A NOTION OF RANK FOR RIGHT CONGRUENCES ON SEMIGROUPS

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Abstract. We introduce a new notion of rank for a semigroup S. The rank is associated with pairs (I, ρ), where ρ is a right congruence and I is a ρ-saturated right ideal. We allow I to be the empty set; in this case the rank of $(\emptyset, \rho)$ is the Cantor-Bendixson rank of ρ in the lattice of right congruences of S, with respect to a topology we entitle the *finite type topology*. If all pairs have rank, then we say that S is *ranked*. Our notion of rank is intimately connected with chain conditions: every right noetherian semigroup is ranked, and every inverse ranked semigroup is weakly right noetherian.

Our interest in ranked semigroups stems from the study of the class $\mathcal{E}_S$ of existentially closed S-sets over a right coherent monoid S. It is known that for such S the set of sentences in the language of S-sets that are true in every existentially closed S-set, that is, the theory T_S of $\mathcal{E}_S$, has the model theoretic property of being stable. Moreover, T_S is superstable if and only if S is weakly right noetherian. We show that T_S satisfies the stronger property of being totally transcendental if and only if S is ranked and weakly right noetherian.

1. Introduction

Let S be a semigroup; we denote by $\mathcal{RC}$ and $\mathcal{RI}$ the lattices of right congruences and of right ideals of S, respectively. The semigroup S is *right noetherian* if the ascending chain condition holds for $\mathcal{RC}$ and *weakly right noetherian* if the ascending chain condition holds for $\mathcal{RI}$. It is well known that a right noetherian semigroup is weakly right noetherian but the converse is certainly not true - one needs only to think of the case where S is a group and $\mathcal{RC}$ is the subgroup lattice of S. In an attempt to provide a property intermediate between right noetherian and weakly right noetherian we introduce a notion of rank for elements of $\mathcal{RC}$, inspired by the model theoretic Morley rank.

To be a little more precise, we define rank on elements of $\mathcal{C}$, where $\mathcal{C}$ is the set of all *congruence pairs* (I, ρ) where $\rho \in \mathcal{RC}$, $I \in \mathcal{RI}$ and I is ρ-saturated. A pair $(I, \rho) \in \mathcal{C}$ defines a complete sublattice

$$\mathcal{C}(I, \rho) = \{\theta \in \mathcal{RC} : (I, \theta) \in \mathcal{C} \text{ and } \rho \cap (I \times I) = \theta \cap (I \times I)\}$$

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of $\mathcal{RC}$. The S-rank $\mathrm{S}(I,\rho)$ of (I,ρ) is the Cantor-Bendixson rank of ρ in $\mathcal{C}(I,\rho)$ under the restriction of the finite type topology. The topology and the calculation of S-rank we give explicitly in Section 3. The Cantor-Bendixson rank of a point in a topological space is an ordinal (or ∞) which measures how far the point is from being isolated. Isolated points have rank 0; a point has rank at least $\alpha+1$ if it cannot be isolated amongst points of rank α. We say that a semigroup S is *ranked* if every element of $\mathcal{C}$ possesses an ordinal S-rank (not ∞).

The motivation behind our study of ranked semigroups comes from an old model theoretic question which we now explain. However, readers who wish to skim over the model theoretic references can safely do so.

Let L_S denote the language of (right) S-sets over a monoid S and let $\sum_S$ be a set of sentences in L_S which axiomatises S-sets. A general result of model theory says that $\sum_S$ has a model companion, denoted by T_S, precisely when the class $\mathcal{E}_S$ of existentially closed S-sets is axiomatisable and in this case, T_S axiomatises $\mathcal{E}_S$. It is known that T_S exists and is stable if and only if S is right coherent. Details of right coherent monoids can be found in [8]; we make no explicit use of this notion here.

Stability properties arose from the question of how many models a theory (a set of sentences of a first order language) has of any given cardinality. The seminal work of Shelah shows that an unstable theory, indeed a non-superstable theory, has 2^λ models of cardinality λ for any $\lambda > |T|$ [17]. The philosophy then is that, in these cases, there are too many models to attempt to classify by means of a sensible structure theorem. It is reasonable therefore for the algebraist to consider for a given class of algebras 'how stable' is the theory associated with it, before embarking on the search for structure or classification theorems.

It is known [12, 6] that T_S is superstable if and only if S is weakly right noetherian. We approached this question in [6] by determining the U-rank of a type p in $S(A)$, where A is an S-set. Associated to p is a congruence pair (I_p,ρ_p); then $\mathrm{U}(p)$ is the foundation rank of (I_p,ρ_p) in the sublattice of $\mathcal{RI}$ consisting of those right ideals that are ρ_p-saturated. Since T_S is superstable if and only if every p has U-rank, it is then easy to see that T_S is superstable if and only if S is weakly right noetherian. The question of under which conditions T_S satisfies the stronger property of being *totally transcendental* proved rather more problematic. For a countable theory T, it is a fact that T is totally transcendental if and only if T is ω-stable [11]. There are, however, uncountable theories T which are not totally transcendental but are κ-stable for all κ with $|T|\leq\kappa$.

A stable theory T is totally transcendental if every type p has Morley rank $\mathrm{M}(p)$. We note it is always the case that $\mathrm{U}(p)\leq\mathrm{M}(p)$. In [6] we we characterised those right coherent monoids S such that every type p has $\mathrm{U}(p)=\mathrm{M}(p)<\infty$. The aim of the last section of this paper is to improve upon this result by showing that T_S is totally transcendental if and only if S is weakly right noetherian and ranked.

The structure of the paper is as follows. In Section 2 we give brief details of foundation rank for a partially ordered set and the connection with the U-rank of a type over an S-set. A fuller account of types is reserved until later in the paper. Section 3 introduces S-rank for elements of $\mathcal{C}$. We show that S is ranked if and only if every pair $(\emptyset, \rho)$ is ranked and investigate the conditions under which an element of $\mathcal{C}$ has rank 0. The property of being ranked is inherited by maximal subgroups, monoid principal factors, and monoid $\mathcal{J}$-classes, as we show in Section 4.

We presented above ranked semigroups as a class having properties related to chain conditions. This is justified in Section 5 where we show that a right noetherian semigroup is ranked, and an inverse ranked semigroup is weakly right noetherian. We also prove that a chain is ranked if and only if it is finite. In a subsequent paper we use this result to show that the bicyclic monoid B is not ranked, whilst being right coherent, weakly right noetherian and having trivial subgroups.

After giving further examples of ranked semigroups in Section 6, Section 7 concentrates on the model-theoretic motivation for our notion of rank. We give the necessary details of Morley rank and show that every type over T_S has Morley rank if and only if S is weakly right noetherian and ranked; moreover, we obtain an upper bound on $\mathrm{M}(p)$ for a type p in terms of S-rank. We can be more precise with our upper bound in the case where S is a group.

A few remarks on notation. Even if not stated explicitly, S will always denote a semigroup with set of idempotents $E(S)$. Where there is more than one semigroup in question we use subscripts for clarity; for example we denote by $\mathcal{RC}_S$ the lattice of right congruences *on* S. In an attempt to streamline we suppress explicit mention of indexing sets. For example, if $\{(a_i, b_i) : i \in I\}$ is a finite set of pairs of elements of S then we write $\langle \{(a_i, b_i)\} \rangle$ for $\langle \{(a_i, b_i) : i \in I\} \rangle$. It is worthwhile pointing out that if $H \subseteq S \times S$, then for any $a, b \in S$, a is related to b via the right congruence generated by H if and only if $a = b$ or there is a finite sequence

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ \ldots, d_l t_l = b$$

where for each i, $(c_i, d_i) \in H \cup H^{-1}$ and $t_i \in S^1$.

This paper is intended to be accessible to readers having familiarity with the basic ideas of semigroup theory. Knowledge of model theory (and in particular of the notion of a type) is not required until the final section, which nevertheless contains the required definitions and gives a more leisurely account than is usual in the stability literature. For background in model theory we recommend [4] and [5] and in semigroup theory [9]. Full accounts of the stability theory we use can be found in the books [1, 2, 10, 13, 14, 15].

2. Foundation rank

We recall the *foundation* rank on a set $\mathcal{S}$ partially ordered by $\leq$. We define subclasses $\mathcal{S}^\alpha$ of $\mathcal{S}$ for each ordinal α by transfinite induction:

(I) $\mathcal{S}^0 = \mathcal{S}$;
(II) $\mathcal{S}^\alpha = \bigcap\{\mathcal{S}^\beta : \beta < \alpha\}$, if α is a limit ordinal;
(III) $x \in \mathcal{S}^{\alpha+1}$ if and only if $x < y$ for some $y \in \mathcal{S}^\alpha$.

We thus obtain a nested sequence of subclasses of $\mathcal{S}$ indexed by the ordinals. The *foundation rank* of $x \in \mathcal{S}$, denoted by $\mathrm{R}(x)$, can now be defined as follows.

If $x \in \mathcal{S}^\alpha$ for all ordinals α, then we write $\mathrm{R}(x) = \infty$. Otherwise, $\mathrm{R}(x) = \alpha$ where α is the (unique) ordinal such that $x \in \mathcal{S}^\alpha \setminus \mathcal{S}^{\alpha+1}$; in this case we say that *x has R-rank.*

The convention that $\alpha < \infty$ for all ordinals α simplifies the statements of the following standard proposition (see for example [14], p. 35).

Proposition 2.1. *(i) For any $x \in \mathcal{S}$ and any ordinal α*

$$R(x) \geq \alpha \text{ if and only if } x \in \mathcal{S}_\alpha.$$

(ii) Let $x, y \in \mathcal{S}$ where $x < y$. If $R(y)$ is an ordinal then $R(x) > R(y)$. Moreover, if $R(x)$ is an ordinal then so is $R(y)$.

(iii) For any $x \in \mathcal{S}$, $R(x)$ is an ordinal if and only if there are no infinite chains of the form

$$x = x_0 < x_1 < \dots .$$

For the first application of foundation rank, consider a right congruence ρ on a semigroup S, that is, $\rho \in \mathcal{RC}$, and put

$$\mathcal{S} = \{J : (J, \rho) \in \mathcal{C}\}.$$

The relation $\leq$ is taken as the usual inclusion order of right ideals. If $J \in \mathcal{S}$ then $\mathrm{R}(J)$ is said to be the *ρ-rank* of J and is written as ρ-$\mathrm{R}(J)$.

Corollary 2.2. *Let $(I, \rho) \in \mathcal{C}$. Then ρ-$R(I)$ is an ordinal if and only if S has the ascending chain condition on ρ-saturated right ideals containing I.*

The significance of the above result for us is the following.

Theorem 2.3. [6] *Let S be a right coherent monoid. For any S-set A and $p \in S(A)$,*

$$U(p) = \rho_p\text{-}R(I_p).$$

Consequently, T_S is superstable if and only if S is weakly right noetherian.

Denoting the identity relation on S by ι, every right ideal I of S is ι-saturated. We let $V(I)$ be ι-$\mathrm{R}(I)$ and refer to this as the *V-rank* of I. Clearly $\mathrm{V}(I) = 0$ if and only if $I = S$ and for any $(J, \rho) \in \mathcal{C}$, ρ-$\mathrm{R}(J) \leq \mathrm{V}(J)$. In view of Proposition 2.1, every right ideal has V-rank if and only if S is weakly right noetherian. In the case where S is a right coherent monoid we aim to use the V-rank of I_p and the S-rank of (I_p, ρ_p) to obtain a bound on the Morley rank of a type p over the theory T_S.

3. S-rank of congruence pairs

We introduce *sandwich rank* for semigroups, ultimately to determine those right coherent monoids S for which T_S is totally transcendental.

Sandwich rank, or S-rank, is defined on elements of $\mathcal{C}$. We first define subsets in $\mathcal{RC}$ which we will refer to as having finite type. Let ν denote a finitely generated right congruence on S and let $K \subseteq S \times S$ be finite. Then

$$[\nu, K] = \{\rho \in \mathcal{RC} : \nu \subseteq \rho \subseteq S \times S \setminus K\};$$

we say that $[\nu, K]$ is a *subset of finite type*. Clearly subsets of finite type are convex and $[\mu, H] \cap [\nu, K] = [\langle \mu \cup \nu \rangle, H \cup K]$, so that finite intersections of subsets of finite type are of finite type. Hence the set of subsets of finite type are a basis for a topology on $\mathcal{RC}$; we call this the *finite type topology*.

Let $(I, \rho) \in \mathcal{C}$. The S-rank of (I, ρ) is the Cantor-Bendixson rank of ρ in $\mathcal{C}(I, \rho)$ equipped with the restriction of the finite type topology. We make this explicit by defining subsets $\mathcal{C}^\alpha$ of $\mathcal{C}$ for each ordinal α, as follows:

(I) $\mathcal{C}^0 = \mathcal{C}$;

(II) if α is a limit ordinal, then

$$\mathcal{C}^\alpha = \bigcap \{\mathcal{C}^\beta : \beta < \alpha\};$$

(III) $(I, \rho) \in \mathcal{C}^{\alpha+1}$ if and only if $(I, \rho) \in \mathcal{C}^\alpha$ and for all subsets of finite type $[\nu, K]$ with $\rho \in [\nu, K]$, there exists $(I, \theta) \in \mathcal{C}$ with

$$\theta \in [\nu, K], \theta \cap (I \times I) = \rho \cap (I \times I), \theta \neq \rho$$

and

$$(I, \theta) \in \mathcal{C}^\alpha.$$

The *S-rank* of $(I, \rho) \in \mathcal{C}$ is $\mathrm{S}(I, \rho)$, where if $(I, \rho) \in \mathcal{C}^\alpha$ for all α then $\mathrm{S}(I, \rho) = \infty$, and otherwise $\mathrm{S}(I, \rho) = \alpha$ where $(I, \rho) \in \mathcal{C}^\alpha \setminus \mathcal{C}^{\alpha+1}$. If $\mathrm{S}(I, \rho) < \infty$, then we say that (I, ρ) *has S-rank*. Notice that for any $(I, \rho) \in \mathcal{C}$ and ordinal α, $\mathrm{S}(I, \rho) \geq \alpha$ if and only if $(I, \rho) \in \mathcal{C}^\alpha$. Clearly, for $(\emptyset, \rho) \in \mathcal{C}$, $\mathrm{S}(\emptyset, \rho)$ is simply the Cantor-Bendixson rank of $\rho \in \mathcal{RC}$ equipped with the finite type topology.

We say that a semigroup S is *ranked* if every element of $\mathcal{C}$ has S-rank. Much of this paper is devoted to investigating ranked semigroups. It is worth remarking that for any finite semigroup S and any $(I, \rho) \in \mathcal{C}$, $\mathrm{S}(I, \rho) = 0$, since $[\rho, (S \times S) \setminus \rho]$ certainly isolates ρ in $\mathcal{C}(I, \rho)$. Thus any finite semigroup is ranked.

We state as a lemma a technique we will use repeatedly.

Lemma 3.1. *Let S and T be semigroups, let $X \subseteq \mathcal{C}_S$ and let $\theta : X \to \mathcal{C}_T$ be a map. If for all ordinals α and $(I, \rho) \in X$,*

$$\mathrm{S}(I, \rho) \geq \alpha \textit{ implies that } \mathrm{S}(I, \rho)\theta \geq \alpha \quad (*),$$

then

$$\mathrm{S}(I, \rho) \leq \mathrm{S}(I, \rho)\theta.$$

Further, to establish $(*)$, *it is enough to show that for any ordinal* α,

$$\mathrm{S}(I,\rho) \geq \alpha + 1 \textit{ implies that } \mathrm{S}(I,\rho)\theta \geq \alpha + 1$$

and apply transfinite induction.

Lemma 3.2. *Let* $I, J \in \mathcal{RI}$ *with* $I \subseteq J$. *Then for any* $(I,\rho), (J,\rho) \in \mathcal{C}$ *we have that* $\mathrm{S}(I,\rho) \geq \mathrm{S}(J,\rho)$.

Proof. We show that for any ordinal α, and $(I,\rho), (J,\rho) \in \mathcal{C}$,

$$\mathrm{S}(J,\rho) \geq \alpha + 1 \text{ implies that } \mathrm{S}(I,\rho) \geq \alpha + 1.$$

Applying Lemma 3.1 with $S = T$, $X = \{(J,\rho) \in \mathcal{C} : (I,\rho) \in \mathcal{C}\}$ and $(J,\rho)\theta = (I,\rho)$, yields the result.

We make the inductive assumption that for any $(J,\rho) \in X$, $\mathrm{S}(J,\rho) \geq \alpha$ implies that $\mathrm{S}(I,\rho) \geq \alpha$, and suppose that $\mathrm{S}(J,\rho) \geq \alpha + 1$ for some ordinal α. Certainly $\mathrm{S}(J,\rho) \geq \alpha$, whence by our inductive assumption, $\mathrm{S}(I,\rho) \geq \alpha$. Let $[\nu, K]$ be a subset of finite type with $\rho \in [\nu, K]$. Since $\mathrm{S}(J,\rho) \geq \alpha + 1$ there exists $(J,\theta) \in \mathcal{C}$ with

$$\theta \in [\nu, k],\ \theta \cap (J \times J) = \rho \cap (J \times J),\ \theta \neq \rho$$

and $S(J,\theta) \geq \alpha$. Clearly $\theta \cap (I \times I) = \rho \cap (I \times I)$. If $a \in I$ and $a\,\theta\,b$, then as $a \in J$ and $(J,\theta) \in \mathcal{C}$ we have that $b \in J$; but then $a\,\rho\,b$ and so $b \in I$ as $(I,\rho) \in \mathcal{C}$. Consequently, $(I,\theta) \in \mathcal{C}$ and our inductive assumption tells us that $\mathrm{S}(I,\theta) \geq \alpha$. We deduce that $\mathrm{S}(I,\rho) \geq \alpha + 1$.

□

Corollary 3.3. *A monoid* S *is ranked if and only if every congruence pair of the form* $(\emptyset, \rho)$ *has S-rank, that is, the Cantor-Bendixson rank of every right congruence exists.*

Finally in this section we give a characterisation of congruence pairs with S-rank 0. To this end we make the following definition.

Let $(I,\rho) \in \mathcal{C}$. Then ρ is *I-finitely generated* if

$$\rho = \langle \rho \cap (I \times I) \cup \{(a_1,b_1), \ldots, (a_n,b_n) : n \in \mathbb{N} \cup \{0\}, a_i, b_i \in S \setminus I\} \rangle.$$

Let L be a lattice and let $x, y \in L$. Then y *covers* x if $x < y$ and there are no elements z with $x < z < y$. We say that x is *finitely covered* in L if there exists a finite set C of covers of x such that if $x < z$ then $c \leq z$ for some $c \in C$. Notice that a maximal element is finitely covered by the empty set and if x is finitely covered by C, then $C = \emptyset$ if and only if x is maximal. Clearly in a finite lattice every element is finitely covered.

Proposition 3.4. *Let* $(I,\rho) \in \mathcal{C}$. *Then* $\mathrm{S}(I,\rho) = 0$ *if and only if* ρ *is* I*-finitely generated and is finitely covered in* $\mathcal{C}(I,\rho)$.

Proof. Suppose first that ρ is I-finitely generated and is finitely covered in $\mathcal{C}(I,\rho)$. By definition, we can find a finite set

$$X = \{(a_i, b_i)\}$$

of pairs of elements in $S \setminus I$ such that

$$\rho = \langle \rho \cap (I \times I) \cup X \rangle,$$

and ρ is finitely covered by $\{\rho_l\}$ in $\mathcal{C}(I,\rho)$.

Consider an arbitrary ρ_l. Since $\rho \subset \rho_l$, there exists a pair $(u_l, v_l) \in \rho_l \setminus \rho$. Let τ_l be the right congruence generated by ρ and (u_l, v_l); as $\rho \subset \tau_l \subseteq \rho_l$, it is easy to see that $\tau_l \in \mathcal{C}(I,\rho)$ and consequently, $\tau_l = \rho_l$ since ρ_l is a cover of ρ in this lattice.

Let

$$\nu = \langle X \rangle, \text{ and } K = \{(u_l, v_l)\},$$

so that $\rho \in [\nu, K]$. We claim that $[\nu, K]$ isolates ρ in $\mathcal{C}(I,\rho)$. For suppose that $\theta \in \mathcal{C}(I,\rho)$ and $\theta \in [\nu, K]$. Then

$$\rho \cap (I \times I) = \theta \cap (I \times I) \subseteq \theta$$

and $X \subseteq \theta$, whence $\rho \subseteq \theta$. It follows that $\rho = \theta$. For otherwise we must have $\rho_l \subseteq \theta$ for some l, since $\{\rho_l\}$ finitely covers ρ, but this would contradict $(u_l, v_l) \notin \theta$. Consequently, $S(I,\rho) = 0$.

Conversely, suppose that $S(I,\rho) = 0$. Then there is a subset of finite type $[\nu, K]$ which isolates ρ in $\mathcal{C}(I,\rho)$. Let $\nu = \langle X \rangle$ where $X = \{(a_i, b_i)\}$ and let $K = \{(u_l, v_l)\}$. Notice first that $a_i \in I$ if and only if $b_i \in I$ since $\nu \subseteq \rho$. Let $Y = X \cap (S \setminus I \times S \setminus I)$ and put $\mu = \langle \rho \cap (I \times I) \cup Y \rangle$. It is easy to see that $\mu \in \mathcal{C}(I,\rho)$ and as $\nu \subseteq \mu \subseteq \rho$ it follows that $\mu \in [\nu, K]$, whence $\rho = \mu$ and ρ is I-finitely generated.

For each (u_l, v_l) let ρ_l be the right congruence generated by ρ and (u_l, v_l), and let C be the subset of $D = \{\rho_l\}$ consisting of those ρ_l minimal in D and lying in $\mathcal{C}(I,\rho)$. We claim that C finitely covers ρ in $\mathcal{C}(I,\rho)$.

Let $\theta \in \mathcal{C}(I,\rho)$ with $\rho \subset \theta$. Since $\nu \subseteq \theta$ and $[\nu, K]$ isolates ρ we must have $\theta \notin [\nu, K]$ and so $(u_l, v_l) \in \theta$ for some $(u_l, v_l) \in K$. Thus $\rho_l \subseteq \theta$ and so $\rho_j \subseteq \theta$ for some minimal ρ_j. It is easy to see that we must have $\rho_j \in \mathcal{C}(I,\rho)$ and so $\rho_j \in C$ as required.

$\square$

Observe that for any $\rho \in \mathcal{RC}$, ρ is S-finitely generated and $\mathcal{C}(S,\rho) = \{\rho\}$ so that ρ is certainly finitely covered in $\mathcal{C}(S,\rho)$. On the other hand, $\mathcal{C}(\emptyset,\rho) = \mathcal{RC}$.

Corollary 3.5. *For any $\rho \in \mathcal{RC}$, $\mathrm{S}(S,\rho) = 0$ and $\mathrm{S}(\emptyset,\rho) = 0$ if and only if ρ is finitely generated and is finitely covered in $\mathcal{RC}$.*

4. Principal factors and subgroups

In Section 6 we proceed to give examples and counterexamples of ranked semigroups. Before doing so we show that if every $(I,\rho) \in \mathcal{C}_S$ has S-rank, then the same is true for every $(I,\rho) \in \mathcal{C}_T$, where T is a maximal subgroup of S, a monoid principal factor, or a monoid $\mathcal{J}$-class. The latter cases will follow from our next result.

Proposition 4.1. *Let S,T be semigroups and let $\psi : S \to T$ be an onto morphism. Let $(I,\rho) \in \mathcal{C}_T$ and let*

$$\rho\psi^{-1} = \{(s,t) \in S \times S : s\psi \, \rho \, t\psi\}.$$

Then

$$(I\psi^{-1}, \rho\psi^{-1}) \in \mathcal{C}_S$$

and

$$\mathrm{S}(I,\rho) \leq \mathrm{S}(I\psi^{-1}, \rho\psi^{-1}).$$

Consequently, if S is ranked, then so is T.

Proof. It is straightforward to verify that $(I\psi^{-1}, \rho\psi^{-1}) \in \mathcal{C}_S$ and $\ker\psi \subseteq \rho\psi^{-1}$. On the other hand if $(J,\theta) \in \mathcal{C}_S$ with $\ker\psi \subseteq \theta$, then $J\psi \in \mathcal{RI}_T$ and if we define a relation $\theta\psi$ on T by

$$s\psi \, \theta\psi \, t\psi \text{ if and only if } s \, \theta \, t,$$

then $\theta\psi \in \mathcal{RC}_T$ and $(J\psi, \theta\psi) \in \mathcal{C}_T$. Moreover, $I = I\psi^{-1}\psi$ and $\rho = \rho\psi^{-1}\psi$.

Proceding by induction, we suppose that $\mathrm{S}(I,\rho) \geq \alpha + 1$. Let $\rho\psi^{-1} \in [\nu, K]$ where $\nu = \langle H \rangle$ for some finite $H \subseteq S \times S$. Putting $\nu' = \langle \ker\psi \cup H \rangle$, we have that $\nu' \subseteq \rho\psi^{-1}$.

With $H = \{(a_i, b_i)\}$ and $K = \{(u_l, v_l)\}$ let $\kappa = \langle \{(a_i\psi, b_i\psi)\} \rangle$ and $G = \{(u_l\psi, v_l\psi)\}$. Notice that $\nu'\psi = \kappa$ and $\rho \in [\kappa, G]$.

Since $\mathrm{S}(I,\rho) \geq \alpha + 1$ there exists $\mu \in \mathcal{RC}_T$ with

$$(I,\mu) \in \mathcal{C}_T, \; \mu \cap (I \times I) = \rho \cap (I \times I), \; \mu \neq \rho$$

and

$$\mathrm{S}(I,\mu) \geq \alpha.$$

We know that $(I\psi^{-1}, \mu\psi^{-1}) \in \mathcal{C}_S$ and by our inductive assumption, $\mathrm{S}(I\psi^{-1}, \mu\psi^{-1}) \geq \alpha$. It is straightforward to verify that

$$\mu\psi^{-1} \in [\nu, K], \mu\psi^{-1} \cap (I\psi^{-1} \times I\psi^{-1}) = \rho\psi^{-1} \cap (I\psi^{-1} \times I\psi^{-1})$$

and

$$\mu\psi^{-1} \neq \rho\psi^{-1}.$$

We conclude that $\mathrm{S}(I\psi^{-1}, \rho\psi^{-1}) \geq \alpha+1$. The proposition follows from Lemma 3.1. □

Corollary 4.2. *Let S be a semigroup and let P be any monoid principal factor. If S is ranked, then so is P.*

Proof. By Proposition 4.1 it is enough to show that P is a a morphic image of S. Let P be the principal factor associated with the $\mathcal{J}$-class of e, where e is the identity of J_e, and denote the Rees congruence class of $s \in S$ by $[s]$.

Define $\psi : S \to P$ by $s\psi = [se]$. Let $s, t \in S$. If $te <_{\mathcal{J}} e$, then $ste, sete <_{\mathcal{J}} e$ also, so that

$$(st)\psi = [ste] = 0 = [sete] = [se][te] = s\psi t\psi.$$

On the other hand, if $te \,\mathcal{J}\, e$, then $te = ete$ since we are supposing e is the identity of J_e. Hence

$$(st)\psi = [ste] = [sete] = [se][te] = s\psi t\psi,$$

and ψ is a (clearly onto) morphism as required. □

Our next lemma is straightforward, and we merely sketch the by now familiar technique.

Lemma 4.3. *Let S be a semigroup and let S^0 be the semigroup obtained by adjoining zero to S. If S^0 is ranked, then so is S.*

Proof. For any $\rho \in \mathcal{RC}_S$ we let

$$\rho^0 = \rho \cup \{(0, 0)\}$$

and note that $\rho^0 \in \mathcal{RC}_{S^0}$.

Proceding by induction, suppose that $\rho \in \mathcal{RC}_S$ and $\mathrm{S}(\emptyset, \rho) \geq \alpha + 1$. If $[\nu, K]$ is a subset of finite type of $\mathcal{RC}_{S^0}$ and $\rho^0 \in [\nu, K]$, then it is easy to see that $\rho \in [\nu', K']$ where $\nu' = \nu \setminus \{(0, 0)\}$ and $K' = K \cap (S \times S)$.

Since $\mathrm{S}(\emptyset, \rho) \geq \alpha + 1$ there exists $\theta \in [\nu', K']$ with $\theta \neq \rho$ and $\mathrm{S}(\emptyset, \theta) \geq \alpha$. Clearly $\theta^0 \in [\nu, K]$, $\theta^0 \neq \rho^0$ and our inductive assumption gives that $\mathrm{S}(\emptyset, \theta^0) \geq \alpha$. Hence $\mathrm{S}(\emptyset, \rho^0) \geq \alpha + 1$ as required. □

Corollary 4.4. *Let S be a semigroup and let J be a monoid $\mathcal{J}$-class of S. If S is ranked, then so is J.*

Proof. From Corollary 4.2 we know that every element of $\mathcal{C}_P$ has S-rank, where P is the principal factor associated with J. Since J is a monoid, P is J with possibly a zero adjoined. The result follows from Lemma 4.3.

□

Finally in this section we aim to show that if every element of $\mathcal{C}_S$ has S-rank, then so does every element of $\mathcal{C}_G$ for any maximal subgroup of S.

Let G be a maximal subgroup of S with identity e. Then G is the $\mathcal{H}$-class of the idempotent e. Let

$$J = eS \text{ and } I = \{s \in S : s <_{\mathcal{R}} e\}.$$

Then I, J are right ideals with $G \subseteq R_e = J \setminus I$. Let $\rho \in \mathcal{RC}_G$ and let $\overline{\rho} = \langle \rho \rangle$ be the right congruence on S generated by ρ. Clearly if $a, b \in S$ and $a \,\overline{\rho}\, b$, then if $a \neq b$ we must have $a, b \in J$.

Lemma 4.5. *Let $I, J, \overline{\rho}$ be as above. Then*
(i) $(I, \overline{\rho}), (J, \overline{\rho}) \in \mathcal{C}_S$:
(ii) if $a \, \overline{\rho} \, b$ and $a, b \in R_e$, then $a \, \mathcal{H} \, b$;
(iii) $\overline{\rho} \cap (G \times G) = \rho$.

Proof. Notice first that as $\rho \subseteq G \times G \subseteq \mathcal{L}$ and $\mathcal{L}$ is a right congruence, certainly $\overline{\rho} \subseteq \mathcal{L}$ so that (ii) holds.

If $a, b \in S$ and $a \, \overline{\rho} \, b$, then $a = b$ or there is a sequence

$$a = x_1 t_1, y_1 t_1 = x_2 t_2, \ldots, y_l t_l = b \quad (*)$$

where $(x_i, y_i) \in \rho$ and $t_i \in S^1$. Suppose that $x_i t_i \, \mathcal{R} \, e$. Since $\mathcal{R}$ is a left congruence,

$$y_i t_i = y_i e t_i = y_i x_i^{-1} x_i t_i \, \mathcal{R} \, y_i x_i^{-1} e \, \mathcal{R} \, e.$$

Thus if $a \in R_e$ we conclude that $b \in R_e$ and consequently, $(I, \overline{\rho}), (J, \overline{\rho}) \in \mathcal{C}_S$.

Certainly $\rho \subseteq \overline{\rho} \cap (G \times G)$. Suppose now that $a, b \in G$ and $a \, \overline{\rho} \, b$. Then $a = b$ (so that $a \, \rho \, b$), or there is a sequence $(*)$. Now

$$e t_1 = x_1^{-1} x_1 t_1 = x_1^{-1} a \in G.$$

Suppose for finite induction that $1 \leq i < l$ and $e t_i \in G$. Then

$$e t_{i+1} = x_{i+1}^{-1} x_{i+1} t_{i+1} = x_{i+1}^{-1} y_i t_i = x_{i+1}^{-1} y_i e t_i \in G.$$

Consequently, replacing $(*)$ with the sequence

$$a = x_1(e t_1), y_1(e t_1) = x_2(e t_2), \ldots, y_l(e t_l) = b,$$

we see that $a \, \rho \, b$. Thus (iii) holds. $\square$

With $I, J, \overline{\rho}$ as above we let

$$\overline{\overline{\rho}} = \overline{\rho} \cup \rho_I,$$

where ρ_I is the Rees right congruence associated with I. Clearly $\overline{\overline{\rho}}$ is a right congruence with

$$(I, \overline{\overline{\rho}}), (J, \overline{\overline{\rho}}) \in \mathcal{C}_S \text{ and } \overline{\overline{\rho}} \cap (R_e \times R_e) = \overline{\rho} \cap (R_e \times R_e),$$

so that in particular,

$$\overline{\overline{\rho}} \cap (G \times G) = \rho.$$

To proceed we require a technical result.

Lemma 4.6. *Let S be a semigroup and let G be a maximum subgroup with identity e. If $u, v \in S$ and $u \, \mathcal{H} \, v \, \mathcal{R} \, e$, then for any $t \in S^1$ with $ut \in G$ we have that $f = t(ut)^{-1} u \in E(S)$,*

$$u \, \mathcal{H} \, v \, \mathcal{L} \, f$$

and $vt \in G$.

Proof. Let u, v, t be as given. Putting $f = t(ut)^{-1}u$, we have that

$$f^2 = t(ut)^{-1}ut(ut)^{-1}u = t(ut)^{-1}u$$

so that f is idempotent. Moreover

$$uf = ut(ut)^{-1}u = eu = u$$

since $e\,\mathcal{R}\,u$, so that $u\,\mathcal{L}\,f$. But then $v\,\mathcal{L}\,f$ so that

$$v = vf = vt(ut)^{-1}u$$

and $v\,\mathcal{R}\,vt$. Certainly $ut\,\mathcal{L}\,vt$ so that $ut\,\mathcal{H}\,vt$ and $vt \in G$.

□

Lemma 4.7. *With ρ, I and $\overline{\overline{\rho}}$ as above,*

$$\mathrm{S}(\emptyset, \rho) \leq \mathrm{S}(I, \overline{\overline{\rho}}).$$

Proof. Proceeding by induction we suppose that $\mathrm{S}(\emptyset, \rho) \geq \alpha + 1$. Let $\overline{\overline{\rho}} \in [\nu, K]$ where $\nu = \langle H \rangle$ for some finite $H = \{(a_i, b_i)\}$.

Let $(a_i, b_i) \in H \cap (R_e \times R_e)$; since $a_i\,\overline{\overline{\rho}}\,b_i$ we know from Lemma 4.5 that $a_i\,\mathcal{H}\,b_i$. We choose $w_i \in S^1$ with $a_i w_i \in G$; from Lemma 4.6 we know that $b_i w_i \in G$ and certainly $a_i w_i\,\overline{\overline{\rho}}\,b_i w_i$. From Lemma 4.5 we have that $a_i w_i\,\rho\,b_i w_i$. Putting

$$\nu' = \langle \{(a_i w_i, b_i w_i) : (a_i, b_i) \in H \cap (R_e \times R_e)\} \rangle$$

we have that $\nu' \subseteq \rho$.

For each $(u_l, v_l) \in K \cap (R_e \times R_e)$ with $u_l\,\mathcal{H}\,v_l$ we choose $t_l \in S^1$ with $u_l t_l, v_l t_l \in G$. If $u_l t_l\,\rho\,v_l t_l$, then using Lemma 4.6 we have that

$$u_l = u_l t_l (u_l t_l)^{-1} u_l\,\overline{\overline{\rho}}\,v_l t_l (u_l t_l)^{-1} u_l = v_l,$$

a contradiction. Thus putting

$$K' = \{(u_l t_l, v_l t_l) : (u_l, v_l) \in K \cap (R_e \times R_e), u_l\,\mathcal{H}\,v_l\}$$

we have that

$$\rho \in [\nu', K'].$$

Since $\mathrm{S}(\emptyset, \rho) \geq \alpha + 1$ there exists $\theta \in [\nu', K']$ with $\theta \neq \rho$ and $\mathrm{S}(\emptyset, \theta) \geq \alpha$. Our inductive hypothesis gives that $\mathrm{S}(I, \overline{\overline{\theta}}) \geq \alpha$. Clearly $\overline{\overline{\theta}} \neq \overline{\overline{\rho}}$, and

$$\overline{\overline{\theta}} \cap (I \times I) = \overline{\overline{\rho}} \cap (I \times I) = I \times I.$$

Let $(a_i, b_i) \in H$; then $a_i\,\overline{\overline{\rho}}\,b_i$. Thus $a_i = b_i$ or $a_i, b_i \in I$, so that certainly $a_i\,\overline{\overline{\theta}}\,b_i$, or else $a_i, b_i \in R_e$. In the latter case $a_i\,\mathcal{H}\,b_i$ and we have a pair $(a_i w_i, b_i w_i) \in \nu'$, so that $a_i w_i\,\theta\,b_i w_i$. Then

$$a_i = a_i w_i (a_i w_i)^{-1} a_i\,\overline{\overline{\theta}}\,b_i w_i (a_i w_i)^{-1} a_i = b_i,$$

calling upon Lemma 4.6. Thus $\nu \subseteq \overline{\overline{\theta}}$.

Finally, let $(u_l, v_l) \in K$. Since $\overline{\overline{\rho}} \subseteq (S \times S) \setminus K$, we either have $u_l \in I, v_l \notin I$ (or vice versa), in which case $(u_l, v_l) \notin \overline{\overline{\theta}}$, or $u_l, v_l \in S \setminus I$. Then $u_l, v_l \in S \setminus J$ and

$u_l \neq v_l$, $u_l \in R_e, v_l \in S \setminus J$ (or vice versa), or $u_l, v_l \in R_e$ and $(u_l, v_l) \notin \overline{\rho}$. In either of the first two cases, $(u_l, v_l) \notin \overline{\overline{\theta}}$. We concentrate on the third. If $(u_l, v_l) \notin \mathcal{H}$, then $(u_l, v_l) \notin \overline{\overline{\theta}}$ by Lemma 4.5. On the other hand if $u_l \,\mathcal{H}\, v_l$ then we have a pair $(u_l t_l, v_l t_l) \in K'$. If $u_l \,\overline{\overline{\theta}}\, v_l$ then $u_l t_l \,\overline{\overline{\theta}}\, v_l t_l$, so that $u_l t_l \,\theta\, v_l t_l$ by Lemma 4.5. This contradiction gives that $\overline{\overline{\theta}} \in [\nu, K]$ and so $\mathrm{S}(I, \overline{\overline{\rho}}) \geq \alpha + 1$ and the result follows. □

Corollary 4.8. *Let S be a semigroup. If S is ranked, then so is every maximal subgroup of S.*

5. Ranked monoids and noetherian conditions

We first show that being right noetherian guarantees ranking.

Proposition 5.1. *Let S be a right noetherian semigroup. Then S is ranked.*

Proof. Let $\rho \in \mathcal{RC}$ and suppose that $\mathrm{S}(\emptyset, \rho) = \infty$. Now ρ is finitely generated by assumption, so that $[\rho, \emptyset]$ is a subset of $\mathcal{RC}$ of finite type and $\rho \in [\rho, \emptyset]$. If every element of $[\rho, \emptyset]$, other than ρ, has S-rank, then choosing an ordinal α strictly greater than the S-ranks of all the elements of $[\rho, \emptyset]$, distinct from ρ, we would have that $\mathrm{S}(\emptyset, \rho) \not\geq \alpha + 1$, a contradiction. Hence we can find $\rho_1 \in [\rho, \emptyset]$ with $\rho_1 \neq \rho$ and $\mathrm{S}(\emptyset, \rho_1) = \infty$. Clearly $\rho \subset \rho_1$. Proceeding in this manner we can find a chain

$$\rho \subset \rho_1 \subset \rho_2 \subset \ldots$$

such that $\mathrm{S}(\emptyset, \rho_i) = \infty$ for all i. This contradicts the fact that S is right noetherian.

□

Our next aim is to show that a ranked inverse semigroup is weakly right noetherian. To this end we have the following.

Lemma 5.2. *Let S be a ranked semigroup. Then S contains no infinite set of pairwise $\mathcal{R}$-incomparable elements.*

Proof. Suppose that

$$\{a_i : i \in \mathbb{N}\}$$

is a set of pairwise $\mathcal{R}$-incomparable elements (where $a_i \neq a_j$ for $i \neq j$). Let

$$\sigma = \langle \{(a_i, a_j) : i, j \in \mathbb{N}\} \rangle.$$

We consider the subset X of $\mathcal{RC}$, consisting of right congruences of the form

$$\langle \nu \cup \{(a_i, a_j) : i, j \geq t\} \rangle),$$

where ν is a finitely generated right congruence contained in σ and $t \in \mathbb{N}$. Notice that if $\rho \in X$, then $\rho \subseteq \sigma$.

Let α be an ordinal and suppose that for any $\rho \in X$

$$\mathrm{S}(\emptyset, \rho) \geq \alpha.$$

Fix $\rho \in X$, so that

$$\rho = \langle \nu \cup \{(a_i, a_j) : i, j \geq t\}\rangle)$$

for some finitely generated $\nu \subseteq \sigma$ and some $t \in \mathbb{N}$.

Let $[\tau, K]$ be an interval of finite type with $\rho \in [\tau, K]$ and let $H = \{(x_k, y_k)\}$ be a finite set of generators for $\langle \nu \cup \tau \rangle$. Since that a_i's are pairwise $\mathcal{R}$-incomparable, we may choose $w \in \mathbb{N}$, $w \geq t$, with x_k, y_k *not* $\mathcal{R}$-related to a_{w+s} for all k and for all $s \geq 0$. Let

$$\theta = \langle H \cup \{(a_i, a_j) : i, j \geq w + 2\}\rangle,$$

so that $\theta \in X$.

Clearly $\theta \subseteq \rho$. However, $\theta \neq \rho$ since $(a_w, a_{w+1}) \in \rho$ but $(a_w, a_{w+1}) \notin \theta$. For if $a_w \, \theta \, a_{w+1}$, then we are forced to have $a_w \leq_{\mathcal{R}} x_k$ or $a_w \leq_{\mathcal{R}} y_k$ for some k. Without loss of generality suppose the former. Since $x_k \, \sigma \, y_k$ we have that $x_k \leq_{\mathcal{R}} a_i$ for some i. But then

$$a_w \leq_{\mathcal{R}} x_k \leq_{\mathcal{R}} a_i$$

forces $i = w$ and $x_k \, \mathcal{R} \, a_w$, a contradiction.

By our inductive assumption we have that $\mathrm{S}(\emptyset, \theta) \geq \alpha$; since $\theta \in [\tau, K]$ this gives that $\mathrm{S}(\emptyset, \rho) \geq \alpha + 1$. Consequently, $\mathrm{S}(\emptyset, \rho) = \infty$ for any $\rho \in X$ (and in particular for $\rho = \sigma$). But this contradicts the fact that S is ranked.

□

We remark that if $\mathcal{M}^0(G; I, \Lambda; P)$ is a ranked Rees matrix semigroup, then from Corollary 4.8 and Lemma 5.2, G is ranked and I is finite. Conversely, we show in the sequel that a Brandt semigroup $\mathcal{B}^0(G, I)$ with G ranked and I finite, is ranked.

A similar technique to that in Lemma 5.2 yields our next result. The partial order concerned is the natural partial order on idempotents, in which $e \leq f$ if and only if $ef = fe = e$.

Lemma 5.3. *Let S be a ranked semigroup. Then S contains no infinite ascending chain*

$$e_1 < e_2 < \ldots$$

of idempotents.

Proof. We proceed as in Lemma 5.2, replacing the elements a_i with the idempotents e_i, and this time choosing $w \geq t$ such that $x_k, y_k \leq_{\mathcal{R}} e_w$ for all k. The proof diverges where we wish to show that $(e_w, e_{w+1}) \notin \theta$.

If $e_w \, \theta \, e_{w+1}$ then we must have a sequence

$$e_{w+1} = c_1 t_1, \; d_1 t_1 = c_2 t_2, \; \ldots, d_l t_l = e_w$$

where for each $1 \leq i \leq l$ we have that $(c_i, d_i) = (x_{k_i}, y_{k_i}), (y_{k_i}, x_{k_i})$ or (e_s, e_t) for some $s, t \geq w + 2$.

If $c_1 = x_{k_1}$, then

$$e_{w+1} \leq_{\mathcal{R}} x_{k_1} \leq_{\mathcal{R}} e_w,$$

so that $e_{w+1}\,\mathcal{R}\,e_w$, which is impossible since $e_w < e_{w+1}$; similarly, we cannot have that $c_1 = y_{k_1}$. Hence we must have that $c_1 = e_s, d_1 = e_t$ for some $s, t \geq w+2$. Then

$$e_{w+1} = e_{w+1}c_1t_1 = e_{w+1}t_1.$$

Suppose for finite induction that $1 \leq i < l$,

$$\{c_i, d_i\} \subseteq \{e_j : j \geq w+2\}$$

and $e_{w+1} = e_{w+1}t_i$. We obtain

$$e_{w+1} = e_{w+1}d_it_i = e_{w+1}c_{i+1}t_{i+1};$$

as above c_{i+1} cannot be below e_w in the $\leq_{\mathcal{R}}$-order. Consequently, $c_{i+1}, d_{i+1} \in \{e_j : j \geq w+2\}$ and so $e_{w+1} = e_{w+1}t_{i+1}$.

Finite induction gives that

$$e_{w+1} = e_{w+1}t_l = e_{w+1}d_lt_l = e_{w+1}e_w = e_w,$$

a contradiction. This tells us that $\theta \neq \rho$ and proceeding as in Lemma 5.2 allows us to deduce that $\mathrm{S}(\emptyset, \sigma) = \infty$, contradicting the fact that S is ranked. □

Corollary 5.4. *Let S be a ranked inverse semigroup. Then S is weakly right noetherian.*

Proof. Since S is inverse, $E(S)$ forms a semilattice, and the natural partial order on $E(S)$ coincides with the $\leq_{\mathcal{R}}$-order.

Let I be a right ideal of S with set of generators X. Since S is inverse we may assume that $X \subseteq E(S)$ and from Lemma 5.3 that X consists of idempotents maximal under the $\leq_{\mathcal{R}}$-order. Thus the elements of X are pairwise incomparable, so that by Lemma 5.2, X is finite. □

In a subsequent paper we show that the Bicyclic monoid B, which is certainly inverse and weakly right noetherian, is not ranked. To achieve this result, we need the dual of Lemma 5.3 in the case where S is a chain.

Let C be a chain (that is, a totally ordered semilattice), and let $\rho \in \mathcal{RC}$. It is easy to see that the ρ-classes are convex subsets, and these therefore form a chain under the induced ordering. On the other hand, if C is partitioned into convex subsets, then the associated equivalence relation is a congruence.

We say that a convex subset of C is *bounded* if it is of the form

$$[a, b] = \{c : a \geq c \geq b\}.$$

Lemma 5.5. *A congruence on a chain C is finitely generated if and only if the corresponding partition contains only finitely many non-trivial (convex) sets, and these are all bounded.*

Proof. Suppose that $\rho \in \mathcal{RC}$ is finitely generated; say

$$\rho = \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle.$$

It is easy to see that we may assume that

$$a_1 > b_1 > a_2 > b_2 > \ldots > a_n > b_n.$$

Let $x, y \in C$ with $x \,\rho\, y$ and $b_i > x > a_{i+1}$. Then $x = y$ or there is a sequence

$$x = c_1 t_1, d_1 t_1 = c_2 t_2, \ldots, d_l t_l = y$$

where for each i, $(c_i, d_i) = (a_{j_i}, b_{j_i})$ or (b_{j_i}, a_{j_i}) and $t_i \in C^1$.

Since $x = c_1 t_1 \in \{c_1, t_1\}$ and $x \notin \{a_1, b_1, \ldots, a_n, b_n\}$, we must have that $x = t_1 < c_1$. It follows that $x < d_1$ also. Now if $i < l$, $x = t_i$ and $x < c_i, d_i$, then $x = d_i t_i = c_{i+1} t_{i+1}$, so that $x = t_{i+1} < c_{i+1}, d_{i+1}$. Consequently, $x = t_l < d_l$ so that $x = d_l t_l = y$. Thus the non-trivial ρ-classes are precisely the bounded subsets

$$[a_1, b_1], \ldots, [a_n, b_n].$$

Conversely, if we are given a congruence ρ such that the only non-trivial classes are

$$[a_1, b_1], \ldots, [a_n, b_n],$$

then it is easy to see that

$$\rho = \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle.$$

□

Lemma 5.6. *Let C be a chain and let $\rho, \nu \in \mathcal{RC}$ with $\nu \subset \rho$, ν finitely generated and ρ* not *finitely generated. Then there exists a non-finitely generated $\theta \in \mathcal{RC}$ with $\nu \subset \theta \subset \rho$.*

Proof. Let ν, ρ be as given and let

$$\nu = \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle$$

where $a_1 > b_1 > \ldots > a_n > b_n$.

Suppose first that all ρ-classes are bounded. From Lemma 5.5, ρ has infinitely many non-trivial classes. We can therefore choose a non-trivial ρ-class $[u, v]$ such that

$$[u, v] \cap \{a_1, b_1, \ldots, a_n, b_n\} = \emptyset.$$

Let θ be the congruence corresponding to the partition obtained from the ρ-classes with $[u, v]$ replaced by $\{u\}, \{c : u > c \geq v\}$. Clearly $\nu \subset \theta \subset \rho$ and θ is not finitely generated, since it has infinitely many non-trivial classes.

Consider now the case where ρ has an unbounded class X. Bearing in mind that $\nu \subset \rho$, we have that

$$X \cap \{a_1, b_1, \ldots, a_n, b_n\} = \{a_i, b_i, \ldots, a_j, b_j\}$$

for some (possibly none) i, j with $i \leq j$. Without loss of generality assume that X is unbounded below. Pick $u \in X$ with $b_j > u$. Now

$$X = \{x \in X : x \geq u\} \cup \{x \in X : u > x\}$$

is a partition of X into two non-empty sets, at least one of which is unbounded. Let θ be the congruence corresponding to the partition obtained from the ρ-classes by dividing X as given. Then $\nu \subset \theta \subset \rho$ and θ is not finitely generated since it has an unbounded class.

□

It is now straightforward to achieve our desired result.

Proposition 5.7. *Let C be a chain. Then C is ranked if and only if it is finite.*

Proof. As remarked before Proposition 3.4, every finite semigroup is ranked.

Suppose that C is ranked and C contains an unbounded subset X. It is easy to see that C then contains an unbounded convex subset and hence, in view of Lemma 5.5, a congruence ρ that is not finitely generated. Using a now familiar technique, assume that for any non-finitely generated $\rho \in \mathcal{RC}$ and an ordinal α we have that $\mathrm{S}(\emptyset, \rho) \geq \alpha$. If $\rho \in [\nu, K]$ where $[\nu, K]$ is of finite type, then by Lemma 5.6, there is a non-finitely generated $\theta \in \mathcal{RC}$ with $\nu \subset \theta \subset \rho$. Consequently, $\theta \in [\nu, K]$ and by assumption, $\mathrm{S}(\emptyset, \theta) \geq \alpha$. Thus $\mathrm{S}(\emptyset, \rho) \geq \alpha + 1$. Transfinite induction yields that $\mathrm{S}(\emptyset, \rho) = \infty$, contradicting the fact that C is ranked.

It is easy to see that a chain with no unbounded subsets is finite. □

6. Examples of ranked monoids

We have seen in the previous section that any right noetherian semigroup is ranked. In particular, any finite semigroup is ranked.

The free commutative monoid on a set X is denoted by $\mathcal{FC}_X^*$; it is worth remarking that $\mathcal{FC}_X^*$ is right coherent [8]. For any word w in $\mathcal{FC}_X^*$ the *content* $c(w)$ of w is the set of letters of X occurring in w.

Proposition 6.1. *The free commutative monoid $\mathcal{FC}_X^*$ on a non-empty set X is ranked if and only if X is finite.*

Proof. If X is finite, then by Rédei's Theorem [7, 16], $\mathcal{FC}_X^*$ is (right) noetherian. By Lemma 5.1, $\mathcal{FC}_X^*$ is ranked.

Conversely, suppose that X is infinite. We say that a congruence ρ is *basic* if ρ has a finite set of generators

$$\{(a_1, b_1), \ldots, (a_n, b_n)\}$$

where $a_1, b_1, \ldots, a_n, b_n$ are distinct elements of X. We show by transfinite induction that $\mathrm{S}(\emptyset, \rho) \geq \alpha$ for every ordinal α and every basic ρ. The statement is true for $\alpha = 0$, and the step at limit ordinals is clear.

Let α be an ordinal and suppose that for all basic congruences ρ, $\mathrm{S}(\emptyset, \rho) \geq \alpha$. Let ρ be basic, say

$$\rho = \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle$$

where $a_1, b_1, \ldots, a_n, b_n$ are distinct elements of X. Suppose that $\rho \in [\nu, K]$ where $[\nu, K]$ is an interval of finite type. Let $C \subseteq X$ be the union of the sets $c(u)$, where a pair (u, v) or (v, u) appears in K. Choose distinct

$$a_{n+1}, b_{n+1} \in X \setminus D$$

where

$$D = C \cup \{a_1, \ldots, b_n\}$$

and let

$$\mu = \langle (a_1, b_1) \ldots, (a_n, b_n), (a_{n+1}, b_{n+1}) \rangle.$$

Clearly $\rho \subseteq \mu$. If $\rho = \mu$, then $a_{n+1} \, \rho \, b_{n+1}$, whence (as $a_{n+1} \neq b_{n+1}$) we have that

$$a_{n+1} = a_j t \text{ or } b_j t$$

for some $j \in \{1, \ldots, n\}$ and $t \in \mathcal{FC}_X^*$, which is impossible. Hence $\rho \subset \mu$.

Let $(u, v) \in K$. If $u \, \mu \, v$ then we must have a sequence

$$u = c_1 t_1, d_1 t_1 = c_2 t_2, \ldots, d_l t_l = v$$

where for each k, $(c_k, d_k) = (a_{i_k}, b_{i_k})$ or (b_{i_k}, a_{i_k}). From $u = c_1 t_1$ we obtain that $c(t_1) \subseteq D$, and $i_1 \neq n + 1$. Supposing for finite induction that $c(t_j) \subseteq D$ and $i_j \neq n + 1$, consider $d_j t_j = c_{j+1} t_{j+1}$. Then

$$c(c_{j+1}), c(t_{j+1}) \subseteq c(d_j) \cup c(t_j) \subseteq D$$

so that $i_{j+1} \neq n + 1$. We conclude that none of the i_k's are $n + 1$, hence $u \, \rho \, v$, a contradiction. Consequently, $(u, v) \notin \mu$ and so $\mu \in [\nu, K]$.

By our inductive assumption we have that $\mathrm{S}(\emptyset, \mu) \geq \alpha$, whence we deduce that $\mathrm{S}(\emptyset, \rho) \geq \alpha + 1$. Consequently, $\mathrm{S}(\emptyset, \rho) = \infty$ for any basic right congruence ρ.

$\square$

From Proposition 4.1 and Proposition 6.1 we deduce the following.

Corollary 6.2. *Any finitely generated commutative monoid is ranked.*

Proposition 3.4 tells us that for a right congruence ρ on a right noetherian monoid S, $\mathrm{S}(\emptyset, \rho) = 0$ if and only if ρ is finitely covered. In $\mathcal{FC}_X^*$ where X is finite, the identity congruence ι is not finitely covered. For if $\iota \subset \rho$ for

$$\rho = \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle \in \mathcal{RC},$$

where we may assume that $a_i \neq b_i$ for each i, pick a word v strictly longer than any a_i or b_i. Clearly

$$\iota \subset \langle (a_1 v, b_1 v) \rangle \subset \rho.$$

Hence ι has no covers, so certainly is not finitely covered.

In the case where $X = \{a\}$, $\mathcal{FC}_X^*$ is isomorphic to $\mathbb{N}^0$ under addition, and we can be precise about the S-rank of elements of $\mathcal{C}$. Let $(I, \rho) \in \mathcal{C}$. If $I \neq \emptyset$, then $I = \{n, n + 1, \ldots\}$ for some n. Let

$$K = \{(i, j) : i, j < n, (i, j) \notin \rho\}.$$

It is easy to see that $[\rho, K]$ isolates ρ in $\mathcal{C}(I, \rho)$. Hence $\mathrm{S}(I, \rho) = 0$. Suppose now that $I = \emptyset$. If $\rho \neq \iota$, then, calling upon standard results for monogenic monoids,

$$\rho = \langle (n, n + r) \rangle$$

for some n, r with $r \geq 1$. Let

$$K = \{(u, v) : 0 \leq u, v \leq n + r - 1, u \neq v\}.$$

Let $\theta \in [\rho, K]$ and suppose that $a\,\theta\,b$. Certainly $a\,\rho\,a'$ and $b\,\rho\,b'$ for some a', b' with $0 \leq a', b' \leq n + r - 1$. Since $a'\,\theta\,b'$ and $(a', b') \notin K$ we must have that $a' = b'$ so that $a\,\rho\,b$. Thus $[\rho, K]$ isolates ρ in $\mathcal{RC}$ and $\mathrm{S}(\emptyset, \rho) = 0$. We showed above that $\mathrm{S}(\emptyset, \iota) \geq 1$, and we deduce that $\mathrm{S}(\emptyset, \iota) = 1$.

In view of Corollary 3.5, when determining S-rank for groups, we need only consider elements of $\mathcal{C}$ of the form $(\emptyset, \rho)$.

For the free cyclic group $\mathbb{Z}$, Proposition 4.1 and Proposition 6.1 certainly yield that $\mathbb{Z}$ is ranked. Since any subgroup of $\mathbb{Z}$ is cyclic, and any non-trivial subgroup $n\mathbb{Z}$ where $n \in \mathbb{N}$ is finitely covered by the subgroups $\frac{n}{p}\mathbb{Z}$ where p is a prime factor of n, it follows from Proposition 3.4 that $\mathrm{S}(\emptyset, \rho) = 0$ for any non-identity congruence ρ. On the other hand the trivial subgroup has no covers, so that $\mathrm{S}(\emptyset, \iota) \geq 1$. Consequently, $\mathrm{S}(\emptyset, \iota) = 1$.

Considering now the group $\mathbb{Z}(p^\infty)$ of all p^n-th roots of unity, we see that $\mathrm{S}(\emptyset, \rho) = 0$ for every non-universal congruence ρ, and $\mathrm{S}(\emptyset, \omega) = 1$. Indeed, let S be a semigroup such that

$$\mathcal{RC} = \{\rho_i : i \in \mathbb{N} \cup \{0\}\} \cup \{\omega\}$$

where

$$\rho_0 = \iota \subset \rho_1 \subset \rho_2 \subset \ldots$$

and each ρ_i is finitely generated. From Proposition 3.4 we see that for each i, $\mathrm{S}(\emptyset, \rho_i) = 0$. But ω cannot be isolated by any interval $[\rho_i, \emptyset]$, so that $\mathrm{S}(\emptyset, \omega) = 1$.

In a subsequent paper we show that a Brandt semigroup $\mathcal{B}^0(G, I)$ where G is a ranked group and I is finite, is ranked.

7. Total transcendence of T_S

We begin with a brief discussion of the model theory required for this section.

Let L be a first order language and let T be a complete theory in L, admitting elimination of quantifiers. The example of interest to us here is $L = L_S$ and $T = T_S$, where S is a right coherent monoid.

The notion of a *type* over T is crucial to our investigations. To define this, it is useful to employ the so-called monster model of a theory. We fix a model $\mathbf{M}$ of T, saturated of cardinality κ for some cardinal κ much bigger than all other cardinals under consideration; $\mathbf{M}$ is the *monster model* of T. We make the convention that all models of T will be elementary substructures of $\mathbf{M}$ with universes of cardinality less than κ and all sets of parameters will be subsets of

$\mathbf{M}$ of cardinality less than κ. Justification of the use of the monster model can be found in [3].

Let A be a subset of $\mathbf{M}$ and let $c \in \mathbf{M}$. Then

$$\mathrm{tp}(c/A) = \{\phi(x) \in L(A) : \mathbf{M} \models \phi(c)\}$$

is a *(complete 1-)type over* A. Clearly $\mathrm{tp}(c/A)$ is a set $p(x)$ of sentences of $L(A, x)$, that is *consistent* with $\mathrm{Th}(\mathbf{M}, a)_{a \in A}$ and is *complete* in the sense that for any formula $\phi(x)$ of $L(A)$, either $\phi(x)$ or $\neg\phi(x)$ is in $p(x)$; we say that $p(x)$ is *realised* by c. Conversely, if $p(x)$ is a set of formulae satisfying these conditions, then the saturation of M gives that $p(x) = \mathrm{tp}(b/A)$ for some $b \in \mathbf{M}$. The *Stone space* $S(A)$ of A is the collection of all types over A and comes equipped with a natural topology, the basic open sets of which are

$$\langle\phi(x)\rangle = \{p \in S(A) : \phi(x) \in p\},$$

where $\phi(x)$ is a formula of $L(A)$. The space $S(A)$ has a basis of clopen sets $\langle\phi(x)\rangle$, and is compact and Hausdorff. Since we are assuming that T has elimination of quantifiers, a routine argument gives that the sets $\langle\theta(x)\rangle$ where $\theta(x)$ is a conjunction of atomic and negated atomic formulae form a basis for the topology of $S(A)$.

For a cardinal κ, T is *κ-stable* if for every subset A of a model of T with $|A| \leq \kappa$ we have $|S(A)| \leq \kappa$. If T is κ-stable for some infinite κ, then T is *stable* and T is *superstable* if T is κ-stable for all $\kappa \geq 2^{|T|}$. If T is not stable, then it is said to be *unstable*. We know that T_S is stable, and superstable if and only if S is weakly right noetherian [12, 6]. To investigate the stronger property of being totally transcendental, we need to define Morley rank for types.

Let A be a subset of $\mathbf{M}$. Subsets $M^\alpha(A)$ of $S(A)$ are defined by induction on the ordinal α as follows:

(I) $M^0(A) = S(A)$;

(II) if α is a limit ordinal, then

$$M^\alpha(A) = \bigcap\{M^\beta(A) : \beta < \alpha\};$$

(III) for any α, $M^{\alpha+1}(A) = M^\alpha(A) \setminus X^\alpha$, where

$$X^\alpha = \{p \in M^\alpha(A) : \text{ for all } B \supseteq A \text{ and all extensions } q \text{ of } p \text{ on } B,$$
$$q \notin M^\alpha(B) \text{ or } q \text{ is isolated in } M^\alpha(B)\}.$$

We may take B to be an L-substructure of a model of T.

For $p \in S(A)$, the *Morley rank* of p is $\mathrm{M}(p)$, where if $p \in M^\alpha(A)$ for all α, then $\mathrm{M}(p) = \infty$ and otherwise, $\mathrm{M}(p)$ is α where $p \in M^\alpha(A) \setminus M^{\alpha+1}(A)$. If $\mathrm{M}(p) < \infty$, then we say that *p has Morley rank*. As for the other ranks met in this paper, it is easy to see that for $p \in S(A)$ and an ordinal α, $p \in M^\alpha(A)$ if and only if $\mathrm{M}(p) \geq \alpha$.

A theory T is *totally transcendental* if and only if for all subsets A of models of T, all types over A have Morley rank. From Proposition 4.27 of [10], T is totally transcendental if and only if all types over the empty set have Morley rank. As mentioned in the Introduction, it is a standard result that for all types p, $\mathrm{U}(p) \leq \mathrm{M}(p)$ [14]; thus a totally transcendental theory is necessarily superstable.

For the remainder of the paper, the theory in question will be T_S where S is a right coherent monoid. The key to our arguments is the following description of types.

If A is an S-set, then an A-*triple* is a triple (I, ρ, f) such that $(I, \rho) \in \mathcal{C}$ and $f : I \to A$ is an S-morphism with $\mathrm{Ker} f = \rho \cap (I \times I)$. We denote the set of all A-triples by $\mathcal{T}(A)$.

Proposition 7.1. [6] *Let p be a type over an S-set A. Let*

$$I_p = \{s \in S : xs = a \in p \text{ for some } a \in A\},$$

$$\rho_p = \{(s,t) \in S \times S : xs = xt \in p\},$$

and

$$f_p : I_p \to A \text{ be defined by } f_p(s) = a \text{ where } xs = a \in p.$$

Then $\mathcal{T}_p = (I_p, \rho_p, f_p)$ is an A-triple and the map

$$p \mapsto \mathcal{T}_p$$

is a bijection from $S(A)$ to $\mathcal{T}(A)$.

We note that for any type p over an S-set A, $\mathrm{M}(p) = 0$ if and only if $I_p = S$, that is $\mathrm{U}(p) = 0$. For if $I_p = S$ and $p \subseteq q$ where $q \in S(B)$, then since $1 \in I_q$, $x = b \in q$ for some $b \in B$ and $\{x = b\}$ isolates q in $S(B)$. Thus $p \notin M^1(A)$ so that $\mathrm{M}(p) = 0$. On the other hand, if $\mathrm{M}(p) = 0$, then by the comment above, $\mathrm{U}(p) = 0$, so that $I_p = S$ by Proposition 4.9 of [6].

In the following proposition we suppose that S is weakly right noetherian and every $(I, \rho) \in \mathcal{C}$ has S-rank, and find an upper bound on the Morley rank of a given type. Our bound can be improved upon in the case where S is a group, as we show in Theorem 7.4. Note that to show that T_S is totally transcendental, it would be enough to argue that every type in $S(\emptyset)$ as Morley rank. We prefer, however, to produce a reasonably tight upper bound on the Morley rank of an arbitrary type. The order on $\mathcal{O} \times \mathcal{O}$ below is lexicographic.

Proposition 7.2. *If S is weakly right noetherian and every $(I, \rho) \in \mathcal{C}$ has S-rank, then T_S is totally transcendental. Moreover,*

$$\mathrm{M}(p) \leq (V(I_p), S(I_p, \rho_p))$$

for any $p \in S(A)$.

Proof. We first note that as $\mathcal{RI}$ and $\mathcal{C}$ are sets, our hypothesis guarantees that we can find ordinals α, β with $\mathrm{V}(I) < \alpha$ and $S(I, \rho) < \beta$ for all $\alpha \in \mathcal{RI}$, $(I, \rho) \in \mathcal{C}$.

We put

$$\mathcal{P} = \{(\gamma, \delta) : \gamma < \alpha, \delta < \beta\};$$

$\mathcal{P}$ is well-ordered by the lexicographic ordering, so there exists an order-isomorphism from $\mathcal{P}$ to a segment of $\mathcal{O}$ assigning to each $(\gamma, \delta) \in \mathcal{P}$ its ordinal $o(\gamma, \delta)$. For ease of notation we omit explicit mention of o.

Our aim is to show by transfinite induction on $(\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$ that for any $p \in S(A)$,

$$\mathrm{M}(p) \leq (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p)).$$

Suppose that $(\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p)) = (0, 0)$. Then $I_p = S$ so that by earlier comments, $\mathrm{M}(p) = (0, 0) = (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$.

Proceeding by induction, suppose that for any $q \in S(B)$ with $(V(I_q), S(I_q, \rho_q)) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$ we have that $\mathrm{M}(q) \leq (\mathrm{V}(I_q), \mathrm{S}(I_q, \rho_q))$.

Let $p \subseteq q \in S(B)$; we show that $\mathrm{M}(q) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$ or q is isolated in $S(B)$ among types of Morley rank equal to or greater than $(\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$.

First, we suppose that $I_p \subset I_q$. Then as $V(I_p) > V(I_q)$ we have that

$$(\mathrm{V}(I_q), \mathrm{S}(I_q, \rho_q)) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$$

and so by our inductive hypothesis, $\mathrm{M}(q)$ exists and

$$M(q) \leq (\mathrm{V}(I_q), \mathrm{S}(I_q, \rho_q)) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p)).$$

To proceed we now consider the case where $I_q = I_p$, so that $(I_q, \rho_q) = (I_p, \rho_p)$. Let $\alpha = S(I_p, \rho_p)$; then there is a subset $[\nu, K]$ of finite type such that $[\nu, K]$ isolates ρ_p in $\mathcal{C}(I_p, \rho_p)$ among those elements θ such that $\mathrm{S}(I_p, \theta) \geq \alpha$.

Let

$$\nu = \langle X \rangle \text{ where } X = \{(a_i, b_i)\}$$

and put

$$K = \{(u_l, v_l)\}.$$

We are supposing that S is weakly right noetherian, so that

$$I_q = I_p = \bigcup s_j S$$

for some finite set $\{s_j\} \subseteq S$. By definition of I_q we must have a formula $xs_j = c_j \in q$ for each j. Putting

$$\phi = \bigwedge xs_j = c_j \wedge xa_i = xb_i \wedge xu_l \neq xv_l$$

we have that

$$q \in \langle \phi \rangle.$$

Suppose now that $r \in \langle \phi \rangle$, so that

$$\rho_r \in [\nu, K] \text{ and } I_p \subseteq I_r.$$

If $I_p \subset I_r$ then

$$(\mathrm{V}(I_r), \mathrm{S}(I_r, \rho_r)) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$$

so that by our inductive hypothesis, $\mathrm{M}(r)$ exists and

$$\mathrm{M}(r) \leq (\mathrm{V}(I_r), \mathrm{S}(I_r, \rho_r)) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p)).$$

Otherwise, $I_r = I_p = I_q$ so that $(I_p, \rho_r) \in \mathcal{C}$. Now for any $s_j t \in I_q$,

$$f_q(s_j t) = c_j t = f_r(s_j t),$$

so that $f_q = f_r$. Consequently,

$$\rho_p \cap (I_p \times I_p) = \rho_q \cap (I_q \times I_q) = \ker f_q = \ker f_r = \rho_r \cap (I_p \times I_p).$$

Thus $\rho_r \in \mathcal{C}(I_p, \rho_p)$ so that either $\rho_r = \rho_p = \rho_q$, in which case $r = q$ by Proposition 7.1, or else $\mathrm{S}(I_p, \rho_r) < \alpha$. In the latter case

$$(\mathrm{V}(I_r), \mathrm{S}(I_r, \rho_r)) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$$

and the inductive hypothesis called upon for a final time gives that $\mathrm{M}(r)$ exists and

$$\mathrm{M}(r) \leq (\mathrm{V}(I_r), \mathrm{S}(I_r, \rho_r)) < (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p)).$$

Thus $\langle \phi \rangle$ isolates q in $S(B)$ among types of Morley rank greater than or equal to $(\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$. Consequently,

$$\mathrm{M}(p) \not> (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$$

so that

$$\mathrm{M}(p) \leq (\mathrm{V}(I_p), \mathrm{S}(I_p, \rho_p))$$

as required.

□

We are now in a position to prove the main result of this section.

Theorem 7.3. *The theory T_S is totally transcendental if and only if S is weakly right noetherian and ranked.*

Proof. If S is weakly right noetherian and ranked, then Corollary 3.3 and Proposition 7.2 give that T_S is totally transcendental.

Conversely, suppose that T_S is totally transcendental, so that $\mathrm{M}(p)$ exists for all types p. Since $\mathrm{U}(p) \leq \mathrm{M}(p)$ every type has U-rank, whence S is weakly right noetherian by [6] or [12].

Consider now $\rho \in \mathcal{RC}$; clearly $\rho = \rho_{p_\rho}$ where $p_\rho = \mathrm{tp}(1\rho/\emptyset)$. Suppose that $\mathrm{S}(\emptyset, \rho) = \infty$ for some $\rho \in \mathcal{RC}$. Amongst all such possible $\rho \in \mathcal{RC}$ pick one with $\mathrm{M}(p_\rho)$ least; say $\mathrm{M}(p_\rho) = \alpha$. From the definition of Morley rank we know that p_ρ is isolated by some open set U in $S(\emptyset)$ among types of Morley rank greater than or equal to α. By earlier remarks we can assume that $U = \langle \phi \rangle$ where ϕ is a conjunction of atomic and negated atomic formulae. We write

$$\phi = \bigwedge_{i,l} x a_i = x b_i \wedge x u_l \neq x v_l$$

where the i, l run over finite (possibly empty) indexing sets.

Let

$$\nu = \langle \{(a_i, b_i)\} \rangle \text{ and } K = \{(u_l, v_l)\},$$

so that $[\nu, K]$ is a subset of $\mathcal{RC}$ of finite type and $\rho \in [\nu, K]$.

We are supposing that $(\emptyset, \rho)$ does not have S-rank. Since $[\nu, K]$ is a *set*, if $\mathrm{S}(\emptyset, \theta)$ exists for all $\theta \in [\nu, K], \theta \neq \rho$, then we could find an ordinal β such that $\mathrm{S}(\emptyset, \theta) < \beta$ for all $\theta \in [\nu, K], \theta \neq \rho$. But this would contradict $\mathrm{S}(\emptyset, \rho) \geq \beta + 1$. Thus there exists some $\theta \in [\nu, K], \theta \neq \rho$ with $\mathrm{S}(\emptyset, \theta) = \infty$. But then $p_\theta \in \langle \phi \rangle$ and $p_\theta \neq p_\rho$, so that $\mathrm{M}(p_\theta) < \mathrm{M}(p_\rho)$, contradicting the minimality of $\mathrm{M}(p_\rho)$. We deduce that $\mathrm{S}(\emptyset, \rho) < \infty$ for every $\rho \in \mathcal{RC}$. Corollary 3.3 gives that $\mathrm{S}(I, \rho) < \infty$ for every $(I, \rho) \in \mathcal{C}$. $\square$

In the case where S is a group, we can obtain an exact value for the Morley rank of a type over T_S.

Theorem 7.4. *Let G be a group. Then T_G is totally transcendental if and only if G is ranked. Moreover, for any type $p \in S(A)$, $\mathrm{M}(p) = 0$ if $I_p = G$ and otherwise,*

$$\mathrm{M}(p) = 1 + \mathrm{S}(\emptyset, \rho_p).$$

Proof. The first statement follows from Theorem 7.3 and the fact that $\emptyset, G$ are the only right ideals of G. As remarked earlier, for any type p, if $I_p = G$, then $\mathrm{M}(p) = 0$.

Suppose now that $I_p = \emptyset$. We show via transfinite induction that for all ordinals α,

$$\mathrm{S}(\emptyset, \rho_p) \geq \alpha \Leftrightarrow \mathrm{M}(p) \geq 1 + \alpha,$$

whence the result follows.

Certainly $\mathrm{S}(\emptyset, \rho_p) \geq 0$, and

$$\mathrm{M}(p) \geq \mathrm{U}(p) = 1 = 1 + 0.$$

Let α be an ordinal and suppose that for all ordinals $\beta < \alpha$ and types q with $I_q = \emptyset$ we have that

$$\mathrm{S}(\emptyset, \rho_q) \geq \beta \Leftrightarrow \mathrm{M}(q) \geq 1 + \beta.$$

Consider first the case where α is a limit ordinal; we remark that $1 + \alpha = \alpha$ and for any $\beta < \alpha$ we have that $1 + \beta < 1 + \alpha = \alpha$. Then, using the inductive hypothesis,

$$\begin{array}{rclll} \mathrm{S}(\emptyset, \rho_p) \geq \alpha & \Leftrightarrow & \mathrm{S}(\emptyset, \rho_p) & \geq & \beta & \text{for all } \beta < \alpha \\ & \Leftrightarrow & \mathrm{M}(p) & \geq & 1 + \beta & \text{for all } \beta < \alpha \\ & \Leftrightarrow & \mathrm{M}(p) & \geq & \beta & \text{for all } \beta < \alpha \\ & \Leftrightarrow & \mathrm{M}(p) & \geq & \alpha = 1 + \alpha. & \end{array}$$

Suppose now that $\alpha = \beta + 1$. If $\mathrm{S}(\emptyset, \rho_p) \geq \alpha$ then certainly $\mathrm{S}(\emptyset, \rho_p) \geq \beta$ so our inductive hypothesis gives that $\mathrm{M}(p) \geq 1 + \beta$. Let $p \in \langle \phi \rangle$ where $\langle \phi \rangle$ is a basic open set of $S(A)$. Since $I_p = \emptyset$ we must have that

$$\phi = \bigwedge_{i,l,h} xa_i = xb_i \wedge xu_l \neq xv_l \wedge xt_h \neq d_h$$

where i, l, h run over finite (possibly empty) indexing sets. Let

$$\nu = \langle \{(a_i, b_i)\} \rangle, K = \{(u_i, v_i)\}$$

so that $[\nu, K]$ is a subset of $\mathcal{RC}$ of finite type and $\rho_p \in [\nu, K]$. Since $\mathrm{S}(\emptyset, \rho_p) \geq \beta+1$ there exists $\theta \in [\nu, K]$ with $\theta \neq \rho_p$ and $S(\emptyset, \theta) \geq \beta$. Let $q = \mathrm{tp}(1\theta/\emptyset)$ and let $r \in S(A)$ be such that $I_r = I_q = \emptyset$ and $\rho_r = \rho_q = \theta$. Certainly $r \in \langle \phi \rangle$, $r \neq p$ and our inductive hypothesis gives that $\mathrm{M}(r) \geq 1+\beta$. Thus $\mathrm{M}(p) \geq (1+\beta)+1 = 1+\alpha$ as required.

Conversely, we suppose that $\mathrm{M}(p) \geq 1 + \alpha = 1 + (\beta + 1)$. Then $\mathrm{M}(p) \geq 1 + \beta$ so that $\mathrm{S}(\emptyset, \rho_p) \geq \beta$. Let $[\mu, H]$ be a subset of $\mathcal{RC}$ of finite type with $\rho_p \in [\mu, H]$. As $\mathrm{M}(p) \geq 1 + \beta + 1$ there is a G-set $B \supseteq A$ and and extension q of p in $S(B)$ with $\mathrm{M}(q) \geq 1+\beta$ and q a limit point in $S(B)$ of types with Morley rank greater than or equal to $1 + \beta$. We cannot have $I_q = G$, since otherwise $\mathrm{M}(q) = 0$; thus $I_q = \emptyset$ and as $p \subseteq q$, $\rho_p = \rho_q$. Putting

$$\mu = \langle \{(c_i, d_i)\} \rangle, H = \{(w_l, z_l)\}$$

we have that $q \in \langle \psi \rangle$ where

$$\psi = \bigwedge_{i,l} xc_i = xd_i \wedge xw_l \neq xz_l.$$

But then there exists $s \in \langle \psi \rangle$ with $s \neq q$ and $\mathrm{M}(s) \geq 1 + \beta$. It follows that $I_s = \emptyset$, $\rho_s \neq \rho_q$ and $\rho_s \in [\mu, H]$. The inductive hypothesis gives that $\mathrm{S}(\emptyset, \rho_s) \geq \beta$. Consequently, $\mathrm{S}(\emptyset, \rho_p) \geq \beta + 1 = \alpha$.

□

In view of Theorem 7.4 and the comments at the end of Section 6, for the free cyclic group $\mathbb{Z}$, any type over the empty set has Morley rank 1, with the exception of the type p with $\rho_p = \iota$, which has Morley rank 2.

Considering now the group $\mathbb{Z}(p^\infty)$ of all p^n-th roots of unity, we see that the Morley rank of any type over the empty set is 1 except for the type p with $\rho_p = \omega$, which has Morley rank 2.

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