NOTES ON RESTRICTION SEMIGROUPS AND RELATED STRUCTURES *FORMERLY* (WEAKLY) LEFT E-AMPLE SEMIGROUPS

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ABSTRACT. The aim of these notes is to provide a single reference source for basic definitions and results concerning classes of semigroups (and, indeed, of semigroupoids) related to those we refer to as *left restriction* or *weakly left E-ample*.

We give the 'York' perspective on these classes of semigroups. We present a comprehensive account of the relations $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}_E$ and show how many classes of interest to us, including that of left restriction semigroups, arise as natural generalisations of inverse semigroups. Little of this material is new, but some parts of it lie in the realms of folklore, hence one reason for this article. We give a list of original sources, but with no claim to comprehensiveness *more references will be added*.

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1. INTRODUCTION

Left restriction semigroups appear in the literature under a plethora of names. They are first seen in the work of Schweizer and Sklar [20, 21, 22, 23] on *function systems*. The latter are one instance of algebras that arise from attempts to find axiomatisations of algebras embedded in the semigroup of partial functions $\mathcal{PT}_X$ on a set X *enriched with additional operations*. Function systems were revisited by Schein in [18], correcting a misconception of [23]. A survey of this material, in the setting of relation algebras, was given by Schein in the first ever Semigroup Forum article [19] and revisited in the more recent article [15] of Jackson and Stokes. A more recent (and not readily available) survey appears in Chapter 2 of the PhD thesis of Chris Hollings [13], where left restriction semigroups are referred to as being *weakly left E-ample*. Left restriction semigroups (under another name)

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appear for the first time as a class in their own right in the work of Trokhimenko [24], where the earliest proof of their representation theorem (Theorem 5.2 below) is given. They appeared also as the *type SL2 γ-semigroups* of Batbedat [2, 3] in the early 1980s. More recently, they have arisen in the work of Jackson and Stokes [14] in the guise of *(left) twisted C-semigroups* and in that of Manes [17] as *guarded semigroups*, motivated by consideration of closure operators and categories, respectively. The work of Manes has a forerunner in the *restriction categories* of Cockett and Lack [4], who were influenced by considerations of theoretical computer science.

The (former) York terminology 'weakly left E-ample' was first used in [7], arrived at from the starting point of the left ample semigroups of Fountain [5, 6] via the route of replacing considerations of the relation $\mathcal{R}^*$ on a semigroup S by those of $\widetilde{\mathcal{R}}$ (hence the 'weakly') and then by those of $\widetilde{\mathcal{R}}_E$ (making reference to a specific set of idempotents E, which may not be the whole of $E(S)$).

The aim of these notes is to give a careful account of the (very) basics of left restriction semigroups, together with a number of related classes. For example, whereas left restriction semigroups have a semilattice of idempotents, some authors have considered analogous classes having a left regular band of idempotents (see, for example [1]), so we wish to extend our article to cover these wider classes. There are essentially two approaches, one involving the relations $\widetilde{\mathcal{R}}_E$ and $\mathcal{R}^*$, and the second by using identities and quasi-identities. Further, results proved for these semigroups are at times arrived at via the employment of corresponding semigroupoids (see, for example, [11]), hence the inclusion of semigroupoids in our discussions.

2. The relations $\mathcal{R}, \mathcal{R}^*$ and $\widetilde{\mathcal{R}}_E$

Although the concepts outlined in this section are given in greater generality than is normally required, we prefer to use the most general setting, that is, the setting of semigroupoids. By doing so we provide an easy reference point for future work.

Let $\mathcal{S}$ be a semigroupoid, which we think of as a set $\mathcal{S}$ together with a second set O and functions $\mathbf{d}, \mathbf{r} : \mathcal{S} \to O$ such that:

(*i*) ab is defined if and only if $\mathbf{r}(a) = \mathbf{d}(b)$, and in this case, $\mathbf{d}(ab) = \mathbf{d}(a)$ and $\mathbf{r}(ab) = \mathbf{r}(b)$;

(*ii*) for all $a, b, c \in \mathcal{S}$ such that $\mathbf{r}(a) = \mathbf{d}(b)$ and $\mathbf{r}(b) = \mathbf{d}(c)$, we have that $(ab)c = a(bc)$;

(*iii*) $O = \text{ im } \mathbf{d} \cup \text{ im } \mathbf{r}$.

For each $\alpha, \beta \in O$ we let

$$M_{\alpha,\beta} = \{a \in \mathcal{S} : \mathbf{d}(a) = \alpha, \mathbf{r}(a) = \beta\},$$

so that

$$M_\alpha = M_{\alpha,\alpha}$$

is (empty or) a semigroup, the *local semigroup at* α. For a set E of idempotents of $\mathcal{S}$ (where e is an idempotent if and only if ee is defined and $e^2 = e$), and for $\alpha \in O$, we put

$$E_\alpha = E \cap M_\alpha.$$

We denote by $E(\mathcal{S})$ the set of all idempotents of $\mathcal{S}$.

An equality $w = v$ of elements in a semigroupoid is interpreted as meaning that both α and β are defined, and are the same element.

Green's relations $\mathcal{R}$ and $\mathcal{L}$ and their associated preorders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ are, of course, well understood for semigroups. Suppose now that $\mathcal{S}$ is a semigroupoid. We define the relation $\leq_{\mathcal{R}}$ on $\mathcal{S}$ by the rule that

$$a \leq_{\mathcal{R}} b \text{ if and only if } a = b \text{ or } a = bu \text{ for some } u \in \mathcal{S}.$$

Notice that $\leq_{\mathcal{R}}$ is a quasi-order, and if $a \leq_{\mathcal{R}} b$, then $\mathbf{d}(a) = \mathbf{d}(b)$. The equivalence relation associated with $\leq_{\mathcal{R}}$ is denoted by $\mathcal{R}$. In a semigroupoid $\mathcal{S}$, as in a semigroup, we have that $a \leq_{\mathcal{R}} e$ where e is idempotent, if and only if $ea = a$. It follows that if $e, f \in \mathcal{S}$ are idempotent, then

$$\begin{aligned} e\,\mathcal{R}\,f \text{ in } \mathcal{S} \;&\Leftrightarrow\; ef = f \text{ and } fe = e \\ &\Leftrightarrow\; e, f \in M_{\mathbf{d}(e)} \text{ and } e\,\mathcal{R}\,f \text{ in } M_{\mathbf{d}(e)}. \end{aligned}$$

The relations $\leq_{\mathcal{L}}$ and $\mathcal{L}$ are defined in a dual manner to $\leq_{\mathcal{R}}$ and $\mathcal{R}$.

Next, we consider the relations $\leq_{\mathcal{R}^*}$ and $\mathcal{R}^*$ on $\mathcal{S}$. We say that for $a, b \in \mathcal{S}$, $a \leq_{\mathcal{R}^*} b$ if and only if

$$xb = yb \Rightarrow xa = ya \text{ and } xb = b \Rightarrow xa = a.$$

for all $x, y \in \mathcal{S}$. Clearly $\leq_{\mathcal{R}^*}$ is a quasi-order on $\mathcal{S}$.

Lemma 2.1. *In any semigroupoid $\mathcal{S}$, we have that*

$$a \leq_{\mathcal{R}} b \Rightarrow a \leq_{\mathcal{R}^*} b.$$

Proof. Suppose that $a, b \in \mathcal{S}$ and $a \leq_{\mathcal{R}} b$. If $a = b$ then clearly $a \leq_{\mathcal{R}^*} b$. On the other hand, if $a = bu$ and $xb = yb$, then as bu is defined we have that $xbu = ybu$ and so $xa = ya$. Also, if $xb = b$ then again $xa = a$. Thus $a \leq_{\mathcal{R}^*} b$.

□

It is worth remarking that if xb is defined for any $x, b \in \mathcal{S}$, then $xb = xb$ so that if $a \leq_{\mathcal{R}^*} b$, then $xa = ya$, and $\mathbf{d}(a) = \mathbf{d}(b)$. Further, if $a \in \mathcal{S}$ and $a \in E(\mathcal{S})$, then it is easy to see that $a \leq_{\mathcal{R}^*} e$ if and only if $ea = a$. Thus for $e, f \in \mathcal{S}$,

$$e \leq_{\mathcal{R}^*} f \text{ if and only if } e \leq_{\mathcal{R}} f.$$

The equivalence relation associated with $\leq_{\mathcal{R}^*}$ is denoted by $\mathcal{R}^*$. In view of Lemma 2.1 we have that

$$a \,\mathcal{R}\, b \text{ implies that } a \,\mathcal{R}^*\, b$$

for any $a, b \in \mathcal{S}$. Further,

$$e \,\mathcal{R}\, f \text{ if and only if } e \,\mathcal{R}^*\, f$$

for all $e, f \in E(\mathcal{S})$. The relations $\leq_{\mathcal{L}^*}$ and $\mathcal{L}^*$ are defined dually.

Let ρ be a relation on a semigroupoid $\mathcal{S}$; we say that ρ is *left compatible* if $a \,\rho\, b$ implies that ca is defined if and only if cb is defined, and in this case, $ca \,\rho\, cb$. *Right compatibility* is defined dually.

Lemma 2.2. *The relations $\leq_{\mathcal{R}}, \mathcal{R}$ and $\mathcal{R}^*$ are left compatible and the relations $\leq_{\mathcal{L}}, \mathcal{L}$ and $\mathcal{L}^*$ are right compatible. For $\leq_{\mathcal{R}^*}$ we have that if $a \leq_{\mathcal{R}^*} b$, then if cb is defined, ca is defined and $ca \leq_{\mathcal{R}^*} cb$, and dually for $\leq_{\mathcal{L}^*}$.*

The relations that will form our main focus are further extensions of the above Green's relations and their generalisations. We stress however, that these relations have been proved (by, for example, the number of contexts in which they arise), to be extremely natural.

Let E be a set of idempotents of a semigroupoid $\mathcal{S}$. Where E is a subsemigroupoid (necessarily a disjoint union of bands), we refer to E as a *subband* of $\mathcal{S}$; where $ef = fe$ (whenever one (equivlalently, both) sides are defined) for all $e, f \in E$ then we say that E is a *subsemilattice* of $\mathcal{S}$. Note that *we do not insist that* $E = E(\mathcal{S})$. However, when this is the case, we may drop mention of the specific set E in our notation and terminology.

For any $a \in \mathcal{S}$, the set of left identities for a in E is denoted by ${}_Ea$, that is,

$${}_Ea = \{e \in E : ea = a\}.$$

Of course, if $e \in {}_Ea$, then $\mathbf{d}(e) = \mathbf{r}(e) = \mathbf{d}(a)$. The relation $\leq_{\widetilde{\mathcal{R}}_E}$ is then defined on $\mathcal{S}$ by the rule that

$$a \leq_{\widetilde{\mathcal{R}}_E} b \text{ if and only if } {}_Eb \subseteq {}_Ea.$$

That is, $a \leq_{\widetilde{\mathcal{R}}_E} b$ if and only if for all $e \in E$,

$$eb = b \text{ implies that } ea = a.$$

Clearly, $\leq_{\widetilde{\mathcal{R}}_E}$ is a quasi-order; the equivalence relation associated with $\leq_{\widetilde{\mathcal{R}}_E}$ is denoted by $\widetilde{\mathcal{R}}_E$.

Lemma 2.3. *For any $a \in S$, if ${}_E a \neq \emptyset$, then it is a subset of $E_{\mathbf{d}(a)}$ that is closed under multiplication. Further, ${}_E a$ is a filter in E under the $\leq_{\mathcal{R}}$-order. Consequently, if $e, f \in E_{\mathbf{d}(a)}$ and $e \,\mathcal{R}\, f$, then $e \in_E a$ if and only if $f \in_E a$.*

Proof. The first statement is clear. Suppose now that $e \in_E a$ and $e \leq_{\mathcal{R}} f$ where $f \in E$. It follows that $fe = e$ and $f \in M_{\mathbf{d}(a)}$. Then

$$a = ea = (fe)a = f(ea) = fa$$

so that $f \in_E a$ as required.

□

Lemma 2.4. *For any $e \in E$,*

$${}_E e = \{f \in E : e \leq_{\mathcal{R}} f\},$$

i.e. the principal filter generated by e.

Proof. For any $f \in E$,

$$f \in {}_E e \Leftrightarrow fe = e \Leftrightarrow e \leq_{\mathcal{R}} f.$$

□

The following is immediate from Lemma 2.4.

Corollary 2.5. *For any $a \in \mathcal{S}$, $a \,\widetilde{\mathcal{R}}_E\, e$ for some $e \in E$ if and only if ${}_E a$ is the principal filter generated by e.*

Lemma 2.6. *For any $a, b \in \mathcal{S}$,*

$$a \leq_{\mathcal{R}^*} b \text{ implies that } a \leq_{\widetilde{\mathcal{R}}_E} b.$$

Further, if $e \in E$, then

$$a \leq_{\mathcal{R}} e \text{ if and only if } a \leq_{\mathcal{R}^*} e \text{ if and only if } a \leq_{\widetilde{\mathcal{R}}_E} e$$

and these conditions are equivalent to $ea = a$.

Proof. Suppose that $a \leq_{\mathcal{R}^*} b$ and $eb = b$ for some $e \in E$. By definition of $\leq_{\mathcal{R}^*}$ we have that $ea = a$. Hence ${}_E b \subseteq_E a$ so that $a \leq_{\widetilde{\mathcal{R}}_E} b$ as required.

We know that $a \leq_{\mathcal{R}} e$ if and only if $ea = a$ if and only if $a \leq_{\mathcal{R}^*} e$. Suppose now that $a \leq_{\widetilde{\mathcal{R}}_E} e$; then as $ee = e$ we certainly have that $ea = a$ so that the remainder of the lemma holds.

□

Corollary 2.7. *For any $e, f \in E$ we have that*

$$e \leq_{\mathcal{R}} f \Leftrightarrow e \leq_{\mathcal{R}^*} f \Leftrightarrow e \leq_{\widetilde{\mathcal{R}}_E} f \Leftrightarrow fe = e,$$

so that consequently,

$$e \,\mathcal{R}\, f \Leftrightarrow e \,\mathcal{R}^*\, f \Leftrightarrow e \,\widetilde{\mathcal{R}}_E\, f \Leftrightarrow fe = e \text{ and } ef = f.$$

We will primarily be concerned with semigroups and semigroupoids in which every $\mathcal{R}^*$-class contains an idempotent, or in which every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E. With this in mind we give the following useful characterisations.

Lemma 2.8. *For any $a \in \mathcal{S}$ and $e \in E(S)$, we have that $a \,\mathcal{R}^*\, e$ if and only if $ea = a$,*

$$xa = ya \text{ implies that } xe = ye$$

and

$$xa = a \text{ implies that } xe = a.$$

Proof. We know from Lemma 2.6 that $a \leq_{\mathcal{R}^*} e$ if and only if $ea = a$. The result is now clear. $\square$

Similarly, we have:

Lemma 2.9. *For any $a \in \mathcal{S}$ and $e \in E$, $a \,\widetilde{\mathcal{R}}_E\, e$ if and only if $ea = a$ and for all $f \in E$, if $fa = a$ then $fe = e$.*

We will later specialise to the case where E is a subsemilattice, or more generally a *left regular subband*. Here, a left regular subband is a subband in which $efe = ef$ whenever $\mathbf{d}(e) = \mathbf{d}(f)$, or, equivalently, E_α is empty or is a left regular band, for any $\alpha \in O$.

Corollary 2.10. *Let E be a left regular subband of $\mathcal{S}$. Then for any $a \in S$, a is $\widetilde{\mathcal{R}}_E$-related to at most one idempotent of E.*

Proof. If $a \,\widetilde{\mathcal{R}}_E\, e \,\widetilde{\mathcal{R}}_E\, f$, where $e, f \in E$, then as noted in Corollary 2.7, we have that $ef = f$ and $fe = e$. In particular, $\mathbf{d}(e) = \mathbf{d}(f)$, so that as $E_{\mathbf{d}(e)}$ is a left regular band, $e = f$. $\square$

3. Generalised left restriction semigroupoids

Let E be a subset of idempotents of a semigroupoid $\mathcal{S}$. We say that $\mathcal{S}$ is *left E-abundant* if every $\mathcal{R}^*$-class contains an idempotent of E and *weakly left E-abundant* if every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E. Although these concepts do have wider uses, we largely focus in these notes on the case where E is a subband and the idempotent in the $\mathcal{R}^*$-class or $\widetilde{\mathcal{R}}_E$-class of a is *unique*.

Many of the classes we consider are defined by identities or quasi-identities. By an *identity* for a semigroupoid (possibly with additional unary and nullary operations) we mean an expression $t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$, where $t(x_1, \ldots, x_n)$ and $s(x_1, \ldots, x_n)$ are terms in a finite set $x_1, \ldots, x_n$ of variables. By saying that such an identity *holds* or *is satisfied* in a semigroupoid $\mathcal{S}$ we mean that for any $a_1, \ldots, a_n \in \mathcal{S}$, $t(a_1, \ldots, a_n)$ is defined if and only if $s(a_1, \ldots, a_n)$ is defined, and where defined, they are equal.

By a *quasi-identity* we mean an expression

$$t_1 = s_1 \wedge \ldots \wedge t_m = s_m \rightarrow s = t$$

where t_i, s_i, t, s are terms in variables $x_1, \ldots, x_n$; this quasi-identity holds if for any $a_1, \ldots, a_n \in \mathcal{S}$, we have for each i that if $t_i(a_1, \ldots, a_n)$ and $s_i(a_1, \ldots, a_n)$ are defined and equal for all $i \in \{1, \ldots, m\}$, then $s(a_1, \ldots, a_n)$ and $t(a_1, \ldots, a_n)$ are defined, and are equal.

Lemma 3.1. *Let $\mathcal{S}$ be weakly left E-abundant, where E is a subband, such that every $\widetilde{\mathcal{R}}_E$-class contains a* unique *idempotent of E. Denote the unique idempotent of E in the $\widetilde{\mathcal{R}}_E$-class of a by a^+. Then E is a left regular subband, and the following identities hold:*

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+x^+ = x^+y^+,$$

$$x^+x^+ = x^+,\ x^+(xy)^+ = (xy)^+.$$

Proof. If $e, f \in E$ and $e\,\mathcal{R}\,f$, then $e\,\widetilde{\mathcal{R}}_E\,f$ by Corollary 2.7, so that $e = f$ by the uniqueness claim. Then E is a disjoint union of left regular bands, as required.

The only identity that needs comment is the final. Let $a, b \in \mathcal{S}$, Observe that $(ab)^+$ is defined if and only if ab is defined and as $\mathbf{d}(a) = \mathbf{d}(ab)$, we have that $a^+(ab)^+$ is defined if and only if ab is defined. Moreover, if ab is defined, then so is a^+ab and $a^+ab = ab$, so that $a^+(ab)^+ = (ab)^+$. Hence $x^+(xy)^+ = (xy)^+$ holds. □

Remark 3.2. *Let $\mathcal{S}$ be weakly left E-abundant, where E is a subset of idempotents of $\mathcal{S}$, such that every $\widetilde{\mathcal{R}}_E$-class contains a* unique *idempotent of E. Denote the unique idempotent of E in the $\widetilde{\mathcal{R}}_E$-class of a by a^+. The clearly for all $a, b \in S$,*

$$a\,\widetilde{\mathcal{R}}_E\,b \text{ if and only if } a^+ = b^+.$$

We also observe from Lemma 3.1 that if $\mathcal{S}$ satisfies the conditions of that Lemma, then for any $a, b \in \mathcal{S}$, such that ab is defined, we have that $(ab)^+ \leq_{\mathcal{R}} a^+$.

A semigroupoid $\mathcal{S}$ together with an additional unary operation $^+$ is *generalised left restriction* if it satisfies the following identities:

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+x^+ = x^+y^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

Proposition 3.3. *Let $\mathcal{S}$ be a generalised left restriction semigroupoid. Put*

$$E = \{a^+ : a \in \mathcal{S}\}.$$

Then E is a subband, $\mathcal{S}$ is weakly left E-abundant such that for any $a \in \mathcal{S}$, the $\widetilde{\mathcal{R}}_E$-class of a contains a unique *idempotent a^+.*

Conversely, if $\mathcal{S}$ is weakly left E-abundant where E is a subband, such that for any $a \in \mathcal{S}$, the $\widetilde{\mathcal{R}}_E$-class of a contains a unique *idempotent a^+ of E, then $\mathcal{S}$ is a generalised left restriction semigroupoid with respect to the additional unary operation $^+$.*

Proof. In view of Lemma 3.1, we need only show the direct part.

From $(x^+y^+)^+ = x^+y^+$ and $x^+x^+ = x^+$, we obtain that E is a subband, and from $x^+y^+x^+ = x^+y^+$, that it is left regular. For any $a \in \mathcal{S}$ we have that $a^+a = a$. On the other hand, if $b^+a = a$, then as a^+a is defined, we have that b^+a^+ is defined and

$$b^+a^+ = b^+(b^+a)^+ = (b^+)^+(b^+a)^+ = (b^+a)^+ = a^+,$$

so that $a \,\widetilde{\mathcal{R}}_E\, a^+$. Since E is left regular, Corollary 2.10 says that a^+ is the unique idempotent of E that is $\widetilde{\mathcal{R}}_E$-related to a. □

We wish to specialise the above to left E-abundant semigroupoids. First, an observation.

Lemma 3.4. *Let $\mathcal{S}$ be a semigroupoid with subset E of idempotents. Then the following conditions are equivalent:*

(i) $\mathcal{S}$ is left E-abundant;

(ii) $\mathcal{S}$ is weakly left E-abundant and $\leq_{\mathcal{R}^*} = \leq_{\widetilde{\mathcal{R}}_E}$;

(iii) $\mathcal{S}$ is weakly left E-abundant and $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$.

Proof. Clearly (ii) implies (iii) and if (iii) holds, then as every $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$-class contains an idempotent of E, we certainly have that every $\mathcal{R}^*$-class contains an idempotent of E, so that $\mathcal{S}$ is left E-abundant.

It remains to prove that (i) implies (ii). To this end, suppose that $\mathcal{S}$ is left E-abundant. Since $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$ from Lemma 2.6, certainly $\mathcal{S}$ is weakly left E-abundant. Let $a, b \in \mathcal{S}$ with $a \leq_{\widetilde{\mathcal{R}}_E} b$. Then $a \,\mathcal{R}^*\, e, b \,\mathcal{R}^*\, f$ for some $e, f \in E$. We have

$$e \,\widetilde{\mathcal{R}}_E\, a \leq_{\widetilde{\mathcal{R}}_E} b \,\widetilde{\mathcal{R}}_E\, f$$

so that $e \leq_{\widetilde{\mathcal{R}}_E} f$ and by Corollary 2.7 we have that $e \leq_{\mathcal{R}^*} f$ and so $a \leq_{\mathcal{R}^*} b$. It follows that $\leq_{\mathcal{R}^*} = \leq_{\widetilde{\mathcal{R}}_E}$.

□

Proposition 3.5. *Let $\mathcal{S}$ be a generalised left restriction semigroupoid such that in addition the quasi-identity*

$$xz = yz \to xz^+ = yz^+$$

holds. Put

$$E = \{a^+ : a \in \mathcal{S}\}.$$

Then E is a subband and $\mathcal{S}$ is left E-abundant such that for any $a \in \mathcal{S}$, the $\mathcal{R}^*$-class of a contains a unique *idempotent a^+ of E.*

Conversely, if $\mathcal{S}$ is left E-abundant where E is a subband, such that for any $a \in \mathcal{S}$, the $\mathcal{R}^*$-class of a contains a unique *idempotent of E, denoted a^+, then $\mathcal{S}$ is a generalised left restriction semigroupoid with respect to the additional unary operation $^+$ and satisfies the quasi-identity*

$$xa = ya \to xa^+ = ya^+.$$

Proof. Suppose first that $\mathcal{S}$ is a generalised left restriction semigroupoid such that the quasi-identity

$$xz = yz \to xz^+ = yz^+$$

holds. From Proposition 3.3 we have that E is a subband and $\mathcal{S}$ is weakly left E-abundant such that for any $a \in \mathcal{S}$, the $\widetilde{\mathcal{R}}_E$-class of a contains a *unique* idempotent a^+. If $xa = a$, then $xa = a^+a$ so that from the given quasi-identity, $xa^+ = a^+a^+ = a^+$. It follows that $a \,\mathcal{R}^*\, a^+$ so that $\mathcal{S}$ is left E-abundant. From Lemma 3.4 we have that $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$ so that the $\mathcal{R}^*$-class of a contains a unique idempotent a^+ of E.

Suppose conversely that $\mathcal{S}$ is left E-abundant where E is a subband, such that for any $a \in \mathcal{S}$, the $\mathcal{R}^*$-class of a contains a *unique* idempotent a^+ of E. Again from Lemma 3.4, we have that $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$ and $\mathcal{S}$ is weakly left E-abundant such that the $\widetilde{\mathcal{R}}_E$-class of a contains a unique idempotent a^+ of E. From Proposition 3.3, $\mathcal{S}$ is generalised left restriction and as $a \,\mathcal{R}^*\, a^+$ for any $a \in \mathcal{S}$, the given quasi-identity holds. □

It is worth specifically detailing the specialisation of the above to semigroups; we see our first varieties and quasi-varieties arising.

Let S be a set equipped with a binary and a unary operation: with some slackness of notation we say that S is a $(2,1)$-*algebra*; we denote the binary operation by juxtaposition and the unary by $^+$. Then S is a *generalised left restriction semigroup* if S is a semigroup under the binary operation, and a generalised left restriction semigroup(oid), that is, if S satisfies the identities:

$$(xy)z = x(yz),\ x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,$$
$$x^+y^+x^+ = x^+y^+,\ x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

Note that the class of generalised left restriction semigroups forms a *variety of algebras*.

Corollary 3.6. *Let S be a generalised left restriction semigroup. Put*

$$E = \{a^+ : a \in \mathcal{S}\}.$$

Then E is a subband, and S is weakly left E-abundant such that for any $a \in S$, the $\widetilde{\mathcal{R}}_E$-class of a contains a unique *idempotent a^+.*

Conversely, if S is a weakly left E-abundant semigroup where E is a subband, such that for any $a \in \mathcal{S}$, the $\widetilde{\mathcal{R}}_E$-class of a contains a unique *idempotent a^+ of E, then S is a generalised left restriction semigroup with respect to the additional unary operation $^+$.*

We adapt the above the the case where $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$.

Corollary 3.7. *Let S be a generalised left restriction semigroup such that in addition the quasi-identity*

$$xa = ya \to xa^+ = ya^+$$

holds. Put

$$E = \{a^+ : a \in \mathcal{S}\}.$$

Then E is a subband and S is left E-abundant such that for any $a \in S$, the $\mathcal{R}^*$-class of a contains a unique *idempotent a^+ of E.*

Conversely, if S is left E-abundant where E is a subband, such that for any $a \in \mathcal{S}$, the $\mathcal{R}^*$-class of a contains a unique *idempotent of E, denoted a^+, then S is a generalised left restriction semigroup with respect to the additional unary operation $^+$ and satisfies the quasi-identity*

$$xa = ya \to xa^+ = ya^+.$$

Note that the semigroups appearing in Corollary 3.7 form a *quasi-variety* of algebras of type $(2, 1)$.

For completeness, a little further terminology. Let E be a subsemilattice in a semigroupoid $\mathcal{S}$. We say that $\mathcal{S}$ is *left E-adequate* if S is left E-abundant, that is, every $\mathcal{R}^*$-class contains an element of E (which by Corollaries 2.7 and 2.10 is unique). If every $\widetilde{\mathcal{R}}_E$-class contains a (necessarily unique) idempotent of E then we say that $\mathcal{S}$ is *weakly left E-adequate* or *left E-semiadequate*. Propositions 3.3 and 3.5 may be adapted as follows.

Corollary 3.8. *Let $\mathcal{S}$ be a semigroupoid equipped with an additional unary operation $^+$. Suppose that $\mathcal{S}$ satisfies the identities*

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

Put

$$E = \{a^+ : a \in \mathcal{S}\}.$$

Then E is a subsemilattice and $\mathcal{S}$ is weakly left E-adequate, such that for any $a \in \mathcal{S}$, $a \widetilde{\mathcal{R}}_E\, a^+$.

Conversely, if $\mathcal{S}$ is weakly left E-adequate where E is a subsemilattice, such that for any $a \in \mathcal{S}$, $a \widetilde{\mathcal{R}}_E\, a^+ \in E$, then $\mathcal{S}$ satisfies the identities

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

In particular, $\mathcal{S}$ is generalised left restriction.

Proof. Suppose first that $\mathcal{S}$ is a semigroupoid satisfying the given identities. Let $a, b \in \mathcal{S}$. Then if a^+b^+ exists, we have that

$$a^+a^+ = a^+,\ b^+b^+ = b^+ \text{ and } a^+b^+ = b^+a^+.$$

Hence $\mathbf{d}(a^+) = \mathbf{r}(a^+) = \mathbf{d}(b^+) = \mathbf{r}(b^+)$. It follows that $a^+b^+a^+$ exists, and $a^+b^+a^+ = a^+a^+b^+ = a^+b^+$. Hence $\mathcal{S}$ is generalised left restriction. From Proposition 3.3 we have that E is a subband, $\mathcal{S}$ is weakly left E-abundant and $a \widetilde{\mathcal{R}}_E\, a^+$ for all $a \in \mathcal{S}$. Clearly then E is a subsemilattice and $\mathcal{S}$ is weakly left E-adequate.

Conversely, suppose that $\mathcal{S}$ is weakly left E-adequate where E is a subsemilattice, such that for any $a \in \mathcal{S}$, $a \widetilde{\mathcal{R}}_E\, a^+ \in E$. Note that from Corollary 2.10, a^+ is the unique idempotent in the $\widetilde{\mathcal{R}}_E$-class of a. Thus by Proposition 3.3, we have that $\mathcal{S}$ is generalised left restriction. In addition, as E is a semilattice, $\mathcal{S}$ satisfies $x^+y^+ = y^+x^+$. □

Corollary 3.9. *Let $\mathcal{S}$ be a semigroupoid equipped with an additional unary operation $^+$. Suppose that $\mathcal{S}$ satisfies the identities*

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+$$

and the quasi-identity

$$xz = yz \to xz^+ = yz^+.$$

Put

$$E = \{a^+ : a \in \mathcal{S}\}.$$

Then E is a subsemilattice and $\mathcal{S}$ is left E-adequate, such that for any $a \in \mathcal{S}$, $a\,\mathcal{R}^*\,a^+$.

Conversely, if $\mathcal{S}$ is left E-adequate where E is a subsemilattice, such that for any $a \in \mathcal{S}$, $a\,\mathcal{R}^*\,a^+ \in E$, then $\mathcal{S}$ satisfies

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+,$$
$$xa = ya \to xa^+ = ya^+.$$

In particular, $\mathcal{S}$ is generalised left restriction.

Again we specialise to the case for semigroups, obtaining now a variety and a quasi-variety of algebras of type $(2, 1)$.

Corollary 3.10. *Let S be an algebra of type $(2, 1)$. Suppose that S satisfies the identities*

$$(xy)z = x(yz),\ x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

Put

$$E = \{a^+ : a \in S\}.$$

Then E is a subsemilattice and S is weakly left E-adequate, such that for any $a \in \mathcal{S}$, $a\,\widetilde{\mathcal{R}}_E\,a^+$.

Conversely, if S is a semigroup under the binary operation, and weakly left E-adequate where E is a subsemilattice, such that for any $a \in S$, $a\,\widetilde{\mathcal{R}}_E\,a^+$, then S satisfies the identities

$$(xy)z = x(yz),\ x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,$$
$$x^+y^+ = y^+x^+,\ x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

In particular, S is generalised left restriction.

Corollary 3.11. *Let S be an algebra of type $(2, 1)$. Suppose that S satisfies the identities*

$$(xy)z = x(yz),\ x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+$$

and the quasi-identity

$$xz = yz \to xz^+ = yz^+.$$

Put

$$E = \{a^+ : a \in S\}.$$

Then E is a subsemilattice and S is left E-adequate, such that for any $a \in S$, $a\,\mathcal{R}^*\,a^+$.

Conversely, if S is a semigroup under the binary operation, and left E-adequate where E is a subsemilattice, such that for any $a \in S$, $a \,\mathcal{R}^*\, a^+ \in E$, then S satisfies the identities

$$(xy)z = x(yz),\ x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,$$

$$x^+y^+ = y^+x^+,\ x^+x^+ = x^+, x^+(xy)^+ = (xy)^+$$

and the quasi-identity

$$xz = yz \rightarrow xz^+ = yz^+.$$

In particular, S is generalised left restriction.

let $\mathcal{S}$ be a generalised left restriction semigroupoid, and let $E = \{a^+ : a \in \mathcal{S}\}$. We say that E is the *distinguished subset of idempotents* of $\mathcal{S}$. We remark that without special signatures, we cannot in general pick out a particular E. We can, however, specialise the results in this section to the case where $E = E(\mathcal{S})$ (or, $E = E(S)$), by adding the quasi-identity

$$x^2 = x \rightarrow x = x^+.$$

Thus, the class of generalised left restriction semigroups S such that $e^+ = e$ for all $e \in E(S)$ is a *quasi-variety.* Whereas varieties of algebras are closed under subalgebra, direct product and morphic image, quasi-varieties are in general only closed under subalgebra and direct product.

The right context for considering all of the classes of semigroups in this section is, we feel, as quasi-varieties (or varieties) of algebras of type $(2,1)$.

For example, let S be a generalised left restriction semigroup with distinguished subset of idempotents E. Let T be a subsemigroup of S that is closed under $^+$, that is, T is a subalgebra of S. Then T is also generalised left restriction, with distinguished subset of idempotents $F = \{a^+ : a \in T\} = E \cap T$. Now, for any $a, b \in T$ we have that

$$a \,\widetilde{\mathcal{R}}_F\, b \text{ in } T \Leftrightarrow a^+ = b^+ \Leftrightarrow a \,\widetilde{\mathcal{R}}_E\, b \text{ in } S.$$

When considering direct products, let $\{S_i : i \in I\}$ be a non-empty set such that each S_i is a generalised left restriction with distinguished subset of idempotents E_i. Consider the $(2,1)$-algebra P, where P is the product

$$P = \Pi_{i \in I} S_i.$$

For any $(a_i) \in P$ we have that $(a_i)^+ = (a_i^+)$. As quasi-varieties are closed under product, P is generalised left restriction, with distinguished set of idempotents

$$E = \{(a_i)^+ : a_i \in S_i\} = \{(a_i^+) : a_i \in S_i\}.$$

Moreover, for $(a_i), (b_i) \in P$,

$$\begin{array}{rcl} (a_i)\,\widetilde{\mathcal{R}}_E\,(b_i) & \Leftrightarrow & (a_i)^+ = (b_i)^+ \\ & \Leftrightarrow & (a_i^+) = (b_i^+) \\ & \Leftrightarrow & a_i^+ = b_i^+ \text{ for all } i \in I \\ & \Leftrightarrow & a_i\,\widetilde{\mathcal{R}}_{E_i}\,b_i \text{ for all } i \in I. \end{array}$$

Finally, suppose that S and T are generalised left restriction semigroups with distinguished subsets E and F of idempotents, respectively, and $\phi : S \to T$ is a $(2,1)$-morphism. Let $a, b \in S$ with $a\,\widetilde{\mathcal{R}}_E\,b$. Then $a^+ = b^+$ so that

$$(a\phi)^+ = a^+\phi = b^+\phi = (b\phi)^+,$$

so that $a\phi\,\widetilde{\mathcal{R}}_F\,b\phi$ in T.

4. The semilattice, congruence and ample conditions

Let $\mathcal{S}$ be a generalised left restriction (g.l.r.) semigroupoid with distinguished subset of idempotents E. We say that $\mathcal{S}$ satisfies the *semilattice condition (e)* if E is a subsemilattice. As in Corollaries 3.8 and 3.10 we know that the class of g.l.r. semigroupoids with (e) is defined (within the class of all semigroupoids with an additional unary operation) by the identities

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+.$$

We now introduce two further conditions that will be crucial. First, although we know that $\mathcal{R}^*$ is always a left congruence, it is not true that $\widetilde{\mathcal{R}}_E$ must be. We say that a g.l.r. semigroupoid satisfies the *left congruence condition (cl)* if $\widetilde{\mathcal{R}}_E$ is a left congruence.

Lemma 4.1. *Let $\mathcal{S}$ be a g.l.r. semigroupoid. Then $\mathcal{S}$ has the left congruence condition if and only if it satisfies the identity*

$$(xy)^+ = (xy^+)^+.$$

Proof. Suppose first that $\mathcal{S}$ has the left congruence condition. For any $a, b \in \mathcal{S}$, certainly ab is defined if and only if ab^+ is defined, so that $(ab)^+$ is defined if and only if $(ab^+)^+$ is defined. Now, if ab is defined, then as $\widetilde{\mathcal{R}}_E$ is a left congruence and $b\,\widetilde{\mathcal{R}}_E\,b^+$, we have that $ab\,\widetilde{\mathcal{R}}_E\,ab^+$ and so $(ab)^+ = (ab^+)^+$.

Conversely, if $\mathcal{S}$ satisfies the given identity and $c\,\widetilde{\mathcal{R}}_E\,d$, then $c^+ = d^+$. Since $\mathbf{d}(c) = \mathbf{d}(c^+) = \mathbf{d}(d^+) = \mathbf{d}(d)$, we have that uc is defined if and only if ud is defined. If uc is defined, then

$$(uc)^+ = (uc^+)^+ = (ud^+)^+ = (ud)^+,$$

so that $uc\,\widetilde{\mathcal{R}}_E\,ud$ and $\widetilde{\mathcal{R}}_E$ is a left congruence. $\Box$

A semigroupoid $\mathcal{S}$ and, in particular, a semigroup, is said to be *left E-Ehresmann* if it is g.l.r. with (e) and (cl). Equivalently, it possesses a subsemilattice of idempotents E such that every $\widetilde{\mathcal{R}}_E$-class contains a (unique) idempotent a^+ of E, and $\widetilde{\mathcal{R}}_E$ is a left congruence (see, for example, [16]). We remark that left E-Ehresmann semigroups form a *variety* of algebras of type $(2,1)$. If we consider *left Ehresmann* semigroups, that is, left $E(S)$-Ehresmann semigroups, then this class forms a *quasi-variety* of algebras, since we must include the quasi-identity $x^2 = x \to x = x^+$ in the set of defining conditions.

The third important property that a g.l.r. semigroupoid may possess is the *ample* or *type A* condition. Let $\mathcal{S}$ be a g.l.r. semigroupoid: then $\mathcal{S}$ satisfies the *left ample condition (al)* if it satisfies the identity

$$xy^+ = (xy^+)^+x.$$

Notice first that if S is an inverse semigroup, so that $a^+ = aa^{-1}$ for all $a \in S$, then

$$(ab^+)^+a = ab^+b^+a^{-1}a = aa^{-1}ab^+ = ab^+,$$

using the fact that idempotents commute.

A g.l.r. semigroupoid is said to be *weakly left quasi-ample* or *wlqa* if it has (al) and $E = E(\mathcal{S})$ (see [1]). If $\mathcal{S}$ is wlqa and has (e) and (cl), then it is *weakly left ample* or *wla* [8]. Note that if $\mathcal{S}$ such that $\mathcal{S}$ is weakly left $E(\mathcal{S})$-abundant, and the $\widetilde{\mathcal{R}}$-class of any $a \in \mathcal{S}$ contains a unique idempotent a^+ and (al) holds, then as in [1] we note that

$$(ef)(ef) = (ef)^+eef = (ef)^+(ef) = ef,$$

so that $E(\mathcal{S})$ is perforce a subband.

However, the most interesting case is of a g.l.r. semigroupoid with (e), (cl) and (al): such a semigroupoid is *left restriction*. Note that the class of left restriction semigroups *forms a variety of algebras of type* $(2,1)$. In this case, the defining set of identities simplifies considerably. We first make a simple observation.

Lemma 4.2. *Let $\mathcal{S}$ be a g.l.r. semigroupoid. Then $\mathcal{S}$ satisfies (AL) and (CL) if and only if it satisfies*

$$xy^+ = (xy)^+x.$$

Proof. Suppose first that $\mathcal{S}$ has (cl) and (al):

$$(xy^+)^+ = (xy)^+ \text{ and } xy^+ = (xy^+)^+x.$$

Then certainly

$$xy^+ = (xy^+)^+x = (xy)^+x.$$

Conversely, suppose that $\mathcal{S}$ satisfies $xy^+ = (xy)^+x$. Then

$$(xy^+)^+ = ((xy)^+x)^+ \leq_{\mathcal{R}} ((xy)^+)^+ = (xy)^+.$$

On the other hand,

$$(xy)^+ = (xy^+y)^+ \leq_{\mathcal{R}} (xy^+)^+.$$

It follows that $(xy^+)^+ \,\mathcal{R}\, (xy)^+$ so that $(xy^+)^+ = (xy)^+$. Clearly then

$$xy^+ = (xy)^+x = (xy^+)^+x.$$

$\square$

Proposition 4.3. *Let $\mathcal{S}$ be a semigroupoid equipped with an additional unary operation $^+$. Then $\mathcal{S}$ is left restriction if and only if $\mathcal{S}$ satisfies:*

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x.$$

Proof. We have that $\mathcal{S}$ is left restriction if and only if it satisfies

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+,$$

together with (cl) and (al)

$$(xy^+)^+ = (xy)^+ \text{ and } xy^+ = (xy^+)x.$$

Suppose that $\mathcal{S}$ is left restriction. For any $a, b \in \mathcal{S}$ we have that $(a^+b)^+$ is defined if and only if $(a^+b^+)^+$ is defined, and in this case

$$(a^+b)^+ = (a^+b^+)^+ = a^+b^+,$$

the first equality from (cl). Thus $(x^+y)^+ = x^+y^+$ holds. By Lemma 4.2, $xy^+ = (xy)^+x$ also holds.

Conversely, suppose that $\mathcal{S}$ satisfies

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x.$$

We show that $\mathcal{S}$ is g.l.r., so that the result follows from Lemma 4.2.

Notice first that for any $a \in \mathcal{S}$, as $a^+a = a$ we have that

$$\mathbf{d}(a^+) = \mathbf{r}(a^+) = \mathbf{d}(a).$$

Let $a \in \mathcal{S}$. Then a^+a^+ is defined and

$$a^+a^+ = (a^+a)^+ = a^+,$$

so that $x^+x^+ = x^+$ holds. Further,

$$a^+ = (a^+)^+a^+ = a^+(a^+)^+ = (a^+a^+)^+ = (a^+)^+,$$

so that $(x^+)^+ = x^+$ holds.

Clearly for $a, b \in \mathcal{S}$, ab is defined if and only if $(ab)^+$ is defined if and only if $a^+(ab)^+$ is defined, and if they are,

$$a^+(ab)^+ = (a^+ab)^+ = (ab)^+,$$

so that $x^+(xy)^+ = (xy)^+$ holds.

If a^+b^+ is defined, then

$$a^+b^+ = a^+(b^+)^+ = (a^+b^+)^+$$

so that $x^+y^+ = (x^+y^+)^+$ holds. Hence $\mathcal{S}$ is g.l.r. with (e). $\square$

The class of *left restriction* semigroups therefore forms a variety of type $(2, 1)$, which may be defined via only four identities (we have taken our four from [4]. Other names for left restriction semigroups are *left twisted C-semigroups* [14] and *weakly left E-ample semigroups* [12]. If a semigroupoid $\mathcal{S}$ is weakly left $E(\mathcal{S})$-ample then we say that $\mathcal{S}$ is *weakly left ample*. The class of weakly left ample semigroups forms a quasi-variety of algebras of type $(2, 1)$.

Recall that a semigroupoid $\mathcal{S}$ is left E-adequate if E is a subsemilattice and every $\mathcal{R}^*$-class contains a, necessarily unique, idempotent of E; we denote the idempotent in the $\mathcal{R}^*$-class of a by a^+. The former convention is that the unique idempotent of E in the $\mathcal{R}^*$-class of $a \in S$ is denoted by $a^\dagger$. A left E-adequate semigroupoid is said to be *left E-ample* if it satisfies condition (al); notice from Lemma 2.2 that $\mathcal{R}^*$ is always left compatible, so we need not mention (cl) explicitly. However, mention of E is also spurious, since in such a semigroupoid, if e is idempotent, then

$$ee^+ = (ee)^+e = e^+e = e.$$

Certainly

$$ee = e^+e$$

and since $e\,\mathcal{R}^*\,e^+$ we have

$$ee^+ = e^+e^+$$

and hence $e = e^+$. Thus for a left E-ample semigroupoid we simply say that it is *left ample*.

We remark here that inverse semigroups are certainly left ample; for any regular semigroup, $\mathcal{R} = \mathcal{R}^*$ and the idempotent in the $\mathcal{R}^*$-class of a is aa^{-1}; we have already noted that (al) holds. Thus $\mathcal{I}_X$, the symmetric inverse semigroup on a (non-empty) set X is left ample. In this case, for any α we have

$$\alpha^+ = \alpha\alpha^{-1} = I_{\text{dom }\alpha}.$$

We denote by E_X the set of idempotents of $\mathcal{I}_X$, so that

$$E_X = \{I_Y : Y \subseteq X\}.$$

Immediately from the fact that left ample semigroups form a quasivariety we have:

Corollary 4.4. *Let S be a $(2,1)$-subalgebra of $\mathcal{I}_X$. Then S is left ample.*

We will see later that all left ample semigroups may be obtained as $(2,1)$-subalgebras of $\mathcal{I}_X$.

Similarly, we show in Section 6 that the *partial transformation semigroup* $\mathcal{PT}_X$ is known to be left restriction, where

$$\alpha^+ = I_{\text{dom } \alpha}.$$

Corollary 4.5. *Let S be a $(2,1)$-subalgebra of $\mathcal{PT}_X$. Then S is left restriction.*

Again, we will see that all left restriction semigroups may be obtained as $(2,1)$-subalgebras of $\mathcal{PT}_X$.

5. The right and two-sided cases

We have thus far presented classes of semigroupoids and semigroups defined using relations $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}_E$, or, via an additional unary operation. We could equally well have presented classes defined dually, using $\mathcal{L}^*$ and $\widetilde{\mathcal{L}}_E$. Indeed, if one takes the point of view (the opposite of ours), that maps should be composed from right to left, Section 6 would reveal this would be the natural route to take. The classes are named by switching 'left' with 'right' and the unary operation employed is denoted by *. Thus, for example, a semigroupoid is *right restriction* if and only if it possesses a unary operation * such that the identities

$$xx^* = x,\ x^*y^* = y^*x^*,\ (xy^*)^* = x^*y^*,\ x^*y = y(xy)^*.$$

Equivalently, a semigroupoid $\mathcal{S}$ is right restriction if it possesses a subsemilattice E of idempotents such that every $a \in \mathcal{S}$ is $\widetilde{\mathcal{L}}_E$-related to an idempotent a^+ of E, and (cr) and (ar) hold, where (cr) and (ar) are the duals of (cl) and (al).

Let $\mathcal{S}$ be a semigroupoid equipped with two unary operations $^+$ and *. Let 'left good' be a class defined previously, using $^+$. We say that $\mathcal{S}$ is 'good' if it is both left good and right good *and* satisfies

$$(x^+)^* = x^+ \text{ and } (x^*)^+ = x^*.$$

Equivalently $\mathcal{S}$ lies in classes defined dually using $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ *for the same set E.*

For example, a semigroupoid $\mathcal{S}$ equipped with two unary operations $^+$ and * is *generalised restriction* if it satisfies

$$x^+x = x,\ (x^+)^+ = x^+,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+x^+ = x^+y^+,$$
$$x^+x^+ = x^+, x^+(xy)^+ = (xy)^+$$

and their left-right duals

$$xx^* = x,\ (x^*)^* = x^*,\ (x^*y^*)^* = x^*y^*,\ x^*y^*x^* = y^*x^*,$$
$$x^*x^* = x^*, (xy)^*y^* = (xy)^*$$

and

$$(x^+)^* = x^+ \text{ and } (x^*)^+ = x^*.$$

Of course, this set of identities may be simplified; we note that

$$E = \{x^+ : x \in \mathcal{S}\} = \{x^* : x \in \mathcal{S}\}$$

is forced to be a subsemilattice. Equivalently, $\mathcal{S}$ is generalised restriction if it possesses a subsemilattice E such that every $\widetilde{\mathcal{R}}_E$-class and every $\widetilde{\mathcal{L}}_E$-class contains a (unique) element of E.

6. The Representation theorems

We now begin to indicate why left restriction semigroups are such a natural concept.

Proposition 6.1. *For any set X, $\mathcal{PT}_X$ is weakly left E_X-ample, where*

$$E_X = \{I_Y : Y \subseteq X\}.$$

Equivalently, $\mathcal{PT}_X$ is left restriction, where

$$\alpha^+ = I_{dom\ \alpha}.$$

Proof. Certainly E_X is a subsemilattice of $\mathcal{PT}_X$. Let $\alpha \in \mathcal{PT}_X$. Then

$$I_{\text{dom }\alpha}\alpha = \alpha.$$

Suppose that $I_Y\alpha = \alpha$. Then for any $x \in \text{dom } \alpha$, $x \in \text{dom } I_Y = Y$, so $\text{dom } \alpha \subseteq Y$ and $I_{\text{dom }\alpha} \leq I_Y$. By Lemma 2.9, $\alpha\, \widetilde{\mathcal{R}}_{E_X}\, I_{\text{dom }\alpha}$.

Notice now that for any $\alpha, \beta \in \mathcal{PT}_X$,

$$\alpha\, \widetilde{\mathcal{R}}_{E_X} \beta \Leftrightarrow \text{dom } \alpha = \text{dom } \beta.$$

Now if $\alpha\, \widetilde{\mathcal{R}}_{E_X}\, \beta$ and $\gamma \in \mathcal{PT}_X$, then

$$\text{dom } \gamma\alpha = (\text{im } \gamma \cap \text{dom } \alpha)\gamma^{-1} = (\text{im } \gamma \cap \text{dom } \beta)\gamma^{-1} = \text{dom } \gamma\beta,$$

so that $\gamma\alpha\, \widetilde{\mathcal{R}}_{E_X}\, \gamma\beta$ and $\widetilde{\mathcal{R}}_{E_X}$ is a left congruence and (cl) holds.

Finally, let $\alpha \in \mathcal{PT}_X$ and $Y \subseteq X$. Then

$$\begin{aligned} \text{dom } (\alpha I_Y)^+\alpha &= (\text{im } (\alpha I_Y)^+ \cap \text{ dom } \alpha)((\alpha I_Y)^+)^{-1} \\ &= \text{dom } \alpha I_Y \cap \text{ dom } \alpha \\ &= \text{dom } \alpha I_Y \end{aligned}$$

and for $x \in \text{dom } \alpha I_Y$,

$$x\alpha I_Y = x\alpha = x(\alpha I_Y)^+\alpha.$$

Therefore $\alpha I_Y = (\alpha I_Y)^+\alpha$ and (cl) holds. □

Thus, any $(2,1)$-subalgebra of $\mathcal{PT}_X$ is left restriction. We now prove the converse.

Theorem 6.2. *Let S be a left restriction semigroup. Define*

$$\phi : S \to \mathcal{PT}_S$$

by

$$s\phi = \rho_s$$

where

$$\textit{dom } \rho_s = Ss^+, \quad x\rho_s = xs \textit{ for all } x \in \textit{ dom } \rho_s.$$

Then (i) im $\rho_s = Ss$;
(ii) ϕ is a $(2,1)$-embedding;
(iii) $s^+\phi = I_{Ss^+}$;
(iv) S is left ample if and only if im $\phi \subseteq \mathcal{I}_S$.

Proof. (i) im $\rho_s = (Ss^+)\rho_s = Ss^+s = Ss$.
(iii) $s^+\phi = \rho_{s^+}$ and dom $\rho_{s^+} = S(s^+)^+ = Ss^+$ and for $xs^+ \in$ dom ρ_{s^+}, we have

$$(xs^+)\rho_{s^+} = xs^+s^+ = xs^+,$$

so that $\rho_{s^+} = I_{Ss^+}$ and (iii) holds.
(ii) We have

$$s^+\phi = I_{Ss^+} = I_{\text{dom } \rho_s} = (\rho_s)^+ = (s\phi)^+$$

so that ϕ is a $^+$-morphism.
If $s\phi = t\phi$, then $Ss^+ = St^+$ so that $s^+ \,\mathcal{L}\, t^+$ and as E commutes, $s^+ = t^+$. Further, $\rho_s = \rho_t$ gives

$$s^+\rho_s = t^+\rho_t$$

so that $s^+s = t^+t$ and $s = t$. Thus ϕ is one-one.
We must show that ϕ is a semigroup morphism. Let $s, t \in S$. If we can show that dom ρ_{st} =dom $\rho_s$dom ρ_t, then it is clear that $(st)\phi = s\phi t\phi$.
Let $x \in$ dom $\rho_s\rho_t$. Then $x \in$ dom ρ_s and $x\rho_s = xs \in$ dom ρ_t. Thus $x \in Ss^+$ and $xs \in St^+$. Using the ample condition we have

$$x(st)^+ = (x(st)^+)^+x = (xst)^+x = (xst^+)^+x =$$
$$= (xs)^+x = (xs^+)^+x = x^+x = x$$

so that $x \in S(st)^+ =$ dom ρ_{st}.
Conversely, let $x \in$ dom ρ_{st}. Then $x(st)^+ = x$ so that $xs^+ = x(st)^+s^+ = x(st)^+ = x$ and $x \in$ dom ρ_s. Now $x\rho_s = xs = x(st)^+s = x(st^+)^+s = xst^+ \in$ dom ρ_t so that $x \in$ dom $\rho_s\rho_t$. This gives that dom $\rho_s\rho_t =$ dom ρ_{st} and ϕ is a semigroup morphism.

(iv) Suppose that im $\phi \subseteq \mathcal{I}_S$. Then im ϕ is a $(2,1)$-subalgebra of $\mathcal{I}_S$, so that S is left ample as in Corollary 4.4.

Conversely, suppose that S is left ample. Consider $s \in S$ and $s\phi = \rho_s$. Let $x, y \in \operatorname{dom} \rho_s$ and suppose that $x\rho_s = y\rho_s$. Then $xs = ys$ but by Lemma 3.4, $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$, so that $xs^+ = ys^+$ and $x = y$. Thus ρ_s is one-one and im $\phi \subseteq \mathcal{I}_S$. □

Putting together Corollaries 4.4, 4.5 and Theorem 6.2 we have the following:

Corollary 6.3. *Left restriction semigroups are precisely the* $(2,1)$-*subalgebras of some* $\mathcal{PT}_X$. *Left ample semigroups are precisely the* $(2,1)$-*subalgebras of some* $\mathcal{I}_X$.

7. The natural partial order on a left restriction semigroup

Let S be a left restriction semigroup with distinguished semilattice E. Then the relation $\leq$ on S defined by the rule that for any $a, b \in S$,

$$a \leq b \text{ if and only if } a = eb$$

for some $e \in E$, is a partial order on S, compatible with multiplication, which restricts to the usual partial order on E.

We begin by noting that if $a \leq b$ with $a = eb$ for some $e \in E$, then $ea = a$ so that $ea^+ = a^+$. Hence

$$a^+b = a^+eb = a^+a = a.$$

Thus

$$a \leq b \text{ if and only if } a = a^+b.$$

It is clear that $\leq$ is reflexive, transitive, right compatible and restricts to the usual order on E. To see that it is anti-symmetric, note that if $a \leq b$ and $b \leq a$, then we have from the above that $a = a^+b$ and $b = b^+a$. From the first equation we have that $b^+a = b^+a^+b = a^+b^+b = a^+b = a$, so that from the second we deduce $a = b$. To see that $\leq$ is left compatible we employ condition (al).

If S is a $(2,1)$-subalgebra of $\mathcal{PT}_X$, then it is easy to see that for $\alpha, \beta \in S$,

$$\alpha \leq \beta \text{ if and only if } \alpha \subseteq \beta;$$

thus demonstrating the importance of $\leq$.

8. The least congruence identifying E

Let S be a semigroup with subset E of idempotents. We denote by σ_E the least semigroup congruence ρ such that $e\,\rho\,f$ for all $e, f \in E$. Thus $a\,\sigma_E\,b$ if and only if $a = b$ or there exist a sequence

$$a = c_1e_1d_1, c_1f_1d_1 = c_2e_2d_2, \dots, c_nf_nd_n = b,$$

where $c_1, d_1, \dots, c_n, d_n \in S^1$ and $(e_1, f_1), \dots, (e_n, f_n) \in E \times E$.

Before stating the next lemma we remark that if E is a left regular band in a semigroup S, and if $ea = fb$ for some $e, f \in E$ and $a, b \in S$, then

$$efa = efea = effb = efb$$

so that $ga = gb$ where $g = ef$.

Lemma 8.1. *Let S be g.l.r. with (al). Then $a\,\sigma_E\,b$ if and only if $ea = eb$ for some $e \in E$.*

Proof. Clearly if $ea = eb$ where $e \in E$ then

$$a = a^+a\,\sigma_E\,ea = eb\,\sigma_E\,b^+b = b.$$

Conversely, suppose that $a\,\sigma_E\,b$; if $a = b$ then $a^+a = a^+b$. On the other hand, suppose there exists a sequence

$$a = c_1e_1d_1, c_1f_1d_1 = c_2e_2d_2, \dots, c_nf_nd_n = b,$$

where $c_1, d_1, \dots, c_n, d_n \in S^1$ and $(e_1, f_1), \dots, (e_n, f_n) \in E \times E$. Then

$$a = (c_1e_1)^+c_1d_1, (c_1f_1)^+c_1d_1 = (c_2e_2)^+c_2d_2, \dots, (c_nf_n)^+c_nd_n = b,$$

so that

$$a = u_1c_1d_1, v_1c_1d_1 = u_2c_2d_2, \dots, v_nc_nd_n = b,$$

for some $u_1, v_1, \dots, u_n, v_n \in E$. By the comment preceding the Lemma,

$$a = g_0c_1d_1, g_1c_1d_1 = g_1c_2d_2, \dots, g_nc_nd_n = b,$$

for some $g_0, \dots, g_n \in E$. Now

$$g_0g_1a = g_0g_1g_0c_1d_1 = g_0g_1c_1d_1 = g_0g_1c_2d_2,$$

so that $ka = kc_2d_2$ where $k = g_0g_1 \in E$. Suppose that $\ell a = \ell c_id_i$ where $i < n$. Then

$$g_i\ell a = g_i\ell c_id_i = g_i\ell g_ic_id_i = g_i\ell g_ic_{i+1}d_{i+1} = g_i\ell c_{i+1}d_{i+1}.$$

By induction we obtain that

$$ha = hc_nd_n$$

for some $h \in E$, so that

$$g_nha = g_nhc_nd_n = g_nhg_nc_nd_n = g_nhb,$$

as required. $\square$

We remark that a g.l.r. semigroup S is properly a $(2, 1)$-algebra; we have defined σ_E as a *semigroup* congruence. However, it is certainly true that if $a\,\sigma_E\,b$, then $a^+\,\sigma_E\,b^+$ (since all idempotents of E are identified by σ_E!), so that σ_E is a $(2, 1)$-congruence. Hence S/σ_E is g.l.r.. Note that the distinguished semilattice of S/σ_E is trivial. We refer to a g.l.r. semigroupoid $\mathcal{S}$ with E_α trivial for all $\alpha \in O$ as *reduced.* Hence S/σ_E is always reduced. Now, if τ is any $(2, 1)$-congruence such that S/τ is reduced, then as

$$a^+\tau = (a\tau)^+ = (b\tau)^+ = b^+\tau$$

for any $a, b \in S$, we have that $e\,\tau\,f$ for all $e, f \in E$ and so $\sigma_E \subseteq \tau$.

We follow standard terminology by referring to a congruence ρ on a semigroup S as a Π congruence, where Π is a property defined for semigroups, if S/ρ has property Π.

Lemma 8.2. [1] *Let S be a g.l.r. semigroup with (al). The relation σ_E is a monoid congruence on S. If $E = E(S)$, then putting $\sigma = \sigma_E$ we have that S/σ is unipotent and consequently σ is the least unipotent monoid congruence on S.*

Proof. Certainly all idempotents of E lie in the same σ_E-class and the class containing E is a left identity for S/σ_E; using (al) it is clear that this class is also a right identity, so that S/σ_E is a monoid.

Suppose now that $E = E(S)$ and $a\,\sigma$ is idempotent in S/σ. Hence $a\ \sigma\ a^2$ so that $ea = ea^2$ for some $e \in E(S)$. Consider the element eae. Using (al) we have

$$(eae)^2 = e(ae)ae = e((ae)^+a)ae$$

and so

$$(eae)^2 = e(ae)^+ea^2e = e(ae)^+eae = e(ae)^+ae = eae.$$

Thus $eae \in E(S)$ and

$$(eae)(eae) = e(ae)^+a$$

so by the comment preceding Lemma 8.1, $a\,\sigma\,eae$ and S/σ is unipotent, as required. □

It is well known that if S is an inverse monoid and $E = E(S)$, then σ is in fact the least *group congruence* on S.

Lemma 8.3. [5] *(i) Let S be a g.l.r. semigroup with (al) for which $E = E(S)$ and such that $\mathcal{R}^* = \widetilde{\mathcal{R}}$. Then S/σ is a right cancellative monoid and σ is the least right cancellative monoid congruence on S.*

(ii) Let S be a left ample semigroup. Then S/σ is a right cancellative monoid and σ is the least right cancellative congruence on S.

Proof. (i) In view of Lemma 8.2, we know that σ is a congruence and S/σ is unipotent. If $ab\,\sigma\,cb$ for some $a, b, c \in S$, then $eab = ecb$ for some $e \in E(S)$, so that as $b\,\mathcal{R}^*\,b^+$ we have $eab^+ = ecb^+$. Since S/σ is a unipotent monoid it follows that $a\,\sigma\,c$ and S/σ is right cancellative as required. If τ is any right cancellative monoid congruence on S, then as a right cancellative monoid is unipotent, we must have that $e\,\tau\,f$ for all $e, f \in E(S)$ so that $\sigma \subseteq \tau$.

(ii) Suppose that τ is a right cancellative congruence on S. Then if $e, f \in E(S)$ we have that

$$(e\tau)(f\tau) = ((ef)\tau)(f\tau)$$

so that by right cancellation,

$$e\tau = (ef)\tau.$$

Dually, $f\tau = (fe)\tau$ so that

$$e\,\tau\,ef = fe\,\tau\,f.$$

It follows that $\sigma_E \subseteq \tau$. □

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