EXTENSIONS AND COVERS FOR SEMIGROUPS WHOSE IDEMPOTENTS FORM A LEFT REGULAR BAND

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Abstract. Proper extensions that are "injective on $\mathcal{L}$-related idempotents" of $\mathcal{R}$-unipotent semigroups, and much more generally of the class of generalised left restriction semigroups possessing the ample and congruence conditions, referred to here as *glrac* semigroups, are described as certain subalgebras of a λ-semidirect product of a left regular band by an $\mathcal{R}$-unipotent or by a glrac semigroup, respectively. An example of such is the generalized Szendrei expansion.

As a consequence of our embedding, we are able to give a structure theorem for proper left restriction semigroups. Further, we show that any glrac semigroup S has a proper cover that is a semidirect product of a left regular band by a monoid, and if S is left restriction, the left regular band may be taken to be a semilattice.

Dedicated to the memory of our friend, Prof. Douglas Munn

1. Introduction

The generalized prefix expansion S^{Pr} of an $\mathcal{R}$-unipotent semigroup [3] was proved to be an idempotent pure extension of S through the second projection η_S which is injective on $\mathcal{L}$-related idempotents. However, not all idempotent pure extensions of an $\mathcal{R}$-unipotent semigroup are of this kind.

The first task of this article is to describe such special extensions by means of a variation of a semidirect product of a left regular band by another $\mathcal{R}$-unipotent semigroup; the variation we are talking of is

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Billhardt's notion of λ-semidirect product, introduced in [2] for inverse semigroups.

More generally we look at a wider class of generalised left restriction semigroups: all semigroups in our class contain a distinguished set of idempotents that must be a left regular band. The class we consider is referred to here for convenience as being that of *glrac* semigroups. The regular members of this class are the $\mathcal{R}$-unipotent semigroups; glrac semigroups generalise $\mathcal{R}$-unipotent semigroups in the same way that left restriction semigroups generalise inverse semigroups. Glrac semigroups were introduced in [1] under another name.

Proper extensions that are "injective on $\mathcal{L}$-related distinguished idempotents", that is, proper $\mathcal{L}_E$-extensions, of glrac ($\mathcal{R}$-unipotent) semigroups, are described as certain $(2,1)$-subalgebras of a λ-semidirect product of a left regular band by another glrac ($\mathcal{R}$-unipotent) semigroup. Although we do not mention it explicitly, the approach of Gomes using categorical constructions (see, for example [9]) underlies our proof. We have merely 'unwound' the categorical notation to produce a more direct approach.

If the left regular band of distinguished idempotents of a glrac semigroup S is a semilattice, that is, S is left restriction, then every proper extension of S is an $\mathcal{L}_E$-extension. It follows that if S is proper, then it embeds into a semidirect product of a semilattice by a monoid. The nature of this embedding allows us to deduce a structure theorem for proper left restriction semigroups, which may be regarded as an analogue of the McAlister P-theorem [17].

McAlister showed that every inverse semigroup has an E-unitary, or proper, cover [16]. In this spirit, we show that every glrac (left restriction) semigroup has a proper cover which is a semidirect product of a left regular band (semilattice) by a monoid. We offer a further covering result in a more general case.

In Section 2 we set the notation and basic results. Section 3 is dedicated to relating proper and idempotent pure extensions. Section 4 presents the generalized prefix expansions and the graph expansions of a glrac semigroup S as examples of proper $\mathcal{L}_E$-extensions of S. Then in Section 5 we obtain the desired embedding of a proper $\mathcal{L}_E$-extension of a glrac semigroup [$\mathcal{R}$-unipotent] semigroup into a λ-semidirect product. We offer two approaches to covering theorems in Section 6. The paper concludes with Section 7 in which we give the promised structure theorem for proper left restriction semigroups.

2. Preliminaries

We start by giving the notation and background results necessary for the rest of the paper.

Given a semigroup S, we denote as usual its set of idempotents by $E(S)$ and the set of inverses of an element a of S by $V(a)$.

We recall that a band B is said to be *left regular* if $efe = ef$, for all $e, f \in B$, which is easily seen to be equivalent to Green's relation $\mathcal{R}$ being trivial. We will be concerned with semigroups S containing a left regular band B of idempotents. If S is regular and $B = E(S)$, this is equivalent to every $\mathcal{R}$-class of S containing a unique idempotent and consequently we say that S is $\mathcal{R}$-*unipotent*.

In [1], a class of semigroups is introduced, referred to in that article as *weakly left quasi-ample* or *wlqa* semigroups. In the light of subsequent work on *restriction semigroups* (see, for example [4]), we now think of wlqa semigroups as a class of *generalised restriction semigroups*. These are semigroups, that need not be regular, containing a left regular band of idempotents; they may be defined in terms of the generalisation $\widetilde{\mathcal{R}}$ of $\mathcal{R}$, or as a quasi-variety of algebras of type $(2, 1)$, that is, possessing a binary and a unary operation. We briefly describe the two approaches; for details, the reader is referred to [12].

Let S be a semigroup and let $E \subseteq E(S)$; we do not presume that $E = E(S)$, but where it does, we may drop explicit mention of E in our notation. The relation $\widetilde{\mathcal{R}}_E$ on S is defined by the rule that for all $a, b \in S$,

$$a \,\widetilde{\mathcal{R}}_E\, b \text{ if and only if } \{e \in E : ea = a\} = \{e \in E : eb = b\}$$

and the relation $\mathcal{R}^*$ on S by the rule that for all $a, b \in S$,

$$a \,\mathcal{R}^*\, b \text{ if and only if } \forall\, x, y \in S^1, (xa = ya \Leftrightarrow xb = yb).$$

Clearly $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}_E$ are equivalences, and it is easy to see that

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$$

with the inclusions being equality if S is regular and $E = E(S)$. Moreover, $\mathcal{R}$ and $\mathcal{R}^*$ are left compatible, but $\widetilde{\mathcal{R}}_E$ may fail to be so. This last fact is shown by a very simple example: the null semigroup with two elements with an adjoined identity. Clearly, if $a \in S$ and $e \in E$, then

$$a \,\widetilde{\mathcal{R}}_E\, e \text{ if and only if } a = ea \text{ and } \forall f \in E\, (a = fa \Rightarrow e = fe).$$

We will be interested in semigroups S in which every $\widetilde{\mathcal{R}}_E$-class contains a *unique* idempotent of E, for some $E \subseteq E(S)$. In this case, we denote by a^+ the idempotent of E in the $\widetilde{\mathcal{R}}_E$-class of a, so that $a \mapsto a^+$ is a unary operation on S and we may regard S as an algebra of type $(2,1)$, that is, possessing a binary and a unary operation. The following result is routine.

Proposition 2.1. *Let S be a semigroup and let $E \subseteq E(S)$ be a band. Suppose that for any $a \in S$, the $\widetilde{\mathcal{R}}_E$-class of a contains a* unique *idempotent a^+ of E. Then S satisfies the identities*

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+x^+ = x^+y^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

Conversely, let S be an algebra of type $(2,1)$ where the binary operation (written as juxtaposition) is associative and the unary operation is written as $^+$, satisfying the above identities. Put

$$E = \{a^+ : a \in S\}.$$

Then E is a subband and for any $a \in S$, the $\widetilde{\mathcal{R}}_E$-class of a contains a unique *idempotent a^+.*

If the above conditions hold, then E is a left regular band.

We remark that the set of identities given in Proposition 2.1 is not minimal, (for example, the fifth identity can be deduced from the first and second), but, rather, given in this form for the sake of clarity. We also note that using the fourth identity, the final identity can be strengthened to say

$$x^+(xy)^+ = (xy)^+ = (xy)^+x^+.$$

We refer to the algebras of type $(2,1)$ satisfying the conditions of Proposition 2.1 as *generalised left restriction semigroups* and E as the *distinguished band* or (where appropriate) *distinguished semilattice*; generating sets and morphisms must respect both operations. Note that this class forms a variety of 'semigroups' each containing a left regular band. Notice also that for S in this class and $a, b \in S$,

$$a\,\widetilde{\mathcal{R}}_E\,b \text{ if and only if } a^+ = b^+.$$

We define the following conditions on a generalised left restriction semigroup:

Condition (A): the identity $(xy^+)^+x = xy^+$ holds (this is the *ample* condition);

Condition (S): the identity $x^+y^+ = y^+x^+$ holds;

and

Condition (C): the identity $(xy^+)^+ = (xy)^+$ holds (this is the *(left) congruence* condition).

As in [12] we see (S) is equivalent to E forming a semilattice, and (C) to $\widetilde{\mathcal{R}}_E$ being a left congruence. To insist that $E = E(S)$ we must add the quasi-identity $x^2 = x \to x = x^+$. Generalised left restriction semigroups satisfying (A), (C) and (S) are called *left restriction* semigroups and have arisen in many contexts; see [12] for details. Wlqa semigroups are precisely generalised left restriction semigroups with (A) and $E = E(S)$. In this article we focus on generalised left restriction semigroups with (A) and (C); for brevity in this article we may refer to them as *glrac* semigroups. These form a very wide class, including all inverse semigroups, left restriction semigroups, and $\mathcal{R}$-unipotent semigroups. In fact, regular glrac semigroups S with $E = E(S)$ are exactly the $\mathcal{R}$-unipotent semigroups. The canonical example falling outside of the latter classes is as follows. We remark that all of our monoid actions are assumed to be unitary. In the example below, the relation $\leq_{\mathcal{L}}$ is the quasi-order associated with Green's relation $\mathcal{L}$, so that $a \leq_{\mathcal{L}} b$ in a semigroup S if and only if $S^1 a \subseteq S^1 b$.

Example 2.2. Let T be a monoid with identity 1_T, acting on the left on a left regular band B by morphisms. Then the semidirect product

$$B * T = B \times T;\ \ (b,s)(c,t) = \big(b(s\cdot c), st\big)$$

is a glrac semigroup with $(b,s)^+ = (b, 1_T)$ and distinguished band

$$B' = \{(b, 1_T) : b \in B\} \cong B.$$

Suppose now that B is a monoid with identity 1_B. Put

$$S_m = B *_m T = \{(b,s) \in B \times T : b \leq_{\mathcal{L}} s \cdot 1_B\}.$$

Then S_m is a glrac monoid with identity $(1_B, 1_T)$.

Proof. The first part of the statement is standard. For the monoid question, notice that if $b, c \in B$ and $b \leq_{\mathcal{L}} c$, then for all $s \in T$ we have that $s \cdot b = s \cdot bc = (s \cdot b)(s \cdot c)$, so that $s \cdot b \leq_{\mathcal{L}} s \cdot c$. If we have $(b,s), (c,t) \in S_m$, then $c \leq_{\mathcal{L}} t \cdot 1_B$, so that $s \cdot c \leq_{\mathcal{L}} s \cdot (t \cdot 1_B) = st \cdot 1_B$, giving $b(s \cdot c) \leq_{\mathcal{L}} st \cdot 1_B$. Thus S_m is closed under multiplication. For any $b \in B$ we have that $b \leq_{\mathcal{L}} 1_T \cdot 1_B$, so that certainly $(b, 1_T) \in S_m$. It follows that S_m is closed under $^+$ and hence a glrac semigroup.

It is easy to see that $(1_B, 1_T)(b,t) = (b,t)$ for any $(b,t) \in S_m$. Now

$$(b,t)(1_B, 1_T) = (b(t \cdot 1_B), t\,1_T) = (b,t)$$

since $b \leq_{\mathcal{L}} t \cdot 1_B$. Thus $(1_B, 1_T)$ is an identity for S_m. $\square$

The following lemma will be used frequently and its proof is straightforward.

Lemma 2.3. *In a glrac semigroup the following identities hold:*

$$(x^+y)^+ = x^+y^+, \ (xy)^+x = xy^+.$$

Lemma 2.4. *(i) Let S be a glrac semigroup. If $s, u, a, b \in S$ are such that $s = u^+ab$, then $s^+ = (s^+a)^+$ and $s^+ab = s$.*

(ii) Let S be a generalised left restriction semigroup with Condition (A). If $S = \langle A \rangle_{(2,1)}$ for some subset A, then

$$S = \{y^+x_1 \dots x_n : n \geq 0, y \in S, x_i \in A\}.$$

Proof. (i) Using the defining identities, the remark following Proposition 2.1 and Lemma 2.3 we have

$$\begin{aligned}(s^+a)^+ &= s^+a^+ = (u^+ab)^+a^+ = u^+(ab)^+a^+ \\ &= u^+(ab)^+ = (u^+ab)^+ = s^+\end{aligned}$$

and

$$s^+ab = (u^+ab)^+ab = u^+(ab)^+(ab) = u^+ab = s.$$

(ii) This follows as in [13, Lemma 4.1]; we do not need Condition (C). □

3. Idempotent pure extensions

In this section, we start by presenting the concept of idempotent pure, and in particular of proper, extension of a glrac semigroup. Then we discuss these concepts in the $\mathcal{R}$-unipotent environment.

Given glrac semigroups K and T, we say that K is an *extension* of T if there exists a surjective ((2,1)-)morphism $\theta : K \twoheadrightarrow T$ from K onto T. We say that the extension is *proper* if it is injective on each $\widetilde{\mathcal{R}}_E$-class, that is, if $\widetilde{\mathcal{R}}_E \cap \operatorname{Ker}\theta = \iota$. We say that θ is $^+$*-idempotent pure* if it is idempotent pure with respect to the idempotents in the image of $^+$, that is, if $a\theta = (a\theta)^+$ implies that $a = a^+$; if the distinguished bands of K and T are $E(K)$ and $E(T)$, respectively, then in this case we say that θ is *idempotent pure*.

As in [10, Proposition 2], we have

Proposition 3.1. *Every proper extension of a glrac semigroup is $^+$-idempotent pure.*

The converse of the above result is not always true, even in the case where $E = E(S)$ is a semilattice: see [6, Example 3]. However, in the regular case we have the following.

Proposition 3.2. *Let K and T be $\mathcal{R}$-unipotent semigroups and let $\theta : K \twoheadrightarrow T$ be a morphism from K onto T. Then θ is idempotent pure if and only if $\mathcal{R} \cap \operatorname{Ker}\theta = \iota$.*

Proof. In view of Proposition 3.1, it remains only to show the converse. Suppose that θ is idempotent pure and let $a, b \in K$ be such that $a \,\mathcal{R}\, b$ and $a\theta = b\theta$. Let $b' \in V(b)$.

We have that $a = bb'a$ and from $(b'b)\theta = b'\theta b\theta = b'\theta a\theta = (b'a)\theta$, it follows that $b'a \in E(S)$, since θ is idempotent pure. Hence $a = be$ and dually, $b = af$ for some $e, f \in E(S)$. Thus

$$b = af = bef = afef = afe = be = a,$$

as required. □

On a semigroup K we denote by $\mathcal{L}_E$ the restriction of the Green's relation $\mathcal{L}$ to $E \subseteq E(K)$.

In [3, Proposition 3.3], it is shown that the Szendrei expansion S^{Pr} of an $\mathcal{R}$-unipotent semigroup S is an idempotent pure extension of S, through the second projection $\eta_S : S^{\mathrm{Pr}} \twoheadrightarrow S$, such that $\mathcal{L}_{E(S^{\mathrm{Pr}})} \cap \operatorname{Ker}\eta_S = \iota$. We notice that this equality is superfluous in case that S is inverse, since it is easy to see that if K and T are inverse, $\theta : K \twoheadrightarrow T$ is idempotent pure if and only if $\mathcal{L} \cap \operatorname{Ker}\theta = \iota$ if and only if $\mathcal{R} \cap \operatorname{Ker}\theta = \iota$. However, not all proper extensions of an $\mathcal{R}$-unipotent semigroup satisfy this property. As an example, let $K = B \times T$ be the direct product of a left regular band B which is not a semilattice by an $\mathcal{R}$-unipotent semigroup T. Clearly K is an $\mathcal{R}$-unipotent semigroup and is an idempotent pure extension of T through the second projection η_T onto the second co-ordinate. However, it is not a $\mathcal{L}_{E(K)}$-extension since for any $e, f \in B$ such that $ef \neq fe$ and any $t \in T$, we have $(ef, t) \,\mathcal{L}_E\, (fe, t)$, since $ef = efe$ and $fe = fef$, and $(ef, t)\eta_T = t = (fe, t)\eta_T$.

Notice also that any $\mathcal{R}$-unipotent idempotent pure extension K of an inverse semigroup T, through a surjective morphism $\theta : K \twoheadrightarrow T$ satisfies $\mathcal{L} \cap \operatorname{Ker}\theta = \iota$ if and only if K is inverse. For, if $\mathcal{L} \cap \operatorname{Ker}\theta = \iota$ and $e, f \in E(K)$, then as $ef \,\mathcal{L}\, fe$ we have that $(ef)\theta = (fe)\theta$ (as T is inverse), so that $ef = fe$ and K is also inverse.

The above observations suggest the study of proper extensions of glrac semigroups that separate $\mathcal{L}$-related idempotents of E. An extension K of a semigroup T, through a surjective morphism $\theta : K \twoheadrightarrow T$, is called an $\mathcal{L}_E$-*extension* if $\mathcal{L}_E \cap \operatorname{Ker}\theta = \iota$ for some subset E. For example, the expansion S^{Pr} of an $\mathcal{R}$-unipotent semigroup S is an idempotent pure $\mathcal{L}_{E(S)}$-extension of S. In fact, we can say more.

Proposition 3.3. *Let K and T be $\mathcal{R}$-unipotent semigroups and let $\theta : K \twoheadrightarrow T$ be an idempotent pure morphism from K onto T. Then $\mathcal{L} \cap \operatorname{Ker}\theta = \iota$ if and only if $\mathcal{L}_{E(S)} \cap \operatorname{Ker}\theta = \iota$.*

Proof. Assume that $\mathcal{L}_E \cap \operatorname{Ker}\theta = \iota$. Let $a, b \in K$ be such that $a\theta = b\theta$ and $a \,\mathcal{L}\, b$. As K is regular, there exist $a' \in V(a)$ and $b' \in V(b)$ such that $a'a = b'b$ [15]. Then as in Proposition 3.2 we have that $ba' \in E(K)$. Now $aa' \,\mathcal{L}_{E(S)}\, ba'$, and by the assumption we get $aa' = ba'$. Thus

$$a = ba'a = bb'b = b.$$

□

Proper extensions of glrac (or even of $\mathcal{R}$-unipotent) semigroups are therefore of two types: the ones that separate $\mathcal{L}$-related idempotents of E and the ones that do not. Here, we are interested in studying the former, as it contains important examples such as the generalized Szendrei expansion and the graph expansion, together with all cases satisfying Condition (S), that is, where E is a semilattice.

In [9] the second author shows that if S is a a proper extension of a left restriction semigroup S with $E = E(S)$, via a morphism $\theta : S \twoheadrightarrow T$, then S embeds into $S(\mathcal{D})$; the latter is a left restriction semigroup constructed from the 'derived T-semigroupoid' of θ, which is a left restriction semigroupoid acted on by T. Gomes then shows that there is a morphism from $S(\mathcal{D})$ to a 'λ-semidirect product' of a semilattice by T, which restricts to an embedding on the image of S. Further, this result is specialised to the inverse case. We remark that in [9] and earlier papers such as [11], left restriction semigroups with $E = E(S)$ are called *weakly left ample*.

Our proof for the corresponding embeddings of glrac semigroups in Section 5 is inspired by the technique of [9], but we omit the category notation and give a complete and direct argument.

4. Examples

We show now that the well known construction of the generalized Szendrei expansion and of the graph expansion of a left restriction semigroup (with $E = E(S)$) [10, 9, 14] extends naturally to that of glrac semigroups, giving rise to proper $\mathcal{L}_E$-extensions.

Let S be a glrac semigroup. We write $\widetilde{R}_{E,s}$ for the $\widetilde{\mathcal{R}}_E$-class of $s \in S$, $\mathcal{P}_{\text{fin}}(S)$ for the set of finite subsets of S, and define

$$S^{\text{Pr}} = \left\{ (A, s) \in \mathcal{P}_{\text{fin}}(S) \times S : s, s^+ \in A \text{ and } A \subseteq \widetilde{R}_{E,s} \right\}$$

with multiplication

$$(A,s)(B,t) = \big((st)^+A \cup sB, st\big)$$

for all (A,s), $(B,t) \in S^{\mathrm{Pr}}$.

In what follows, have in mind that if $s \in S$ and $a \in \widetilde{R}_{E,s}$ then $s^+a = a$, and so if $A \subseteq \widetilde{R}_{E,s}$, then $s^+A = A$.

Proposition 4.1. *Let S be a glrac semigroup [monoid]. Then S^{Pr} is a glrac semigroup [monoid] where the unary operation $^+$ is defined by $(A,a)^+ = (A,a^+)$ for every $(A,a) \in S^{\mathrm{Pr}}$, so the distinguished subset is*

$$E' = \{(A,a^+) \in S^{\mathrm{Pr}} : A \subseteq \widetilde{R}_{E,a}\}.$$

Moreover, if $E = E(S)$, then $E' = E(S^{\mathrm{Pr}})$ and if in addition, $\mathcal{R}^* = \widetilde{\mathcal{R}}$ in S, then $\mathcal{R}^* = \widetilde{\mathcal{R}}$ in S^{Pr}.

Proof. It follows as in [10, Proposition 3] that S^{Pr} is a semigroup having a set E' of idempotents, such that $(A,a)\,\widetilde{\mathcal{R}}_{E'}\,(A,a^+)$ for every $(A,a) \in S^{\mathrm{Pr}}$.

Given (A,a^+), $(B,b^+) \in S^{\mathrm{Pr}}$, as $a^+b^+a^+ = a^+b^+$ and $b^+B = B$, we get

$$\begin{aligned}(A,a^+)(B,b^+)(A,a^+) &= (a^+b^+A \cup a^+B, a^+b^+)(A,a^+)\\ &= (a^+b^+A \cup a^+b^+B, a^+b^+)\\ &= (a^+b^+A \cup a^+B, a^+b^+)\\ &= (A,a^+)(B,b^+)\end{aligned}$$

so that E' is a left regular band and consequently, every $\widetilde{\mathcal{R}}_{E'}$-class contains a unique idempotent of E'. Defining $(A,a)^+ = (A,a^+)$, Conditions (C) and (A) follow as in [10]. Observe that it suffices to require that S is generalised left restriction with (A) to guarantee that S^{Pr} has the same properties.

The final statements are easy to check. $\square$

The glrac semigroup S^{Pr} is called the *generalized Szendrei expansion* of S. Notice that in the $\mathcal{R}$-unipotent case this is the expansion studied in [3].

Proposition 4.2. *Let S be a glrac semigroup. Then the mapping $\eta_S : S^{\mathrm{Pr}} \twoheadrightarrow S$, $(A,s) \mapsto s$, satisfies $\widetilde{\mathcal{R}}_{E'} \cap \operatorname{Ker}\eta_S = \iota = \mathcal{L}_{E'} \cap \operatorname{Ker}\eta_S$. The semigroup S^{Pr} is therefore a proper $\mathcal{L}_{E'}$-extension of S.*

Proof. In the semigroup S^{Pr}, we have that $(A,s)^+ = (A,s^+)$, and so by an earlier comment, $(A,s)\ \widetilde{\mathcal{R}}_{E'}\ (B,t)$ if and only if $A = B$. Hence $\widetilde{\mathcal{R}}_{E'} \cap \operatorname{Ker}\ \eta_S = \iota$ holds.

Given $(A, a^+), (B, b^+) \in E'$ with $(A, a^+) \; \mathcal{L}_{E'} \cap \operatorname{Ker} \eta_S \; (B, b^+)$, we get $a^+ = b^+$ and $(A, a^+) = (A, a^+)(B, b^+)$. Thus $A = a^+A \cup a^+B = A \cup B$, whence $B \subseteq A$. Similarly $A \subseteq B$ and so $A = B$. $\square$

In the case where $\widetilde{\mathcal{R}}_{E'} = \mathcal{R}^*$, the projection η_S satisfies the stronger property $\mathcal{L} \cap \operatorname{Ker} \eta_S = \iota$. We remark that glrac semigroups with $E = E(S)$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}$ are called *left quasi-ample* or *lqa* in [1].

Corollary 4.3. *Let S be a lqa semigroup. Then the semigroup S^{Pr} is a proper $\mathcal{L}$-extension of S through η_S, that is, η_S is injective on $\mathcal{L}$-classes.*

Proof. In view of the last proposition, it remains to show that $\mathcal{L} \cap \operatorname{Ker} \eta_S = \iota$. Let $(A, s), (B, t) \in S^{\mathrm{Pr}}$ be such that $(A, s) \; \mathcal{L} \cap \operatorname{Ker} \eta_S \; (B, t)$. Then $s = t$ and $(A, s) \; \mathcal{L} \; (B, s)$. Now either $(A, s) = (B, s)$ or there exists $(X, x) \in S^{\mathrm{Pr}}$ such that $(A, s) = (X, x)(B, s)$. In this case, $s = xs$ and so $s^+ = xs^+$, since $s \, \mathcal{R}^* \, s^+$. Also $A = (xs)^+X \cup xB$, whence $A \supseteq xB = xs^+B = s^+B = B$. Similarly $A \subseteq B$ and so $A = B$. $\square$

Next, we consider the graph expansion of a glrac semigroup S generated by a subset A (as a (2,1)-algebra), and obtain another proper $\mathcal{L}_E$-extension of S.

The *graph expansion* $\mathcal{M}(A, S)$ of S is defined as follows: on the associated Cayley graph Γ with set of vertices $V(\Gamma) = S$ and edges (s, a, sa), where $s \in S$ and $a \in A$, define the natural action of S by

$$t \cdot v = tv \text{ and } t \cdot (s, a, sa) = (ts, a, tsa)$$

for any $t \in S$, $v \in S$ and $a \in A$; then let

$$\begin{aligned} \mathcal{M}(A, S) = \{(\Delta, s) : & \Delta \text{ is a finite subgraph of } \Gamma, \; s, s^+ \in V(\Delta) \\ & \Delta \text{ is } s^+\text{-rooted and } V(\Delta) \subseteq \widetilde{R}_{E,s}\} \end{aligned}$$

where s-rooted means, there is a path from s in Δ to any vertex of Δ. The multiplication is given by, for all $(\Delta, s), (\Sigma, t) \in \mathcal{M}(A, S)$,

$$(\Delta, s)(\Sigma, t) = ((st)^+\Delta \cup s\Sigma, st).$$

In view of Lemma 2.4, for any $s \in S$ there exist $u \in S$ and $y_1, \ldots, y_m \in A$, $m \in \mathbb{N}^0$, such that $s = u^+y_1 \ldots y_m$. Again by that Lemma, we have that $s^+y_1 \ldots y_m = s$ and for for $i = 1, \ldots, m$, we get $(s^+y_1 \ldots y_i)^+ = s^+$. So, in Γ, we have

$$\Delta(s) : s^+ \bullet \xrightarrow{\;y_1\;} \underset{s^+y_1}{\bullet} \xrightarrow{\;y_2\;} \underset{s^+y_1y_2}{\bullet} \xrightarrow{\;y_3\;} \bullet \cdots\cdots \bullet \xrightarrow{\;y_m\;} \bullet \, s^+y_1 \ldots y_m = s$$

and $(\Delta(s), s) \in \mathcal{M}(A, S)$, showing that the projection onto the second co-ordinate is onto. Most of the next proposition follows as in [9, Theorem 4.1] and Proposition 4.1.

Proposition 4.4. *Let S be a glrac semigroup generated by a subset A. Then $\mathcal{M}(A,S)$ is a proper glrac semigroup where the unary operation ${}^+$ is defined by $(\Delta,s)^+ = (\Delta,s^+)$ so the distinguished subset is*

$$E' = \big\{(\Delta,s^+) : (\Delta,s^+) \in \mathcal{M}(A,S)\big\}.$$

Moreover, the mapping $\eta_S : \mathcal{M}(A,S) \twoheadrightarrow S$, $(\Delta,s) \mapsto s$, satisfies $\widetilde{\mathcal{R}}_{E'} \cap \operatorname{Ker}\eta_S = \iota = \mathcal{L}_{E'} \cap \operatorname{Ker}\eta_S$. The semigroup $\mathcal{M}(A,S)$ is therefore a proper $\mathcal{L}_{E'}$-extension of S.

Moreover, if $E = E(S)$, then $E' = E(\mathcal{M}(A,S))$ and if in addition, $\mathcal{R}^* = \widetilde{\mathcal{R}}$ in S, then $\mathcal{R}^* = \widetilde{\mathcal{R}}$ in $\mathcal{M}(A,S)$.

5. Embeddings into λ-semidirect products

Extending the Billhardt's idea of λ-semidirect product, we show that a proper $\mathcal{L}_E$-extension of a glrac semigroup T is embeddable into a λ-semidirect product of a left regular band by T. In particular, this result holds for idempotent pure $\mathcal{L}_{E(S)}$-extensions of an $\mathcal{R}$-unipotent semigroup S. As commented in Section 3, our proof is inspired from the categorical constructions of the forerunners of this article, but here we argue directly.

Let B be a left regular band and let T be a glrac semigroup. Assume that T acts on B by endomorphisms. For convenience we may write the action of an element of T on B as juxtaposition. Define the *λ-semidirect product* $B *_\lambda T$ of B by T, as follows:

$$B *_\lambda T = \big\{(b,t) \in B \times T : \ t^+ b = b\big\}$$

with the product given by

$$(a,s)(b,t) = \big((st)^+ a \cdot sb,\ st\big)$$

for all $(a,s),(b,t) \in B *_\lambda T$.

Proposition 5.1. *Let B be a left regular band acted upon by a glrac semigroup T. Then $B *_\lambda T$ is also a glrac semigroup with $(b,t)^+ = (b,t^+)$ for any $(b,t) \in B *_\lambda T$, so the distinguished subset is*

$$E' = \{(b,t^+) : (b,t) \in B *_\lambda T\}.$$

If $E = E(T)$ then $E' = E(B *_\lambda T)$, and if $\mathcal{R}^* = \widetilde{\mathcal{R}}$ in T, then the same is true in $B *_\lambda T$. Finally, if T is $\mathcal{R}$-unipotent, then so is $B *_\lambda T$.

Proof. The first part follows the proof of [9, Proposition 3.9] requiring simply to show that

$$E' = \{(b,t^+) : t^+ b = b\}$$

is a left regular band. The next statements follow as for those in Propositions 4.1 and 4.4.

If T is regular, then for any $(b,t) \in B *_\lambda T$ we have that $(t'b,t') \in B *_\lambda T$, where $t' \in V(t)$. It is easy to check that $(t'b,t') \in V(b,t)$, so that T is regular, hence $\mathcal{R}$-unipotent. $\square$

Let B be a left regular band acted upon by a glrac semigroup T. The λ-*wreath product* $B \circledast_\lambda T$, of B by T, is the λ-semidirect product $B^T *_\lambda T$ of B^T by T, where B^T denotes the left regular band of all maps from T into B with the operation defined by $x(fg) = (xf)(xg)$, for all $f,g \in B^T$ and $x \in T$, and the left action of T on B^T is given by $x(tf) = (xt)f$, for all $f \in B^T$ and $t,x \in T$.

Before giving the main result of this section we need a preliminary lemma.

Lemma 5.2. *Let B be a left regular band and let $\mathcal{B}$ denote the set of right ideals of B. Then $\mathcal{B}$ is a left regular band under multiplication in the power semigroup.*

Proof. Let $I,J \in \mathcal{B}$. Then $(IJ)B = I(JB) \subseteq IJ$, so that $IJ \in \mathcal{B}$. Clearly $IJI \subseteq IJ$. If $x \in I, y \in J$, then as B is left regular, $xy = xyx \in IJI$ so that consequently, $IJ = IJI$. $\square$

Theorem 5.3. *Let S be a glrac semigroup which is a proper $\mathcal{L}_F$-extension of a glrac semigroup T. Then S is embeddable into a λ-semidirect product of a left regular band by T.*

Proof. Let F and E be the distinguished subsets of S and T, respectively and suppose that $\theta : S \twoheadrightarrow T$ is such that $\widetilde{\mathcal{R}}_F \cap \operatorname{Ker}\theta = \iota = \mathcal{L}_F \cap \operatorname{Ker}\theta$. Notice that $E = F\theta$, since θ is onto and respects the unary operation. For each $a^+ \in E$, let

$$F_{a^+} = \{m^+ \in F : a^+(m^+\theta) = a^+\}.$$

As θ is onto, $F_{a^+} \neq \emptyset$ and it is clear that it is a subband of F, hence left regular. Put

$$\overline{F}_{a^+} = \{(a^+,u) : u \in F_{a^+}\} \text{ with operation } (a^+,u)(a^+,v) = (a^+,uv)$$

so that each $\overline{F}_{a^+}$ is also a left regular band; where there is no danger of ambiguity, we identity $\overline{F}_{a^+}$ with F_{a^+}. Let $\mathcal{B}_{a^+}$ denote the set of right ideals of $\overline{F}_{a^+}$, so that by Lemma 5.2, $\mathcal{B}_{a^+}$ is a left regular band, and let

$$\mathcal{B} = \bigcup_{a^+ \in E} \mathcal{B}_{a^+} \cup \{0\}$$

be the 0-direct union of all the bands $\mathcal{B}_{a^+}$. Again, $\mathcal{B}$ is a left regular band. We show that S is embedded into $\mathcal{B}^T *_\lambda T$.

For $m \in S$ let $f_m \in \mathcal{B}^T$ be defined by

$$\begin{array}{rcl} f_m : T & \to & \mathcal{B} \\ t & \mapsto & \{\big((t(m\theta))^+, (nm)^+\big) : t(m\theta)^+ = n\theta\}. \end{array}$$

First, $(nm)^+\theta = (n\theta m\theta)^+ = (t(m\theta)^+ m\theta)^+ = (t(m\theta))^+$ so that $(t(m\theta))^+(nm)^+\theta = (t(m\theta))^+$ and $tf_m \subseteq \overline{F}_{(t(m\theta))^+}$. Notice also that $(n\theta)^+ = (t(m\theta))^+$. Dropping the first co-ordinate, if $(nm)^+ \in tf_m$ and $e \in F_{(t(m\theta))^+}$, then $(nm)^+e = (nm)^+e(nm)^+ = ((nm)^+enm)^+$ and

$$((nm)^+en)\theta = (nm)^+\theta e\theta n\theta = (t(m\theta))^+e\theta n\theta = (t(m\theta))^+n\theta = n\theta = t(m\theta)^+$$

so that $(nm)^+e \in tf_m$ and $tf_m \in \mathcal{B}_{(t(m\theta))^+}$. Observe that as θ is a $(2,1)$-morphism and Condition (C) holds for S and T, it follows easily that $f_m = f_{m^+}$ for all $m \in S$.

We now define

$$\begin{array}{rcl} \psi : S & \hookrightarrow & \mathcal{B}^T *_\lambda T \text{ by} \\ m\psi & \mapsto & (f_m, m\theta). \end{array}$$

Observe that for any $t \in T$,

$$t((m\theta)^+f_m) = (t(m^+\theta))f_m =$$

$$\{\big((t(m^+\theta)m\theta)^+, (nm)^+\big) : n\theta = t(m^+\theta)(m\theta)^+\} = tf_m$$

so that $(m\theta)^+f_m = f_m$ and $\operatorname{Im} \psi \subseteq \mathcal{B}^T *_\lambda T$.

To argue that ψ preserves the binary operation, let $m, n \in S$. Then

$$m\psi n\psi = (f_m, m\theta)(f_n, n\theta) = (((mn)\theta)^+f_m \cdot (m\theta)f_n, (mn)\theta)$$

and we must show this equals $(mn)\psi = (f_{mn}, (mn)\theta)$. For any $t \in T$ we have $tf_{mn} \in \mathcal{B}_{(t(mn)\theta)^+}$. Now

$$\big(t((mn)\theta)^+m\theta\big)^+ = \big(t(m\theta n\theta)^+m\theta\big)^+ = \big(t(m\theta)(n\theta)^+\big)^+ = (t(mn)\theta)^+$$

so that $t\big(((mn)\theta)^+f_m\big) = (t((mn)\theta)^+)f_m \in \mathcal{B}_{(t(mn)\theta)^+}$. Clearly $t(m\theta f_n) = (tm\theta)f_n \in \mathcal{B}_{(t(mn)\theta)^+}$ so that $t\big(((mn)\theta)^+f_m \cdot m\theta f_n\big) \in \mathcal{B}_{(t(mn)\theta)^+}$ also.

Dropping mention of the first co-ordinate, we have that

$$tf_{mn} = \{(kmn)^+ : t((mn)\theta)^+ = k\theta\} = A$$

and $t\big(((mn)\theta)^+f_m \cdot m\theta f_n\big) = B$ where

$$B = \{(hm)^+ : t((mn)\theta)^+ = h\theta\}\{(\ell n)^+ : t(m\theta)(n\theta)^+ = \ell\theta\}.$$

We must show that $A = B$.

Let $(kmn)^+ \in A$ where $t((mn)\theta)^+ = k\theta$. Then $(kmn)^+ = (km)^+(kmn)^+$ and

$$(km)\theta = t((mn)\theta)^+m\theta = t(m\theta)(n\theta)^+$$

so that $(kmn)^+ \in B$.

Conversely, let $(hm)^+(\ell n)^+ \in B$, where $t((mn)\theta)^+ = h\theta$ and $t(m\theta)(n\theta)^+ = \ell\theta$. Since F is left regular we have

$$(hm)^+\ell\,\widetilde{\mathcal{R}}_F\,(hm)^+\ell^+ = (hm)^+\ell^+(hm)^+\,\widetilde{\mathcal{R}}_F\,(hm)^+\ell^+hm$$

so that $\big((hm)^+\ell\big)^+ = \big((hm)^+\ell^+hm\big)^+$. Also

$$\begin{array}{rcl} ((hm)^+\ell)\theta & = & ((t((mn)\theta)^+(m\theta))^+t(m\theta)(n\theta)^+ \\ & = & ((t(m\theta)(n\theta)^+)^+t(m\theta)(n\theta)^+ \\ & = & t(m\theta)(n\theta)^+ \end{array}$$

and so

$$\begin{array}{rcl} ((hm)^+\ell^+hm)\theta & = & \big(((hm)^+\ell)^+hm\big)\theta \\ & = & \big(((hm)^+\ell)\theta\big)^+(hm)\theta \\ & = & (t(m\theta)(n\theta)^+)^+t((mn)\theta)^+m\theta \\ & = & (t((mn)\theta)^+)^+t((mn)\theta)^+m\theta \\ & = & t((mn)\theta)^+m\theta \\ & = & t(m\theta(n\theta)^+)^+m\theta \\ & = & t(m\theta)(n\theta)^+ \end{array}$$

and so $(hm)^+\ell = (hm)^+\ell^+hm$ as the extension is proper.

We now have that $(hm)^+(\ell n)^+ = ((hm)^+\ell n)^+ = ((hm)^+\ell^+hmn)^+$ and

$$((hm)^+\ell^+h)\theta = (t(mn)\theta)^+t((mn)\theta)^+ = t((mn)\theta)^+$$

so that $(hm)^+(\ell n)^+ \in A$.

We have shown that $A = B$, hence ψ is a semigroup morphism. Considering the unary operation, we have

$$(m\psi)^+ = (f_m, m\theta)^+ = (f_m, m^+\theta) = (f_{m^+}, m^+\theta) = m^+\psi,$$

since $f_m = f_{m^+}$, by an earlier comment. Hence ψ is a $(2,1)$-morphism.

To see that ψ is an embedding, suppose that $m\psi = n\psi$, so that $f_m = f_n$ and $m\theta = n\theta$. Then $(m\theta)^+f_m = (m\theta)^+f_n$ and so in particular,

$$\{(km)^+ : (m\theta)^+(m\theta)^+ = k\theta\} = \{(\ell n)^+ : (m\theta)^+(n\theta)^+ = \ell\theta\}.$$

With $k = m^+$ we obtain $m^+ = (\ell n)^+$ for some ℓ with $(m\theta)^+(n\theta)^+ = \ell\theta$. By Proposition 3.1 we have that $\ell = \ell^+$ so that $m^+ = \ell^+n^+$ and together with the dual we obtain $m^+\,\mathcal{L}_F\,n^+$. Since $m^+\theta = n^+\theta$ and $\mathcal{L}_F \cap \operatorname{Ker}\theta = \iota$, we obtain $m^+ = n^+$ and so as the extension is proper, $m = n$. Thus ψ is an embedding. $\square$

We have shown that a proper $\mathcal{L}_F$-extension S (where F is the distinguished subset of S) of T is isomorphic to a (2,1)-subalgebra of a λ-semidirect product $\mathcal{B} \circledast_\lambda T$, namely its image under ψ. What else can we say about im ψ? First notice that for any $t \in T$ there exists some $(f, t) \in \operatorname{im}\psi$, since θ is onto. Secondly, as θ is injective on $\mathcal{L}$-related idempotents of F and F is a left regular band, we conclude that, for any $a^+ \in T$, the set of elements u of F such that $u\theta = a^+$ forms a semilattice. Indeed given $u, v \in F$ such that $u\theta = v\theta = a^+$, we get $(uv)\theta = (vu)\theta$ and as $uv \,\mathcal{L}\, vu$ in F it follows that $uv = vu$. Thus the elements of $F\psi$ with second component is a^+, form a semilattice. Consequently, $F\psi$ is a left regular band which is a disjoint union of these semilattices. In fact, the converse is easily proved to be also true.

Corollary 5.4. *A glrac semigroup S with distinguished subset F is a proper $\mathcal{L}_F$-extension of a glrac semigroup T if and only if it is, up to a (2,1)-isomorphism, a (2,1)-subalgebra K of a λ-semidirect product $\mathcal{B} \circledast_\lambda T$ of a left regular band B by T such that the second projection $\pi : K \to T$, $(b,t) \mapsto t$, is onto and for every $a^+ \in T$, the set $K^+ \cap a^+\pi^{-1}$ is a semilattice.*

The next result is immediate from Proposition 3.2 and Theorem 5.3.

Corollary 5.5. *If an $\mathcal{R}$-unipotent semigroup S is an idempotent pure $\mathcal{L}_{E(S)}$-extension of an $\mathcal{R}$-unipotent semigroup T then S is embeddable into a λ-semidirect product of a left regular band by T.*

Observe that an E-unitary $\mathcal{R}$-unipotent semigroup S that is not inverse, is an idempotent pure extension of the group $G = S/\sigma$ through the canonical morphism $\sigma^\natural : S \twoheadrightarrow G$ and so $\mathcal{L}_{E(S)} \cap \operatorname{Ker} \sigma^\natural \neq \iota$. By [18] we know that S embeds into a semidirect product of a left regular band by the group G, but this statement does not follow from our theorem so our construction does not apply here. It is therefore natural to ask if it is possible to obtain a different embedding of S into some $B^T *_\lambda T$ in a way that Szendrei's result may follow when T is a group. More generally, it remains open whether any proper [idempotent pure] extension S of a glrac [$\mathcal{R}$-unipotent] semigroup T is embeddable into a λ-semidirect product of a left regular band by T.

6. Covering theorems for generalised left restriction semigroups

Let S be a semigroup with subset E of idempotents. We denote by σ_E the least semigroup congruence ρ such that $e \,\rho\, f$ for all $e, f \in E$. We note that if S is generalised left restriction, then σ_E is also a $(2,1)$-congruence, and if S is a monoid, a $(2,1,0)$-congruence. It is well known that if S is $\mathcal{R}$-unipotent and $E = E(S)$, then σ_E is the least group congruence σ on S (see above) and is given by the rule that for all $a, b \in S$, $a \,\sigma_E\, b$ if and only if $ea = eb$ for some $e \in E(S)$ [5]. Effectively it is the fact that inverse semigroups have Condition (A) that allows us to deduce this. Indeed the following is true.

Lemma 6.1. [12] *Let S be a generalised left restriction semigroup with Condition (A). Then $a\,\sigma_E\,b$ if and only if $ea = eb$ for some $e \in E$.*

We say that a generalised left restriction semigroup T is *reduced* if the distinguished subset is trivial. It is clear that in this case if T has Condition (A) then T is a monoid and $E = \{1\}$; consequently, T is left restriction. Obviously, for any generalised left restriction semigroup, S/σ_E is reduced.

We say that a generalised left restriction semigroup with Condition (A) is *proper* if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$. If S and T are generalised left restriction with (A) and $\theta : T \twoheadrightarrow S$ is such that $a^+\theta = b^+\theta$ implies that $a^+ = b^+$, then we say that T is a *cover* of S; if T is proper it is a *proper cover*. In this section we aim to show that every generalised left restriction semigroup S with (A) has a proper cover, and if S also has (C), that is, if S is glrac, then the cover may be taken to be a semidirect product of a left regular band and a monoid (regarded as a reduced left restriction semigroup).

Lemma 6.2. *Let $B * T$ and $B *_m T$ be as in Example 2.2. Then for any $(b, s), (c, t) \in S$,*

$$(b, s)\,\sigma_{B'}\,(c, t) \text{ if and only if } s = t.$$

Consequently, $B * T$ and $B *_m T$ are proper.

Proof. Since $B' = \{(b, 1) : b \in B\}$ is the distinguished subset, it is clear from Lemma 6.1 that if $(b, s)\,\sigma_{B'}\,(c, t)$, then $s = t$. On the other hand if $(b, s), (c, s) \in B * T$, then

$$(bc, 1)(b, s) = (bcb, s) = (bc, s) = (bc, 1)(c, s),$$

so that $(b, s)\,\sigma_{B'}\,(c, s)$ as required.

It follows that $B * T$ is proper and it is then immediate that $B *_m T$ is also. □

For our first covering result, we utilise the fact that if S is a glrac monoid, then S acts by morphisms on its distinguished band B, in a natural way. We use the technique given in a special case in [8]. We denote the free monoid and the free semigroup on a set X by X^* and X^+, respectively.

Theorem 6.3. *Let S be a glrac monoid with distinguished band B. Then S has a proper glrac cover of the form $B *_m T$, which is finite if S is finite.*

Further, S has a proper lqa cover of the form $B *_m X^*$; let B'' be its distinguished subset. Then $B'' = E(B *_m X^*)$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}$ in $B *_m X^*$.

Proof. Choose any set of generators of S, so that $S = \langle X \rangle_{(2,1,0)}$ and put $T = \langle X \rangle_{(2,0)}$, that is, T is the *submonoid* generated by X.

Define an action of T on B by

$$t \cdot b = (tb)^+.$$

For any $s, t \in T$ and $b, c \in B$:

$$1 \cdot b = (1b)^+ = b^+ = b$$

and

$$s \cdot (t \cdot b) = s \cdot (tb)^+ = (s(tb)^+)^+ = (stb)^+ = st \cdot b,$$

so that we have an action; further,

$$s \cdot (bc) = (sbc)^+ = ((sb)^+ sc)^+ = (sb)^+ (sc)^+ = (s \cdot b)(s \cdot c),$$

so the action is by morphisms.

From Lemma 6.2, we have that $B *_m T$ is proper glrac with distinguished subset $B' \cong B$. Define $\theta : B *_m T \to S$ by $(b, s)\theta = bs$. Then for any $(b, s), (c, t) \in B *_m T$, we have that

$$\big((b, s)(c, t)\big)\theta = (b(s \cdot c), st)\theta = b(sc)^+ st = bsct = (b, s)\theta\,(c, t)\theta;$$

further, as $b = b(s \cdot 1)$ we get

$$\big((b, s)\theta\big)^+ = (bs)^+ = bs^+ = b(s \cdot 1) = b = b\,1 = (b, 1)\theta = (b, s)^+\theta.$$

It is clear that θ preserves the identity, so that θ is a $(2, 1, 0)$-morphism.

From Lemma 2.4, we have that

$$S = \{y^+ x_1 \dots x_n : y, x_i \in X, i \geq 0\},$$

so that $S = BT$. Let $m = bt \in S$, where $b \in B$ and $t \in T$. Then $m = bt^+ t$ and $bt^+ \leq_{\mathcal{L}} t^+ = t \cdot 1$, so that $(bt^+, t) \in B *_m T$ and $(bt^+, t)\theta = m$. Hence θ is onto. Clearly θ separates idempotents of B', so that $B *_m T$ is a proper cover of S. We remark that B is a semilattice if and only if B' is a semilattice.

We now let $\psi : X^* \to T$ be the natural morphism, so that X^* acts on B via $w \circ b = (w\psi) \cdot b$. Then $B *_m X^*$ is glrac, and it is easy to check that $\varphi : B *_m X^* \to S$ given by $(b, w)\varphi = b(w\psi)$ is an onto morphism separating the distinguished idempotents of $B *_m X^*$.

Clearly every idempotent of $B *_m X^*$ is distinguished. Notice that if M is reduced, that is, with singleton distinguished subset, then the notion of a λ-semidirect product $B *_\lambda M$ simplifies to that of semidirect product. Thus from Proposition 5.1, $\widetilde{\mathcal{R}} = \mathcal{R}^*$ in $B * X^*$ and hence the same is true for $B *_m X^*$. $\square$

It is known from [1] that a *finite* generalised left restriction semigroup with (A) has a *finite* proper cover. Our next result extends this to infinite monoids (the case for semigroups being a consequence).

Theorem 6.4. *Let S be a generalised left restriction monoid with Condition (A). Then S has a proper cover $\widehat{S}$, where $\widehat{S}$ is a subdirect product of S and X^*. Consequently, if S has (C) or (S), then so does $\widehat{S}$.*

Proof. Let B be the distinguished subset of S; we follow the method first given in [7], but without using the categorical machinery, and with an adjustment for the fact that B may not be a semilattice. Let $S = \langle X \rangle_{(2,1,0)}$ and for convenience we write the elements of X^+ as bracketed tuples. For each $\underline{x} = (x_1, x_2, \ldots, x_n) \in X^+$ we put

$$S_{\underline{x}} = Bx_1Bx_2B \ldots Bx_nB$$

and

$$S_1 = B.$$

Then $S_{\underline{x}}S_{\underline{y}} = S_{\underline{x}\,\underline{y}}$. Let

$$\widehat{S} = \{(s, \underline{x}) \in S \times X^* : s \in S_{\underline{x}}\}.$$

As S and X^* are generalised left restriction monoids with (A), so is $S \times X^*$ where $(s, \underline{x})^+ = (s^+, 1)$. We show that $\widehat{S}$ is a $(2, 1, 0)$-subalgebra of $S \times X^*$, so that $\widehat{S}$ is also generalised left restriction with (A).

For $(s, \underline{x}), (t, \underline{y}) \in \widehat{S}$,

$$(s, \underline{x})(t, \underline{y}) = (st, \underline{x}\,\underline{y}) \in \widehat{S},$$

since $st \in S_{\underline{x}}S_{\underline{y}} = S_{\underline{x}\,\underline{y}}$. As $(b, 1) \in \widehat{S}$ for all $b \in B$, $\widehat{S}$ contains the set of distinguished idempotents B' of $S \times X^*$, so that $\widehat{S}$ is a subalgebra as claimed.

Suppose now that $(s, \underline{x}), (t, \underline{y}) \in \widehat{S}$ and $(s, \underline{x})\, \widetilde{\mathcal{R}}_{B'} \cap \sigma\, (t, \underline{y})$. Then $s^+ = t^+$ and $(e, 1)(s, \underline{x}) = (e, 1)(t, \underline{y})$ for some $e \in B$. It follows that $\underline{x} = \underline{y}$. If $\underline{x} = \underline{y} = 1$, then $s, t \in B$, so that $s = s^+ = t^+ = t$. On the other hand, if $\underline{x} = \underline{y} = (x_1, \ldots, x_n)$, then we have that

$$s = e_0x_1e_1 \ldots e_{n-1}x_ne_n,\ t = f_0x_1f_1 \ldots f_{n-1}x_nf_n,$$

for $e_i, f_i \in B, 1 \leq i \leq n$.

Using (A) repeatedly we deduce that $s = bx_1 \ldots x_n$ for some $b \in B$. But then $bs = s$ so that $bs^+ = s^+$ and

$$s = s^+s = s^+bx_1 \ldots x_n = s^+bs^+x_1 \ldots x_n = s^+x_1 \ldots x_n$$

by left regularity of the band B. Similarly, $t = t^+x_1 \ldots x_n$, but $s^+ = t^+$ and so $s = t$. Thus $\widehat{S}$ is proper.

Let θ be the projection onto the first co-ordinate. It is clear that θ separates the idempotents of B'. To see that θ is onto, let $s \in S$. By Lemma 2.4 we have that $s = b \in B$ or $s = bx_1 \ldots x_n$ for some $x_i \in X$. In the former case, $(s, 1) \in \widehat{S}$ and in the latter, $(s, (x_1, \ldots, x_n)) \in \widehat{S}$. Thus θ is onto (and $\widehat{S}$ is a subdirect product of S and X^*.) This concludes the proof that $\widehat{S}$ is a proper cover of S. □

We end with some remarks concerning the semigroup case. Let S be a generalised left restriction semigroup with Condition (A). Then the monoid S^1 is a generalised left restriction monoid with (A), so that there is a monoid cover $\widehat{S}$ and an onto $(2,1,0)$-morphism $\theta : \widehat{S} \twoheadrightarrow S$ that separates distinguished idempotents. Then $S\theta^{-1}$ is a $(2,1)$-subalgebra of $\widehat{S}$, moreover $S\theta^{-1}$ is proper and $\theta|_{S\theta^{-1}} : S\theta^{-1} \to S$ is onto.

If in addition S has (C), then the same argument (but, simplified) as in Theorem 6.3, gives that if $S = \langle X \rangle_{(2,1)}$, then S has a cover of the form $B * T^1$, where T is the subsemigroup generated by X, and also of the form $B * X^*$.

7. A structure theorem for left restriction semigroups

The aim of this final section is to give a structure theorem for proper left restriction semigroups in terms of monoids (regarded as reduced left restriction semigroups) acting on semilattices. The original motivation is the celebrated result of McAlister for E-unitary (proper) inverse semigroups [17], a result which has been much extended. The immediate predecessor of our theorem below is the description of proper weakly left ample semigroups given in [11]. Here we do not use the categorical machinery.

Let $\mathcal{X}$ be a semilattice with subsemilattice $\mathcal{Y}$ and suppose that $\mathcal{Y}$ has an upper bound $\varepsilon \in \mathcal{X}$, so that if $\varepsilon \in \mathcal{Y}$ then ε is the maximum element of $\mathcal{Y}$. Let M be a monoid acting (on the left) on $\mathcal{X}$ by morphisms, satisfying:

(a) $\forall t \in M,\ \exists a \in \mathcal{Y}, \quad a \leq t\varepsilon$,

(b) $\forall a, b \in \mathcal{Y},\ \forall t \in M, \quad a \leq t\varepsilon \Rightarrow a \wedge tb \in \mathcal{Y}$.

A triple $(M, \mathcal{X}, \mathcal{Y})$ satisfying the above conditions is said to be a *strong $\mathcal{M}$-triple*. Given a strong $\mathcal{M}$-triple $(M, \mathcal{X}, \mathcal{Y})$, define

$$\mathcal{M}(M, \mathcal{X}, \mathcal{Y}) = \{(a, t) \in \mathcal{Y} \times M : a \leq t\varepsilon\}$$

with multiplication given by

$$(a, t)(b, h) = (a \wedge tb, th).$$

By observing that $\mathcal{M}(M,\mathcal{X},\mathcal{Y})$ is a subalgebra of the semidirect product $\mathcal{X} * M$, and using Lemma 6.2, it is a routine matter to verify the following result.

Lemma 7.1. *Let $(M,\mathcal{X},\mathcal{Y})$ be a strong $\mathcal{M}$-triple. Then $\mathcal{M}(M,\mathcal{X},\mathcal{Y})$ is a proper left restriction semigroup with distinguished semilattice*

$$\mathcal{Y}' = \{(y,1) : y \in \mathcal{Y}\} \simeq \mathcal{Y}$$

and such that

$$\mathcal{M}(M,\mathcal{X},\mathcal{Y})/\sigma_{\mathcal{Y}'} \simeq M.$$

Moreover, if $\varepsilon \in \mathcal{Y}$ then $\mathcal{M}(M,\mathcal{X},\mathcal{Y})$ is a monoid with identity $(\varepsilon,1)$.

We call $\mathcal{M}(M,\mathcal{X},\mathcal{Y})$ as in Lemma 7.1 a *strong $\mathcal{M}$-semigroup* [*monoid*]. Using Theorem 5.3 we can now give the promised structure theorem.

Theorem 7.2. *A semigroup [monoid] is proper left restriction if and only if it is isomorphic to a strong $\mathcal{M}$-semigroup [monoid].*

Proof. In view of Lemma 7.1, we need only prove the direct implication.

Let S be a proper left restriction semigroup, with distinguised semilattice F. Then $\sigma_F^\natural : S \to S/\sigma_F$ is a proper $\mathcal{L}_F$-morphism (as $\mathcal{L}_F$ is trivial), so that by Theorem 5.3, S is embeddable into a λ-semidirect product of a left regular band by $M = S/\sigma_F$.

We need to examine the proof of Theorem 5.3. First, since M is reduced, there is only one a^+ in M, the identity $1 = u^+\sigma_F$ holds for any $u \in S$, and $F_1 = F$. Now, any right ideal of F is an ideal, so that $\mathcal{B} = \mathcal{B}_1 \cup \{0\}$, where $\mathcal{B}_1$ is the set of ideals of F.

Notice that if I, J are ideals of F, then $IJ = I \cap J$, so that $\mathcal{B}_1$ is a semilattice, and $I \leq J$ if and only if $I \subseteq J$. Further, $\mathcal{B}$ and $\mathcal{B}^M$ are also semilattices.

Observe that in this case, for $m \in S$ and $t \in M$ we have $tf_m = \{(nm)^+ : t = n\sigma_F\}$. Let $\mathcal{Y} = \{f_m : m \in S\} = \{f_{m^+} : m^+ \in F\}$; since ψ is an embedding, it is easy to see that $\mathcal{Y}$ is a subsemilattice of $\mathcal{B}^M$ isomorphic to F.

Define $\varepsilon \in \mathcal{B}^M$ by $t\varepsilon = \{m^+ : m\sigma_F = t\}$. Then if $e \in F$ we have that

$$m^+e = em^+ = (em)^+$$

and $(em)\sigma_F = m\sigma_F = t$, so that $m^+e \in t\varepsilon$ and $t\varepsilon$ is an ideal of F. We show that ε is an upper bound for $\mathcal{Y}$.

Let $m^+ \in F$. For any $t \in M$ we have that

$$tf_{m^+} = \{(nm^+)^+ : t = n\sigma_F\}.$$

Then $(nm^+)\sigma_F = n\sigma_F = t$, so that $(nm^+)^+ \in t\varepsilon$. Thus $tf_{m^+} \subseteq t\varepsilon$ for all $t \in M$, so that $f_m \leq \varepsilon$ as required.

To show that (a) holds, let $t \in M$: say $t = m\sigma_F$. Let us prove that $f_{m^+} \leq t\epsilon$. For any $r \in M$ we have

$$rf_{m^+} = \{(nm^+)^+ : n\sigma_F = r\} \text{ and } r(t\varepsilon) = (rt)\varepsilon = \{s^+ : s\sigma_F = rt\}.$$

For $(nm^+)^+ \in rf_{m^+}$ with $n\sigma_F = r$ we have that $(nm^+)^+ = (nm)^+$ and $(nm)\sigma_F = rt$, so that $(nm^+)^+ \in r(t\varepsilon)$. Hence $f_{m^+} \leq t\varepsilon$ and (a) holds.

Suppose now that $f_{m^+} \in \mathcal{Y}$ and $f_{m^+} \leq t\varepsilon$ for some $t \in M$. That is, for any $r \in M$ we have that

$$\{(um^+)^+ : u\sigma_F = r\} \subseteq \{s^+ : s\sigma_F = rt\}.$$

Taking $r = 1$ and $u = m^+$, we have that $m^+ = s^+$ for some s with $s\sigma_F = t$. Then $s\psi = (f_s, s\sigma_F) = (f_{s^+}, t) = (f_{m^+}, t)$. If in addition we have that $f_{n^+} \in \mathcal{Y}$, then as $(sn^+)\psi = s\psi n^+\psi$, we have that

$$(f_{sn^+}, (sn^+)\sigma_F) = (f_{m^+}, t)(f_{n^+}, 1)$$

and so $f_{m^+} \wedge tf_{n^+} = f_{sn^+} \in \mathcal{Y}$, giving that (b) holds.

We have shown that $(M, \mathcal{B}^T, \mathcal{Y})$ is a strong $\mathcal{M}$-triple. Let $\mathcal{M} = \mathcal{M}(M, \mathcal{B}^T, \mathcal{Y})$. From the argument for (a) we have that im $\psi \subseteq \mathcal{M}$. On the other hand, if $(f_m, t) \in \mathcal{M}$ we have that $f_m \leq t\varepsilon$ so that as above we obtain $(f_m, t) = s\psi$ for some $s \in S$. Hence im $\psi = \mathcal{M}$ and S is isomorphic to $\mathcal{M}$ as required.

Notice that if S is a monoid with identity 1_S, then $f_{1_S} = \varepsilon \in \mathcal{Y}$. $\square$

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