathcal{N}$,

$$\overline{\psi(1_Y)(\alpha)}^+=\overline{\alpha}^+=1_Y\,\overline{\alpha}^+$$

and for $e\in Y\setminus\{1_Y\}$,

$$\overline{\psi(e)(1_T)}^+=\overline{(1_T,e,1_T)}^+=e^+=e=e1_Y=e1_T^+=e\overline{(1_T)}^+$$

and if $\alpha \neq (1_T)$,

$$\overline{\psi(e)(\alpha)}^+ = \begin{cases} \overline{(t_0, ee_1, t_1, \ldots, e_n, t_n)}^+ & \text{if } t_0 = 1_T \\ \overline{\alpha}^+ & \text{if } t_0 \neq 1_T \text{ and } \overline{\alpha}^+ \leq e \\ \overline{(1_T, e\overline{\alpha}^+, t_0, e_1, \ldots, e_n, t_n)}^+ & \text{else} \end{cases}$$

$$= \begin{cases} (1_T e e_1 t_1 \ldots e_n t_n)^+ & \text{if } t_0 = 1_T \\ \overline{\alpha}^+ & \text{if } t_0 \neq 1_T \text{ and } \overline{\alpha}^+ \leq e \\ (1_T e\overline{\alpha}^+ t_0 e_1 \ldots e_n t_n)^+ & \text{else} \end{cases}$$

$$= \begin{cases} (e 1_T e_1 t_1 \ldots e_n t_n)^+ & \text{if } t_0 = 1_T \\ \overline{\alpha}^+ & \text{if } t_0 \neq 1_T \text{ and } \overline{\alpha}^+ \leq e \\ (e\overline{\alpha}^+ t_0 e_1 \ldots e_n t_n)^+ & \text{else} \end{cases}$$

$$= \begin{cases} e(1_T e_1 t_1 \ldots e_n t_n)^+ & \text{if } t_0 = 1_T \\ e\overline{\alpha}^+ & \text{if } t_0 \neq 1_T \text{ and } \overline{\alpha}^+ \leq e \\ e\overline{\alpha}^+ (t_0 e_1 \ldots e_n t_n)^+ & \text{else} \end{cases}$$

$$= e\overline{\alpha}^+.$$

Finally, suppose now that $x = s_0 f_1 \ldots f_m s_m \in T * Y$, where $s_0, \ldots, s_m \in T$ and $f_1, \ldots, f_m \in Y$. If $m = 0$, then

$$\overline{\psi(x)(\alpha)}^+ = s_0 \cdot \overline{\alpha}^+ \leq s_0 \cdot 1_Y = s_0^+ = x^+.$$

Suppose inductively that $0 < i \leq m$ and

$$\overline{\psi(s_i f_{i+1} \ldots f_m s_m)(\alpha)}^+ \leq (s_i f_{i+1} \ldots f_m s_m)^+.$$

Then

$$\begin{aligned} \overline{\psi(s_{i-1} f_i s_i f_{i+1} \ldots f_m s_m)(\alpha)}^+ &= \overline{\psi(s_{i-1})\psi(f_i)\psi(s_i f_{i+1} \ldots f_m s_m)(\alpha)}^+ \\ &= s_{i-1} \cdot \overline{\psi(f_i)\psi(s_i f_{i+1} \ldots f_m s_m)(\alpha)}^+ \\ &= s_{i-1} \cdot \big(f_i(\overline{\psi(s_i f_{i+1} \ldots f_m s_m)(\alpha)}^+\big) \\ &\leq s_{i-1} \cdot \big(f_i(s_i f_{i+1} \ldots f_m s_m)^+\big) \\ &= (s_{i-1} f_i s_i \ldots f_m s_m)^+. \end{aligned}$$

Hence $\overline{\psi(x)(\alpha)}^+ \leq x^+$ by finite induction. Therefore, $H \subseteq \ker \psi$ as required. $\square$

In view of Lemma 2.5, we can define a semigroup morphism

$$\psi^* : (T * Y)/\sim \longrightarrow \mathcal{T}^*(\mathcal{N}) \text{ by } \psi^*([x]) = \psi(x).$$

Suppose now that $x \in T * Y$ and $x \sim \overline{\alpha} \sim \overline{\beta}$ for some $\alpha, \beta \in \mathcal{N}$. Then

$$\psi^*([x]) = \psi^*([\overline{\alpha}]) = \psi^*([\overline{\beta}])$$

so that in particular,

$$\psi(\overline{\alpha})(1_T) = \psi(\overline{\beta})(1_T).$$

Lemma 2.6. *For any $\alpha \in \mathcal{N}$, $\psi(\overline{\alpha})(1_T) = \alpha$.*

Proof. Let $\alpha = (t_0, e_1, \dots, e_n, t_n)$. If $n = 0$, then

$$\psi(\overline{\alpha})(1_T) = \psi(t_0)(1_T) = (t_0) = \alpha.$$

Suppose now that $\alpha = (t_0, e_1, \dots, e_n, t_n)$ where $n > 0$.
If $t_n = 1_T$, then

$$\begin{array}{rcl} \psi(\overline{\alpha})(1_T) & = & \psi(t_0e_1 \dots t_{n-1}e_n1_T)(1_T) \\ & = & \psi(t_0e_1 \dots t_{n-1}e_n)\psi(1_T)(1_T) \\ & = & \psi(t_0e_1 \dots t_{n-1}e_n)(1_T) \\ & = & \psi(t_0e_1 \dots t_{n-1})\psi(e_n)(1_T) \\ & = & \psi(t_0e_1 \dots t_{n-1})(1_T, e_n, 1_T) \\ & = & \psi(t_0e_1 \dots t_{n-1})(1_T, e_n, t_n) \end{array}$$

and if $t_n \neq 1_T$,

$$\begin{array}{rcl} \psi(\overline{\alpha})(1_T) & = & \psi(t_0e_1 \dots t_{n-1}e_nt_n)(1_T) \\ & = & \psi(t_0e_1 \dots t_{n-1})\psi(e_n)(t_n) \\ & = & \psi(t_0e_1 \dots t_{n-1})(1_T, e_n, t_n) \end{array}$$

where at the last step we use the fact that $e_n < t_n^+$.
Suppose inductively that $0 < i < n$ and

$$\psi(\overline{\alpha})(1_T) = \psi(t_0e_1 \dots t_i)(1_T, e_{i+1}, t_{i+1}, \dots, e_n, t_n).$$

Notice that $e_i < \overline{(t_i, e_{i+1}, \dots, e_n, t_n)}^+$ so that

$$e_i\overline{(t_i, e_{i+1}, \dots, e_n, t_n)}^+ = e_i < \overline{(t_i, e_{i+1}, \dots, e_n, t_n)}^+.$$

Then

$$\begin{array}{rcl} \psi(\overline{\alpha})(1_T) & = & \psi(t_0e_1 \dots t_{i-1}e_it_i)(1_T, e_{i+1}, t_{i+1}, \dots, e_n, t_n) \\ & = & \psi(t_0e_1 \dots t_{i-1})\psi(e_i)\psi(t_i)(1_T, e_{i+1}, t_{i+1}, \dots, e_n, t_n) \\ & = & \psi(t_0e_1 \dots t_{i-1})\psi(e_i)(t_i, e_{i+1}, t_{i+1}, \dots, e_n, t_n) \\ & = & \psi(t_0e_1 \dots t_{i-1})(1_T, e_i, t_i, e_{i+1}, t_{i+1}, \dots, e_n, t_n). \end{array}$$

By finite induction we obtain that

$$\begin{array}{rcl} \psi(\overline{\alpha})(1_T) & = & \psi(t_0)(1_T, e_1, t_1, \dots, e_n, t_n) \\ & = & (t_0, e_1, t_1, \dots, e_n, t_n) \\ & = & \alpha. \end{array}$$

□

We can now conclude that if $\alpha, \beta \in \mathcal{N}$ and $\overline{\alpha} \sim \overline{\beta}$, then $\alpha = \beta$, so that each equivalence class of $\sim$ contains a unique $\overline{\alpha}$ for $\alpha \in \mathcal{N}$, by Lemma 2.3.

Let $\mathcal{P} = \mathcal{P}(T, Y) = (T * Y)/\sim$, and let $\nu : T * Y \to \mathcal{P}$ be the natural morphism associated with $\sim$.

Lemma 2.7. *(i) Regarding T as a subsemigroup of $T*Y$, we have that $\nu|_T : T \to T'$ is an isomorphism; (ii) regarding Y as a subsemigroup of $T*Y$, we have that $\nu|_Y : Y \to Y'$ is an isomorphism.*

Proof. (i) We need only prove that $\nu|_T$ is injective. If $t_1, t_2 \in T$ and $[t_1] = [t_2]$, then we have

$$\overline{(t_1)} = t_1 \sim t_2 = \overline{(t_2)}$$

and as (t_1) and (t_2) are normal forms, $t_1 = t_2$.

(ii) Again, we need only prove that $\nu|_Y$ is injective. If $e_1, e_2 \in Y$ and $[e_1] = [e_2]$, then if $e_1 = 1_Y$ and $e_2 \neq 1_Y$, we have that

$$\overline{(1_T)} = 1_T \sim 1_Y = e_1 \sim e_2 = \overline{(1_T, e_2, 1_T)}$$

so that $(1_T) = (1_T, e_2, 1_T)$, which is impossible. Thus $e_1 = e_2 = 1_Y$, or $e_1, e_2 \in Y \setminus \{1_Y\}$. In the latter case,

$$\overline{(1_T, e_1, 1_T)} = e_1 \sim e_2 = \overline{(1_T, e_2, 1_T)}$$

so that $(1_T, e_1, 1_T) = (1_T, e_2, 1_T)$ and $e_1 = e_2$ as required. $\square$

We note that clearly $\mathcal{P}$ is a monoid with identity $[1_T]$. To show that $\mathcal{P}$ is left Ehresmann, we must first define a unary operation on $\mathcal{P}$. For $x \in T*Y$ we denote by $n(x)$ the unique element of $T*Y$ in normal form such that $x \sim n(x)$.

Lemma 2.8. *Let $^+$ be defined on $\mathcal{P}$ by $[x]^+ = [x^+]$. Then $^+$ is a well defined unary operation.*

Proof. Let $x \in T*Y$; we show that $x^+ = n(x)^+$; consequently, if $x \sim y$, then

$$x^+ = n(x)^+ = n(y)^+ = y^+,$$

so that $^+$ is well defined.

We may assume that $x = t_0 e_1 \dots e_n t_n$ for some $t_0, \dots, t_n \in T$ and $e_1, \dots, e_n \in Y \setminus \{1_Y\}$. If $n = 0$, then $x = n(x)$ so there is nothing to show.

Suppose inductively that $n > 0$ and

$$t_1 e_2 \dots e_n t_n \sim s_0 f_1 \dots f_m s_m$$

where $(s_0, f_1, \dots, f_m, s_m) \in \mathcal{N}$ and is such that

$$(t_1 e_2 \dots e_n t_n)^+ = (s_0 f_1 \dots f_m s_m)^+.$$

Notice that

$$x^+ = (t_0 e_1 t_1 e_2 \dots e_n t_n)^+ = t_0 e_1 \cdot (t_1 e_2 \dots e_n t_n)^+ = t_0 e_1 \cdot (s_0 f_1 \dots f_m s_m)^+$$

and

$$x = t_0 e_1 t_1 e_2 \dots e_n t_n \sim t_0 e_1 s_0 f_1 \dots f_m s_m.$$

Suppose first that $s_0 = 1_T$. If $m = 0$, then

$$x \sim t_0 e_1 s_0$$

and $(t_0, e_1, s_0) \in \mathcal{N}$. Hence $n(x) = t_0e_1s_0$. Also, we have

$$x^+ = t_0e_1 \cdot s_0^+ = (t_0e_1s_0)^+ = n(x)^+.$$

If $m > 0$, then

$$x \sim t_0e_1f_1s_2 \dots f_ms_m$$

and as $e_1f_1 \leq f_1 < (s_2 \dots f_ms_m)^+$, we have that $(t_0, e_1f_1, s_2, \dots, f_m, s_m) \in \mathcal{N}$ and

$$x^+ = t_0e_1 \cdot (f_1 \dots f_ms_m)^+ = (t_0e_1f_1 \dots f_ms_m)^+.$$

Assume now that $s_0 \neq 1_T$. Let $e = (s_0f_1 \dots f_ms_m)^+$. We have

$$x \sim t_0e_1es_0f_1 \dots f_ms_m;$$

if $e_1e < e$, then $(t_0, e_1e, s_0, f_1, \dots, f_m, s_m)$ is a normal form and

$$\begin{array}{rcl} x^+ &=& t_0e_1 \cdot (s_0f_1 \dots f_ms_m)^+ \\ &=& t_0e_1 \cdot e^2 \\ &=& t_0e_1e \cdot (s_0f_1 \dots f_ms_m)^+ \\ &=& (t_0e_1es_0f_1 \dots f_ms_m)^+ \\ &=& n(x)^+. \end{array}$$

Finally, if $e_1e = e$, then

$$x \sim t_0es_0f_1 \dots f_ms_m \sim t_0s_0f_1 \dots f_ms_m$$

where $(t_0s_0, f_1, \dots, f_m, s_m) \in \mathcal{N}$. Then

$$x^+ = t_0e_1 \cdot e = (t_0e_1e)^+ = (t_0e)^+ = t_0 \cdot (s_0f_1 \dots f_ms_m)^+ = (t_0s_0f_1 \dots f_ms_m)^+$$

so that $x^+ = n(x)^+$ as required. $\square$

Note that from Lemmas 2.7 and 2.8, if $[x], [y] \in \mathcal{P}$, then $[x]^+ = [y]^+$ if and only if $x^+ = y^+$ in $T * Y$.

Lemma 2.9. *With respect to $^+$ defined above, $\mathcal{P}$ is a left Ehresmann monoid.*

Proof. Notice that the image of $^+$ is given by

$$\{[x]^+ : x \in T * Y\} = \{[x^+] : x \in T * Y\} = \{[e] : e \in Y\} = Y',$$

and by Lemma 2.7, $Y \cong Y'$.

Let $[x] \in \mathcal{P}$; as $x \sim x^+x$ we have that

$$[x]^+[x] = [x^+][x] = [x^+x] = [x].$$

On the other hand, if $[e][x] = [x]$ where $e \in Y$, then $ex \sim x$ so that

$$x^+ = (ex)^+ = ex^+$$

and so

$$[x]^+ = [x^+] = [ex^+] = [e][x^+] = [e][x]^+.$$

Consequently, $[x] \, \widetilde{\mathcal{R}}_{Y'} \, [x]^+$.

To show that $\widetilde{\mathcal{R}}_{Y'}$ is a left congruence, suppose that $[x],[y] \in \mathcal{P}$ with $[x]^+ = [y]^+$. This tells us that $x^+ = y^+$ so that for any $[z] \in \mathcal{P}$,

$$(zx)^+ = z \cdot x^+ = z \cdot y^+ = (zy)^+$$

and so $([z][x])^+ = ([z][y])^+$ as required. □

Certainly $\mathcal{P} = \langle T' \cup Y' \rangle_{(2)}$; we must show that $\mathcal{P}$ has uniqueness of T'-normal forms. This follows from the next lemma.

Lemma 2.10. *Let $t_0 \dots, t_n \in T$ and $e_1, \dots, e_n \in Y$. Then*

$$[t_0][e_1][t_1] \dots [e_n][t_n]$$

is in T'-normal form if and only if

$$t_0 e_1 t_1 \dots e_n t_n$$

is in normal form.

Proof. Since $\nu|_T : T \to T'$ and $\nu|_Y : Y \to Y'$ are isomorphisms, certainly $[t_i] \neq 1_{\mathcal{P}}$ if and only if $t_i \neq 1_T$, for $0 \leq i \leq n$ and $[e_j] \neq 1_{\mathcal{P}}$ if and only if $e_j \neq 1_Y$, for $1 \leq j \leq n$.

Further,

$$\begin{aligned} [e_i] < ([t_i] \dots [e_n][t_n])^+ &\Leftrightarrow [e_i] < [t_i \dots e_n t_n]^+ \\ &\Leftrightarrow [e_i] < [(t_i \dots e_n t_n)^+] \\ &\Leftrightarrow e_i < (t_i \dots e_n t_n)^+ \end{aligned}$$

completing the proof of the lemma. □

With the exception of (*iv*), the remaining assertions of Theorem 2.2 follow from the results of Section 3 of [1], in which we analyse left Ehresmann monoids possessing U-normal forms for a submonoid U.

Lemma 2.11. *Let T act on Y by morphisms. Then $\mathcal{P}$ is hedged.*

Proof. Notice that if T acts by morphisms on Y, then, as certainly Y acts on itself by morphisms, $\phi : T * Y \to \mathcal{O}_Y^*$ has image contained in the (dual of the) endomorphism monoid $\mathcal{E}_Y^*$. Consequently, for any $[x] \in \mathcal{P}$ and $[e],[f] \in Y'$, where $e, f \in Y$, we have

$$[x] \cdot ([e][f]) = ([x][e][f])^+ = [(xef)^+] = [x \cdot ef] = [(x \cdot e)(x \cdot f)]$$

$$= [(xe)^+(xf)^+] = [xe]^+[xf]^+ = ([x][e])^+([x][f])^+ = ([x] \cdot [e])([x] \cdot [f])$$

as required. □

□

3. Covers and a structure theorem

We can now present a structure theorem for left Ehresmann monoids which have uniqueness of T-normal forms with respect to a submonoid T. As a corollary, we show that every left Ehresmann monoid M where $M = \langle T \cup E \rangle_{(2)}$ for a submonoid T, has a cover of the form $\mathcal{P}(T, E)$.

Theorem 3.1. *Let M be a left Ehresmann monoid. Suppose that T is a submonoid of M such that $M = \langle E \cup T \rangle_{(2)}$ and $\theta : U \to T$ is a monoid morphism from a monoid U onto T. Then M has a U-proper cover $\mathcal{P} = \mathcal{P}(U, E)$.*

Proof. As remarked in Section 1, T acts on E by order-preserving maps via $t \cdot e = (te)^+$. It follows that U acts on E by order-preserving maps via $u \circ e = u\theta \cdot e$. Let $\mathcal{P} = (U * E)/\sim$ be constructed as in Theorem 2.2.

Let $\psi : U * E \to M$ be given by $u\psi = u\theta$ and $e\psi = e$. Then $1_U\psi = 1_T = 1_M$ and $1_E\psi = 1_E = 1_M$ so that $(1_U, 1_E) \in \ker\psi$. Suppose now that $x \in U * E$; without loss of generality we may assume that $x = u_0 e_1 \dots e_n u_n$ where $u_0 \dots, u_n \in U$ and $e_1, \dots, e_n \in E$. Then

$$\begin{array}{rcl} x^+ & = & x \circ 1_E = (u_0 e_1 \dots e_n u_n) \circ 1_E \\ & = & u_0 e_1 \dots e_n \circ (u_n \circ 1_E) \\ & = & u_0 e_1 \dots e_n \circ (u_n\theta)^+ \\ & = & u_0 e_1 \dots u_{n-1} \circ e_n (u_n\theta)^+ \\ & = & u_0 e_1 \dots e_{n-1} \circ (u_{n-1}\theta e_n (u_n\theta)^+)^+ \\ & = & u_0 e_1 \dots e_{n-1} \circ (u_{n-1}\theta e_n u_n\theta)^+ \\ & \vdots & \\ & = & (u_0\theta e_1 \dots e_{n-1} u_{n-1}\theta e_n u_n\theta)^+ \\ & = & (x\psi)^+. \end{array}$$

It follows that

$$(x^+ x)\psi = x^+\psi x\psi = x^+(x\psi) = (x\psi)^+ x\psi = x\psi$$

so that $\sim \subseteq \ker\psi$ and there is an induced semigroup morphism $\overline{\psi} : \mathcal{P} \to M$ given by $[x]\overline{\psi} = x\psi$. Since $U\theta \cup E$ generates M as a semigroup, it is clear that $\overline{\psi}$ is onto. It is also clear that $\overline{\psi}$ is a monoid morphism which separates the idempotents of $E' = \{[e] : e \in E\}$.

To see that $\overline{\psi}$ respects $^+$, we note that for $[x] \in \mathcal{P}$ with $x = u_0 e_1 \dots e_n u_n$ as above,

$$[x]^+\overline{\psi} = [x^+]\overline{\psi} = x^+\psi = x^+ = (x\psi)^+ = [x]\overline{\psi}^+.$$

□

Corollary 3.2. *Let M be a left Ehresmann monoid with set of generators X (as a $(2,1,0)$-algebra). Then M has a left adequate X^*-proper cover $\mathcal{P} = \mathcal{P}(X^*, E)$.*

Proof. From Corollary 1.12 of [1], we have that $M = \langle E \cup T \rangle_{(2)}$, where $T = \langle X \rangle_{(2,1)}$. Let $\iota : X \to T$ be inclusion, so that ι lifts to a morphism from X^* onto T. Now call upon Theorem 3.1 to construct $\mathcal{P}$. □

Corollary 3.3. *The variety generated by the quasi-variety of left adequate monoids is the variety of left Ehresmann monoids.*

Corollary 3.4. *Let M be a left Ehresmann monoid. Suppose that T is a submonoid of M such that $M = \langle E \cup T\rangle_{(2)}$. Then M has a T-proper cover $\mathcal{P} = \mathcal{P}(T, E)$. Moreover, the covering morphism is an isomorphism if and only if M has uniqueness of T-normal forms.*

Proof. With $U = T$ and θ the identity map, we see from Theorem 3.1 that $\overline{\psi} : \mathcal{P} \to M$ is a covering morphism. Notice that from the proof of that theorem, for any $x \in T * E$ we have that x^+ *in* $T * E$ coincides with $(x\psi)^+$, that is, with x^+ *in* M. It follows that for any $t_0, t_1, \dots, t_n \in T$ and $e_1, e_2, \dots, e_n \in E$, we have that $(t_0, e_1, \dots, e_n, t_n) \in \mathcal{N}$ if and only if $t_0e_1 \dots e_nt_n$ is an element of M in T-normal form.

The onto morphism $\overline{\psi}$ is an isomorphism if and only if it is one-one. Given the fact that any element of $\mathcal{P}$ has a unique normal form, $\overline{\psi}$ is one-one if and only if for any $(t_0, e_1, \dots, e_n, t_n), (s_0, f_1, \dots, f_m, s_m) \in \mathcal{N}$,

$$[t_0e_1 \dots e_nt_n]\overline{\psi} = [s_0f_1 \dots f_ms_m]\overline{\psi}$$

implies that

$$(t_0, e_1, \dots, e_n, t_n) = (s_0, f_1, \dots, f_m, s_m).$$

That is, $\overline{\psi}$ is one-one if and only if

$$t_0e_1 \dots e_nt_n = s_0f_1 \dots f_ms_m$$

in M implies that

$$(t_0, e_1, \dots, e_n, t_n) = (s_0, f_1, \dots, f_m, s_m),$$

that is, if and only if M has uniqueness of T-normal forms. □

4. The free left Ehresmann monoid

Since left Ehresmann monoids form a non-trivial variety, containing the non-trivial quasi-variety of left adequate monoids, the free left Ehresmann monoid $\mathcal{FLEM}(X)$ and the free left adequate monoid $\mathcal{FLA}d\mathcal{M}(X)$ exist, for any non-empty set X. In this section we use the construction of Section 2 to give an explicit description of $\mathcal{FLEM}(X)$. We find a semilattice E_X constructed from X and define an action of X^* on E_X via order-preserving maps. We then show that $\mathcal{P} = (X^* * E_X)/\sim$ constructed as in Theorem 2.2 is the free left Ehresmann monoid for which we seek. As X^* is certainly cancellative and has no left units other than 1, it follows that $\mathcal{P}$ is also the free left adequate monoid on X. An alternative description of $\mathcal{FLEM}(X)$ appears in [13].

A few words concerning background. It is shown in [4] and [8] that the semilattice of idempotents of the free left ample monoid $\mathcal{FLA}m(X)$ on X is a subsemilattice of the semilattice of idempotents of the free inverse monoid on X.

Moreover, it follows from [6] that $\mathcal{FLA}m(X)$ coincides with the free left restriction monoid on X. One way of describing the semilattice of idempotents F_X of $\mathcal{FLA}m(X)$ is as follows.

First, if Z is a set partially ordered by $\leq$, then for any subset A of Z we denote by min A the set of minimal elements of A. We recall that

$$\{A \subseteq Z : 0 < |A| < \infty,\ \min A = A\}$$

is then a semilattice under the operation

$$A \wedge B = \min\,(A \cup B).$$

On X^* we define a partial order by $u \leq v$ if $u = vw$ for some $w \in X^*$, that is, if v is a prefix of u. Then

$$F_X = \{A \subseteq X^* : 0 < |A| < \infty,\ \min A = A\},$$

and the operation of meet on F_X is given in the standard way by

$$A \wedge B = \min A \cup B.$$

The free monoid X^*, which, by Theorem 5.1 of [1] is embedded in both $\mathcal{FLEM}(X)$ and $\mathcal{FLA}m(X)$, acts on F_X by morphisms where

$$w \cdot A = \{wa : w \in A\}.$$

From universal considerations it is easy to see that F_X must be a morphic image of E_X. We now show how to construct E_X from X.

Let $U_0 = \{1\}$, and let $Y_0 = \{\{1\}\}$, where 1 is the identity of X^*. We now put

$$U_1 = \{(x, A) : x \in X, A \in Y_0\} \cup U_0 = \{(x, \{1\}) : x \in X\} \cup \{1\}$$

and extend the trivial partial order on U_0 to U_1 by declaring $(x, \{1\}) \leq 1$, for all $x \in X$. Next, we put

$$Y_1 = \{W \subseteq U_1 : W = \min W, 0 < |W| < \infty\}.$$

Note that $\{1\} \in Y_1$ and $Y_0 \subseteq Y_1$. Then Y_1 is a semilattice under the operation

$$A \wedge B = \min\,(A \cup B).$$

Suppose inductively that $n \geq 2$ and $U_0, Y_0, U_1, Y_1, \ldots, U_{n-1}, Y_{n-1}$ have been defined such that:

– for $1 \leq j \leq n-1$,

$$U_j = \{(x, A) : x \in X, A \in Y_{j-1}\} \cup \{1\};$$

– for $1 \leq j \leq n-1$, the partial order on U_{j-1} is extended to U_j by declaring 1 to be the greatest element and

$$(x, A) \leq (x, B) \text{ if and only if } A \leq B \text{ in } Y_{j-1};$$

– for $0 \leq j \leq n-1$,

$$Y_j = \{W \subseteq U_j : W = \min W, 0 < |W| < \infty\};$$

– for $0 \leq j \leq n-1$, each Y_j is a semilattice where

$$A \wedge B = \min(A \cup B);$$

– for $0 \leq j \leq n-1$, the element 1 is the greatest in U_j, so that $\{1\}$ is the greatest element of Y_j;

– $U_0 \subseteq U_1 \subseteq \ldots U_{n-1}$ and $Y_0 \subseteq Y_1 \subseteq \ldots Y_{n-1}$.

We now let

$$U_n = \{(x, A) : x \in X, A \in Y_{n-1}\} \cup \{1\}$$

so that certainly $U_{n-1} \subseteq U_n$, and we define $\leq$ on U_n by the rule that for any $\alpha, \beta \in U_n$,

$$\alpha \leq \beta \Leftrightarrow \begin{cases} \beta = 1 \text{ or} \\ \alpha = (x, A), \beta = (x, B) \text{ and } A \leq B \text{ in } Y_{n-1}. \end{cases}$$

Since Y_{n-1} is a semilattice, it is clear that $\leq$ is a partial order; moreover as Y_{n-2} is a subsemilattice of Y_{n-1}, the relation $\leq$ extends the partial order in U_{n-1}.

We now let

$$Y_n = \{W \subseteq U_n : W = \min W, 0 < |W| < \infty\}$$

so that Y_n becomes a semilattice under

$$A \wedge B = \min\,(A \cup B),$$

having subsemilattice Y_{n-1}, and and $\{1\}$ as greatest element.

We may now put $U_X = \bigcup_{i \in \mathbb{N}^0} U_i$ and $E_X = \bigcup_{i \in \mathbb{N}^0} Y_i$. Notice that E_X has greatest element $\{1\}$.

We now define an action of X^* on the left of E_X as follows: for $x \in X$ and $A \in E_X$ we put

$$x \cdot A = \{(x, A)\}.$$

Lemma 4.1. *For $A, B \in E_X$ with $A \leq B$, and $x \in X$, we have that $x \cdot A \leq x \cdot B$.*

Proof. From the ordering in U_X, if $A \leq B$, then $(x, A) \leq (x, B)$. Hence

$$\begin{aligned} x \cdot A \wedge x \cdot B &= \{(x, A)\} \wedge \{(x, B)\} \\ &= \min\,\{(x, A), (x, B)\} \\ &= \{(x, A)\} \\ &= x \cdot A, \end{aligned}$$

so that $x \cdot A \leq x \cdot B$ as required. □

Thus we have a map from X to $\mathcal{OP}^*(E_X)$, the monoid of order-preserving maps on E_X with composition from right-to-left. From the freeness property of X^*, we have a morphism from X^* to $\mathcal{OP}^*(E_X)$ and hence an action of X^* on the left of E_X via order-preserving maps.

From the order preserving action of X^* on the left of the semilattice E_X, we can construct

$$\mathcal{P}_X = \mathcal{P}(X^*, E_X) = (X^* * E_X)/\sim$$

as in Section 2. Notice that as X^* is cancellative and has no units other than 1, $\mathcal{P}_X$ is left adequate. Our aim is to show that $\mathcal{P}_X = \mathcal{FLEM}(X)$.

Let $\theta : X \to M$, where M is a left Ehresmann monoid. We wish to prove there exists a unique morphism $\widetilde{\widetilde{\theta}} : \mathcal{P}_X \to M$ such that $\iota\widetilde{\widetilde{\theta}} = \theta$, where $\iota : X \to \mathcal{P}_X$ is given by $x\iota = [x]$. As X^* is the free monoid on X, θ extens to a unique monoid morphism, which we also denote by θ, from X^* to M. We define a map $\overline{\theta}$ from E_X to E inductively. First, we put $1\overline{\theta} = 1$ to get a map $\overline{\theta} : U_0 \to E$. Suppose now that $n \geq 1$ and $\alpha\overline{\theta}$ is defined for all $\alpha \in U_{n-1}$. Let $(x, A) \in U_n \setminus U_{n-1}$, so that

$$x \in A, A \in Y_{n-1} \setminus Y_{n-2} \text{ and } A \subseteq U_{n-1}.$$

If $A = \{\alpha_1, \ldots, \alpha_m\}$, let

$$A\overline{\theta} = \alpha_1\overline{\theta} \ldots \alpha_m\overline{\theta} \in E.$$

Next, we define $\overline{\theta} : U_n \to E$ by

$$(x, A)\overline{\theta} = (x\theta A\overline{\theta})^+$$

We now define a map, which we again denote by $\overline{\theta}$, from E_X to E by

$$A\overline{\theta} = \Pi_{\beta \in A}\beta\overline{\theta}.$$

Lemma 4.2. *Let X, M, θ and $\overline{\theta}$ be as above. Then $\overline{\theta} : E_X \to E$ is a monoid morphism.*

Proof. We are given that $1\overline{\theta} = 1$ so that also

$$\{1\}\overline{\theta} = 1\overline{\theta} = 1\theta = 1.$$

First we show that for any $n \in \mathbb{N}^0$, if $\overline{\theta}$ preserves order in U_n, then $\overline{\theta} : Y_n \to E$ is a monoid morphism. For, under this assumption, suppose that $A = \{\alpha_1, \ldots, \alpha_r\}, B = \{\beta_1, \ldots, \beta_s\} \in Y_n$. Without loss of generality assume that for some $u, v \geq 0$ (with at least one of $u, v \geq 1$), we have that

$$A \wedge B = \{\alpha_1, \ldots, \alpha_u, \beta_1, \ldots, \beta_v\}.$$

Then

$$(A \wedge B)\overline{\theta} = \alpha_1\overline{\theta} \ldots \alpha_u\overline{\theta}\beta_1\overline{\theta} \ldots \beta_v\overline{\theta}.$$

Suppose that $u < r$; for any $p \in \{u + 1, \ldots, r\}$ we have that $\beta_q \leq \alpha_p$ for some $q \in \{1, \ldots, v\}$. Then as $\overline{\theta}$ preserves order in U_n, we have that $\beta_q\overline{\theta} \leq \alpha_p\overline{\theta}$, whence

$$(A \wedge B)\overline{\theta} = \alpha_1\overline{\theta} \ldots \alpha_u\overline{\theta}\alpha_p\overline{\theta}\beta_1\overline{\theta} \ldots \beta_v\overline{\theta},$$

and as this is true for any such p, we obtain

$$(A \wedge B)\overline{\theta} = \alpha_1\overline{\theta} \ldots \alpha_r\overline{\theta}\beta_1\overline{\theta} \ldots \beta_v\overline{\theta}.$$

On the other hand, if $v < s$, then for any $p \in \{v + 1, \ldots, s\}$, in a similar fashion we may add $\beta_p\theta$ to the expression for $(A \wedge B)\overline{\theta}$ to obtain

$$(A \wedge B)\overline{\theta} = \alpha_1\overline{\theta} \ldots \alpha_r\overline{\theta}\beta_1\overline{\theta} \ldots \beta_s\overline{\theta} = A\overline{\theta}B\overline{\theta}.$$

Let us prove that $\overline{\theta}$ preserves order in U_n, for $n \in \mathbb{N}^0$. This is clear for $n \in \{0,1\}$. Hence $\overline{\theta} : Y_1 \to E$ is a monoid morphism.

Suppose now that $n \geq 1$ and $\overline{\theta}$ preserves order in U_n, so that $\overline{\theta} : Y_n \to E(M)$ is a monoid morphism. We show that $\overline{\theta}$ preserves the order in U_{n+1}. First, for any $(x, A) \in U_{n+1}$,

$$(x, A)\overline{\theta} \leq 1 = 1\overline{\theta}.$$

The other inequalities in U_{n+1} are all of the form $(x, A) \leq (x, B)$, where $A, B \in Y_n$ and $A \leq B$. By our inductive assumption and the remark above, $A\overline{\theta} \leq B\overline{\theta}$. In the comments following Lemma 1.7, if we take T to be M, then we see that

$$\begin{aligned} (x, A)\overline{\theta} &= (x\theta A\overline{\theta})^+ \\ &= x\theta \cdot A\overline{\theta} \\ &\leq x\theta \cdot B\overline{\theta} \\ &= (x\theta B\overline{\theta})^+ \\ &= (x, B)\overline{\theta}. \end{aligned}$$

Since $\overline{\theta}$ preserves order in U_{n+1} we therefore have a morphism $Y_{n+1} \to E$. By induction, we have that $\overline{\theta} : U_X \to E$ is order preserving and $\overline{\theta} : E_X \to E$ is a monoid morphism. □

Lemma 4.3. *With notation as above,*

$$(w \cdot A)\overline{\theta} = (w\theta A\overline{\theta})^+$$

for any $A \in E_X$ and $w \in X^*$.

Proof. We argue by induction on the length $|w|$ of w. If $|w| = 0$, then $(w \cdot A)\overline{\theta} = (1 \cdot A)\overline{\theta} = A\overline{\theta} = (1\theta A\overline{\theta})^+$. If $|w| = 1$, then

$$(w \cdot A)\overline{\theta} = \{(w, A)\}\overline{\theta} = (w, A)\overline{\theta} = (w\theta A\overline{\theta})^+.$$

Suppose now that the result is true for words of length n and $w = xv$ where $x \in X$ and $|v| = n$. Then

$$\begin{aligned} (w \cdot A)\overline{\theta} &= (x \cdot (v \cdot A))\overline{\theta} \\ &= (x\theta(v \cdot A)\overline{\theta})^+ && \text{as above} \\ &= (x\theta(v\theta A\overline{\theta})^+)^+ && \text{by inductive assumption} \\ &= (x\theta v\theta A\overline{\theta})^+ \\ &= ((xv)\theta A\overline{\theta})^+ \\ &= (w\theta A\overline{\theta})^+. \end{aligned}$$

Hence result by induction. □

Thus far we have monoid morphisms $\theta : X^* \to M$ and $\overline{\theta} : E_X \to E$. From the universal property of the free semigroup product, we can construct a semigroup morphism $\widetilde{\theta} : X^* * E_X \to M$ extending both θ and $\overline{\theta}$.

Lemma 4.4. *With notation as above, $\sim \subseteq Ker\, \widetilde{\theta}$.*

Proof. We show that the generating set H of $\sim$ is contained in ker $\widetilde{\theta}$. To avoid confusion here we write 1_{X^*} for the identity of X^* and $1_{E_X} = \{1_{X^*}\}$ for the identity of E_X. Since

$$1_{X^*}\widetilde{\theta} = 1_{X^*}\theta = 1 = \{1_{X^*}\}\overline{\theta} = \{1_{X^*}\}\widetilde{\theta} = 1_{E_X}\widetilde{\theta},$$

we have that $(1_{X^*}, 1_{E_X}) \in \text{Ker } \widetilde{\theta}$.

Consider now $\alpha = w_0 e_1 \ldots e_n w_n \in X^* \times E_X$ where $w_i \in X^*$ and $e_j \in E_X$, for $0 \leq i \leq n$ and $1 \leq j \leq n$. Using Lemma 4.3 we have

$$\begin{array}{rcl}
\alpha^+\widetilde{\theta} & = & \alpha^+\overline{\theta} = (\alpha \cdot 1_{E_X})\overline{\theta} \\
& = & (w_0 \cdot (e_1(w_1 \cdot (\ldots (w_{n-1} \cdot (e_n(w_n \cdot \{1_{X^*}\}))) \ldots))))\overline{\theta} \\
& = & (w_0\theta(e_1(w_1 \cdot (\ldots (w_n \cdot \{1_{X^*}\}) \ldots)))\overline{\theta})^+ \\
& = & (w_0\theta e_1\overline{\theta}(w_1 \cdot (\ldots (w_n \cdot \{1_{X^*}\}) \ldots))\overline{\theta})^+ \\
& = & (w_0\theta e_1\overline{\theta}(w_1\theta(e_2(\ldots (w_n \cdot \{1_{X^*}\}) \ldots))\overline{\theta})^+)^+ \\
& = & (w_0\theta e_1\overline{\theta}w_1\theta(e_2(\ldots (w_n \cdot \{1_{X^*}\}) \ldots))\overline{\theta})^+ \\
& = & (w_0\theta e_1\overline{\theta}w_1\theta e_2\overline{\theta}(\ldots (w_n \cdot \{1_{X^*}\}) \ldots)\overline{\theta})^+ \\
& = & (w_0\theta e_1\overline{\theta}w_1\theta e_2\overline{\theta} \ldots e_n\overline{\theta}(w_n \cdot \{1_{X^*}\})\overline{\theta})^+ \\
& = & (w_0\theta e_1\overline{\theta} \ldots e_n\overline{\theta}(w_n\theta\, 1)^+)^+ \\
& = & (w_0\theta e_1\overline{\theta} \ldots w_n\theta)^+ \\
& = & (\alpha\widetilde{\theta})^+.
\end{array}$$

If α begins/ends with an element of E_X we can add 1_{X^*} to the front and/or back of α to obtain a new element β of $X^* * E_X$ that both begins and ends with an element of X^*. By a remark in Section 2, $\alpha^+ = \beta^+$ and certainly $\alpha\widetilde{\theta} = \beta\widetilde{\theta}$. It follows that for *any* $\alpha \in X^* * Y_X$ we have that $\alpha^+\widetilde{\theta} = (\alpha\widetilde{\theta})^+$.

Since $\widetilde{\theta}$ is a semigroup morphism, we consequently have that

$$(\alpha^+\alpha)\widetilde{\theta} = \alpha^+\widetilde{\theta}\alpha\widetilde{\theta} = (\alpha\widetilde{\theta})^+\alpha\widetilde{\theta} = \alpha\widetilde{\theta},$$

so that $(\alpha^+\alpha, \alpha) \in \text{Ker } \widetilde{\theta}$. □

From Lemma 4.4, it follows that $\widetilde{\theta}$ induces a semigroup morphism $\widetilde{\widetilde{\theta}}$ from $\mathcal{P}_X$ to M, given by $[\alpha]\widetilde{\widetilde{\theta}} = \alpha\widetilde{\theta}$. Now $[1_{X^*}] = [1_{E_X}]$ is the identity of $\mathcal{P}_X$ and clearly

$$[1_{X^*}]\widetilde{\widetilde{\theta}} = 1_{X^*}\widetilde{\theta} = 1_{X^*}\theta = 1,$$

so that $\widetilde{\widetilde{\theta}}$ is a monoid morphism. From the above, for any $\alpha \in X^* * E_X$, we have that

$$([\alpha]\widetilde{\widetilde{\theta}})^+ = (\alpha\widetilde{\theta})^+ = \alpha^+\widetilde{\theta} = [\alpha^+]\widetilde{\widetilde{\theta}} = [\alpha]^+\widetilde{\widetilde{\theta}},$$

so that $\widetilde{\widetilde{\theta}}$ is a $(2, 1, 0)$-morphism.

Moreover, if $\iota : X \to \mathcal{P}_X$ is the map given by $x\iota = [x]$, then

$$x\iota\widetilde{\widetilde{\theta}} = [x]\widetilde{\widetilde{\theta}} = x\widetilde{\theta} = x\theta.$$

Theorem 4.5. *The left adequate monoid $\mathcal{P}_X = \mathcal{P}(X^*, E_X)$ is the free left Ehresmann monoid on X.*

Proof. It remains to show that for any left Ehresmann monoid M, and any $\theta : X \to M$, the morphism $\widetilde{\widetilde{\theta}} : \mathcal{P}_X \to M$ is the unique morphism ϕ from $\mathcal{P}_X$ to M such that $\iota\phi = \theta$. This follows from the fact that $X\iota$ generates $\mathcal{P}_X$, as we now show.

Clearly, for any $w = x_1 \dots x_n \in X^*$, where $x_i \in X$,

$$[w] = [x_1 \dots x_n] = [x_1] \dots [x_n] = (x_1\iota) \dots (x_n\iota).$$

Certainly $[1_{X^*}] \in \langle X\iota \rangle_{(2,1,0)}$, the $(2,1,0)$-algebra generated by X. If we can show that $[A] \in \langle X\iota \rangle_{(2,1,0)}$ for all $A \in E_X$, then it will follow immediately that $\langle X\iota \rangle_{(2,1,0)} = \mathcal{P}_X$.

Certainly $[\{1_{X^*}\}] \in \langle X\iota \rangle_{(2,1,0)}$. Suppose for induction that $[A] \in \langle X\iota \rangle_{(2,1,0)}$ for all $A \in Y_n$.

Let $B \in Y_{n+1}$, so that $B \subseteq U_{n+1}$ and

$$B = Z_1 \wedge \dots \wedge B_m$$

for some singleton subsets B_i of U_{n+1}, $1 \leq i \leq m$.

If for some i, we have that $B_i \subseteq U_n$, then $B_i \in Y_n$, so that $[B_i] \in \langle X\iota \rangle_{(2,1,0)}$ by our inductive assumption. On the other hand, if $B_i \in U_{n+1} \setminus U_n$, then

$$B_i = \{(x, A)\}$$

for some $x \in X$ and $A \in Y_n$, so that again, $[A] \in \langle X\iota \rangle_{(2,1,0)}$. Now in $X^* * E_X$ we have that

$$(xA)^+ = xA \cdot \{1_{X^*}\} = x \cdot A = \{(x, A)\},$$

so that

$$[B_i] = [\{(x, A)\}] = [(xA)^+] = [xA]^+ = (x\iota[A])^+ \in \langle X\iota \rangle_{(2,1,0)}.$$

It follows that

$$[B] = [B_1 \wedge \dots \wedge B_m] = [B_1] \dots [B_m] \in \langle X\iota \rangle_{(2,1,0)}.$$

By induction, $[A] \in \langle X\iota \rangle_{(2,1,0)}$ for any $A \in E_X$, as required. □

From Theorem 2.2 we immediately have the following.

Corollary 4.6. *For any non-empty set X, the free left Ehresmann monoid coincides with the free left adequate monoid on X.*

Writing any element of $\mathcal{P}_X$ as a $(2,1,0)$-term over $X\iota$, and abbreviating the identity by 1 and $x\iota$ by x, for any $x \in X$, leads to a great simplification in the description of elements of E_X and hence of $\mathcal{P}_X$. For example,

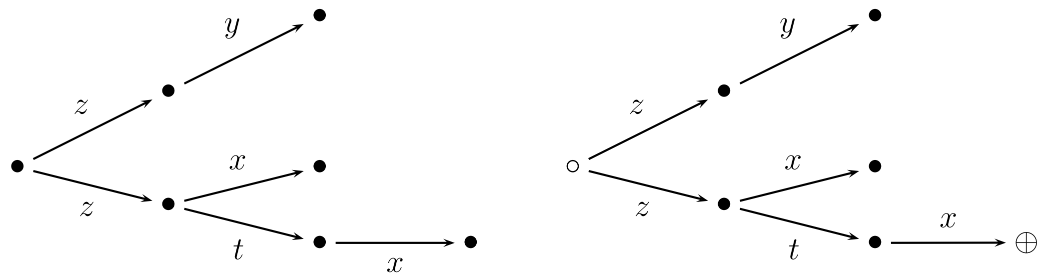
$$\begin{array}{rcl} A & = & [\{(y,\{1\})\}] = (y1)^+ = y^+, \\ B & = & [\{(z,\{(y,\{1\})\})\}] = (zy)^+, \\ C & = & \big[\big\{(z,\{(y,\{1\})\}),(z,\{(x,\{1\}),(t,\{(x,\{1\})\})\})\big\}\big] = (zy)^+(z(x^+(tx)^+))^+ \end{array}$$

and

$$D = \big[\big\{(z,\{(y,\{1\}),(x,\{1\}),(t,\{(x,\{1\})\})\})\big\}\big] = (z(y^+x^+(tx)^+))^+.$$

Notice that if the action of X^* on E_X were by morphisms, then C would be equal to D; but, the action is not by morphism, and this is essentially what has led us to a very careful approach to the ordering in E_X.

Subsequent to our description of E_X and $\mathcal{P}_X$, the authors learnt of an alternative approach to $\mathcal{FLEM}(X)$, due to Kambites [12], in which the idempotents correspond to trees labelled by the elements of X. To illustrate, the tree on the left below



is a picture of the element C of E_X above. Kambites developes a notion of 'pruning' of trees via labelled graph morphisms to define an ordering on E_X; it is straightforward to check that his ordering coincides with ours. The remarkable insight of [12] is the realisation that not only can the idempotents of E_X be represented by labelled trees, but also, by distinguishing two vertices, the elements themselves. By considering branching points, one may see how an element of $\mathcal{P}_X$ in normal form corresponds to a labelled tree. The element of $\mathcal{P}(X^*,E_X)$ depicted by the tree on the right above is $(zy)^+zx^+tx$, or in X^*-normal form,

$$I_{X^*}\big((zy)^+(zx^+(tx)^+)^+\big)z(x^+(tx)^+)tx.$$

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