Karl Auinger
Gracinda M.S. Gomes
Victoria Gould
Benjamin Steinberg

# An application of a theorem of Ash to finite covers

**Abstract.** The technique of *covers* is now well established in semigroup theory. The idea is, given a semigroup $S$, to find a semigroup $\widehat{S}$ having a better understood structure than that of $S$, and an onto morphism $\theta$ of a specific kind from $\widehat{S}$ to $S$. With the right conditions on $\theta$, the behaviour of $S$ is closely linked to that of $\widehat{S}$. If $S$ is finite one aims to choose a finite $\widehat{S}$. The celebrated results for inverse semigroups of McAlister in the 1970's form the flagship of this theory.

Weakly left quasi-ample semigroups form a *quasivariety* (of algebras of type $(2,1)$), properly containing the classes of groups, and of inverse, left ample, and weakly left ample semigroups. We show how the existence of finite proper covers for semigroups in this quasivariety is a consequence of Ash's powerful theorem for pointlike sets. Our approach is to obtain a cover $\widehat{S}$ of a weakly left quasi-ample semigroup $S$ as a subalgebra of $S\times G$, where $G$ is a group. It follows immediately from the fact that weakly left quasi-ample semigroups form a quasivariety, that $\widehat{S}$ is weakly left quasi-ample. We can then specialise our covering results to the quasivarieties of weakly left ample, and left ample semigroups. The latter have natural representations as $(2,1)$-subalgebras of partial (one-one) transformations, where the unary operation takes a transformation $\alpha$ to the identity map in the domain of $\alpha$. In the final part of this paper we consider representations of weakly left quasi-ample semigroups.

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## 1. Introduction

The relation $\widetilde{\mathcal{R}}$ is defined on a semigroup $S$ by the rule that $a\,\widetilde{\mathcal{R}}\,b$ if and only if

$$ea = a \Leftrightarrow eb = b$$

for all $e \in E(S)$. It is easy to see that the restriction of $\widetilde{\mathcal{R}}$ to regular elements coincides with Green's relation $\mathcal{R}$. In general $\mathcal{R}$ is strictly contained in $\widetilde{\mathcal{R}}$, as

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may be seen by considering unipotent monoids, that is, monoids in which the only idempotent is the identity. For details of Green's relations, and other standard semigroup theoretic techniques, we recommend [8] to the reader.

We say that a semigroup is *weakly left quasi-ample*, abbreviated wlqa, if every $\widetilde{\mathcal{R}}$-class contains a *unique* idempotent and, denoting the idempotent in the $\widetilde{\mathcal{R}}$-class of $a \in S$ by $a^+$, the *ample identity*

$$ae = (ae)^+a$$

for all $a \in S, e \in E(S)$ is satisfied. If the idempotents of a wlqa semigroup $S$ commute and $\widetilde{\mathcal{R}}$ is a left congruence, then $S$ is *weakly left ample*, abbreviated wla. Such semigroups are $(2,1)$-subalgebras of semigroups of partial transformations, where the binary operation is composition of mappings (from left to right) and the unary operation sends $\alpha$ to $\alpha^+$ where $\alpha^+$ is the identity map in the domain of $\alpha$ [7, 10, 11]. Wlqa semigroups are 'almost' inverse images of wla semigroups such that the inverse image of each idempotent is a left zero semigroup. The use of 'almost' is clarified in the final section; certainly the statement is true if $\widetilde{\mathcal{R}}$ is a left congruence.

A *cover* of a wlqa semigroup $S$ is a wlqa semigroup $\widehat{S}$, together with an onto idempotent separating $(2,1)$-morphism $\theta$ from $\widehat{S}$ to $S$. In the case where $S$ is a monoid, we require that $\widehat{S}$ is also a monoid and $\theta$ preserves the identity, that is, it is a $(2,1,0)$-morphism.

Let $S$ be a wlqa semigroup. We show that the idempotents of $S$ form a left regular band and that the relation $\sigma$ on $S$, defined by the rule $a\,\sigma\,b$ if and only if $ea = eb$ for some $e \in E(S)$, is the least unipotent monoid congruence on $S$. The semigroup $S$ is *proper* if $\widetilde{\mathcal{R}} \cap \sigma$ is trivial. The structure of proper wla semigroups may be determined by unipotent monoids and semilattices [4]. The authors anticipate that an analogous result for proper wlqa semigroups would involve unipotent monoids and left regular bands. In earlier papers [4, 6] it is shown that every wla semigroup $S$ has a proper cover, which may be chosen to be finite if $S$ is finite. Here we show that the latter result may be deduced for the wider class of wlqa semigroups from Ash's celebrated theorem for pointlike sets [1].

There is a "standard" method of constructing covers for semigroups and monoids, using the notion of relational morphism. If $S$ and $G$ are semigroups, then a relation $\phi$ from $S$ to $G$ is a *relational morphism* if for all $s, t \in S$, $s\phi \neq \emptyset$ and $s\phi\, t\phi \subseteq (st)\phi$. In the monoid case, we insist that $1 \in 1\phi$. This condition is superfluous if $G$ is a finite group; moreover in this case $1 \in e\phi$ for each $e \in E(S)$.

It is well known (see [15]) that a relation $\phi : S \to G$ is a relational morphism if and only if the *graph* of $\phi$, that is,

$$R = \{(s,g) \mid g \in s\phi\}$$

is a subsemigroup of $S \times G$ with the projection $\theta : R \to S$ being surjective. If $G$ is a finite group, then from a comment above

$$E(R) = \{(e,1) : e \in E(S)\}$$

and certainly, $\theta$ is idempotent separating. It follows from the fact that the class of wlqa semigroups forms a quasivariety that $R$ is wlqa; since $\theta$ preserves $^+$ we have that $\widehat{S} = R$ is a cover of $S$. In order that this cover be proper we need a relational morphism $\phi$ which separates any two distinct elements $s$ and $t$ for which $s \,\widetilde{\mathcal{R}}\, t$ holds. In other words, we have to show that no set $\{s,t\}$ of two distinct elements of $S$ with $s^+ = t^+$ is pointlike. This will be established in Section 3 by the use of Ash's theorem [1].

In Section 2 we sketch the necessary background on wlqa semigroups and monoids and related structures. Details may be found on the homepage of the third author at `http:www-users.york.ac.uk/~varg1/` under '(Weakly) left $E$-ample semigroups'. The final section considers representations of wlqa semigroups by partial maps.

## 2. Weakly left quasi-ample semigroups and monoids

The class of wlqa semigroups is a *quasivariety* of algebras of type $(2,1)$. The binary operation is semigroup multiplication; the unary operation is $a \mapsto a^+$.

The proofs of the following two results are straightforward. Note that we do not claim the sets of quasi-identities are minimal, but aim for transparency.

LEMMA 2.1. *Let $S$ be an algebra of type $(2,1)$. Then $S$ is a wlqa semigroup with $a^+$ denoting the (unique) idempotent in the $\widetilde{\mathcal{R}}$-class of $a$ if and only if $S$ satisfies the quasi-identities*

$$(xy)z = x(yz),$$

$$(a^+)^2 = a^+,\ a^+a = a,\ (e^2 = e \wedge ea = a) \Rightarrow ea^+ = a^+,$$

$$(e^2 = e \wedge ea^+ = a^+ \wedge a^+e = e) \Rightarrow e = a^+,$$

$$ab^+ = (ab^+)^+a.$$

If the idempotents of a semigroup commute, then it is immediate that every element is $\widetilde{\mathcal{R}}$-related to at most one idempotent.

COROLLARY 2.2. *Let $S$ be an algebra of type $(2,1)$. Then $S$ is a wla semigroup with $a^+$ denoting the (unique) idempotent in the $\widetilde{\mathcal{R}}$-class of $a$ if and only if $S$ satisfies the quasi-identities*

$$(xy)z = x(yz),$$
$$(a^+)^2 = a^+,\ a^+a = a,\ (e^2 = e \wedge ea = a) \Rightarrow ea^+ = a^+,$$
$$(e^2 = e \wedge f^2 = f) \Rightarrow ef = fe,$$
$$ab^+ = (ab^+)^+a,$$
$$a^+ = b^+ \Rightarrow (ca)^+ = (cb)^+.$$

The corresponding classes of monoids are, of course, quasivarieties of algebras of type $(2,1,0)$ obtained by adding the quasi-identity $1\,x = x = x\,1$ to the above lists.

An inverse semigroup is wla, where $a^+ = aa^{-1}$. Recall that a band $B$ is *left regular* if it satisfies the identity $ef = efe$. Equivalently, $B$ is a semilattice of left zero semigroups [14]. A band is left regular if and only if $\mathcal{R}$ is trivial [14]; clearly then, a left regular band is wlqa. Moreover, if $S$ is a left regular band $B$ of unipotent monoids such that $E(S)$ is a subsemigroup (necessarily isomorphic to $B$), then $S$ is wlqa, but not wla unless $B$ is a semilattice. On the other hand, if $S$ is regular and $E(S)$ is a left regular band, that is, $S$ is $\mathcal{R}$-unipotent (see for example [16]), then $S$ is wlqa. For in this case, as commented in the Introduction, $\widetilde{\mathcal{R}}$ coincides with $\mathcal{R}$, so that it remains only to check that the ample identity holds. Let $a, b \in S$ and let $a^{-1}$ be an inverse of $a$. Then, using the fact that $E(S)$ is a band, $(ab^+a^{-1})ab^+ = a(a^{-1}a)b^+(a^{-1}a)b^+ = a(a^{-1}a)b^+ = ab^+$. Since $ab^+a^{-1}$ is idempotent it follows that $ab^+a^{-1} = (ab^+)^+$. Hence

$$(ab^+)^+a = (ab^+a^{-1})a = a(a^{-1}a)b^+(a^{-1}a) = a(a^{-1}a)b^+ = ab^+,$$

using the fact that $E(S)$ is left regular. Numerous further examples of wlqa semigroups may be found in the references.

LEMMA 2.3. *Let $S$ be a wlqa semigroup. Then $E(S)$ is a left regular band.*

PROOF. Let $e, f \in E(S)$. Using the ample condition,

$$(ef)^2 = (ef)(ef) = (ef)^+e(ef) = (ef)^+(ef) = ef,$$

so that $E(S)$ is a band. Further,

$$efe = (ef)^+e = ef,$$

since $ef \in E(S)$. ∎

If $S$ is a wla semigroup the relation $\sigma$ defined on $S$ by the rule that for any $a, b \in S$,

$$a \,\sigma\, b \Leftrightarrow ea = eb$$

for some $e \in E(S)$, is the least unipotent monoid congruence on $S$ [5]. We extend this result to wlqa semigroups.

PROPOSITION 2.4. *Let $S$ be a wlqa semigroup. Then the relation $\sigma$ defined above is the least unipotent monoid congruence on $S$.*

PROOF. Clearly, if $\sigma$ is a unipotent monoid congruence, it must be the least such.

We begin by showing that if $a, b \in S$ and $ea = fb$ for some $e, f \in E(S)$, then $a \,\sigma\, b$. To see this, recall that $E(S)$ is left regular so that

$$(ef)a = (efe)a = (ef)(ea) = (ef)(fb) = (ef)b.$$

Clearly $\sigma$ is reflexive, symmetric and right compatible. If $a, b, c \in S$ and $a \,\sigma\, b$, then we have that $ea = eb$ for some $e \in E(S)$. The ample condition gives that

$$(ce)^{+}ca = cea = ceb = (ce)^{+}cb,$$

whence $\sigma$ is left compatible.

To see that $\sigma$ is transitive, suppose that $a, b, c \in S$ and $e, f \in E(S)$ with $ea = eb$ and $fb = fc$. Then

$$efa = efea = efeb = efb = efc,$$

so that $a \,\sigma\, c$ and $\sigma$ is a congruence.

For any $e, f \in E(S)$ we have

$$(ef)e = ef = (ef)f,$$

so that $e \,\sigma\, f$. Using the ample condition, it is easy to see that $\sigma$ is a monoid congruence. Finally, if $a \in S$ with $a\sigma$ idempotent, then there exists $e \in E(S)$ with $ea = ea^2$. Now

$$(eae)^2 = (eae)ae = ((eae)^{+}(ea))ae = (eae)^{+}ea^2e = (eae)^{+}eae = eae,$$

so that $eae, eae(ae)^{+} \in E(S)$. Moreover,

$$(eae)(ae)^{+}a = (eae)(ae) = (eae)(eae),$$

so by the comment at the beginning of this proof, $a \,\sigma\, eae$. Thus $S/\sigma$ is a unipotent monoid. ∎

If $S$ is finite, perforce $\sigma$ is the least group congruence on $S$. This is also the case when $S$ is $\mathcal{R}$-unipotent [3]. As for wla semigroups, we say that a wlqa semigroup is *proper* if $\widetilde{\mathcal{R}} \cap \sigma$ is the identity relation.

## 3. Finite proper covers

The relation $\leq_{\widetilde{\mathcal{R}}}$ is defined on a semigroup $S$ by the rule that

$$a \leq_{\widetilde{\mathcal{R}}} b \text{ if and only if for all } e \in E(S), eb = b \text{ implies that } ea = a.$$

Clearly $\leq_{\widetilde{\mathcal{R}}}$ is a quasiorder, with associated equivalence relation $\widetilde{\mathcal{R}}$. Consequently we have:

LEMMA 3.1. *Let $S$ be a semigroup. For all $a, b, c \in S$*
*(i) $ab \leq_{\widetilde{\mathcal{R}}} a$;*
*(ii) if $abc \,\widetilde{\mathcal{R}}\, a$, then $ab \,\widetilde{\mathcal{R}}\, a$.*

The following minor observation will be used repeatedly.

LEMMA 3.2. *Let $S$ be a wlqa semigroup, $a \in S$ and $e \in E(S)$. Then*

$$ae \,\widetilde{\mathcal{R}}\, a \text{ if and only if } ae = a.$$

PROOF. If $ae \,\widetilde{\mathcal{R}}\, a$, then using the ample identity,

$$a = a^+ a = (ae)^+ a = ae.$$

■

LEMMA 3.3. *Let $S$ be a wlqa semigroup. Then*

$$N = \{s \in S : xs \,\widetilde{\mathcal{R}}\, x \text{ implies that } xs = x\}$$

*is a subsemigroup and $E(S) \subseteq N$.*

PROOF. From Lemma 3.2, $E(S) \subseteq N$. If $s, t \in N$ and $xst \,\widetilde{\mathcal{R}}\, x$, then by Lemma 3.1, we have $xs \,\widetilde{\mathcal{R}}\, x$ so that as $s \in N$, $xs = x$. Now $xt \,\widetilde{\mathcal{R}}\, x$, so that as $t \in N$ we have $x = xt = xst$. Thus $N$ is a subsemigroup. ■

Recall that for an element $s$ of a semigroup $S$, $s'$ is a *weak inverse* of $s$ if $s'ss' = s'$; we let $W(s)$ denote the set of weak inverses of $s$. Notice that if $s' \in W(s)$, then $ss', s's \in E(S)$. A *weak conjugate* of $t \in S$ is an element of the form $sts'$ or $s'ts$.

LEMMA 3.4. *Let $S$ be a wlqa semigroup. If $a, b \in W(c)$ for some $a, b, c \in S$, and $xa \,\widetilde{\mathcal{R}}\, x \,\widetilde{\mathcal{R}}\, xb$, then $xa = xb$.*

PROOF. From $xaca = xa \,\widetilde{\mathcal{R}}\, x$ we have that $x \,\widetilde{\mathcal{R}}\, xac$, so that $x = xac$ by Lemma 3.2. Now $xb = xacb \,\widetilde{\mathcal{R}}\, xa$ so that again by Lemma 3.2, $xa = xacb = xb$. ■

Recall from [9] that a subset $X$ of a semigroup $S$ is *pointlike* (with respect to finite groups) if for every relational morphism $\phi : S \to G$ to a finite group $G$, there exists $g \in G$ such that $X \subseteq g\phi^{-1}$. We begin with the well known characterisation of pointlike sets (see Theorem 1.2 of [9]) which is a consequence of Ash's theorem (Theorem 2.1 in [1]) and the fact that in a semigroup in which the idempotents form a band $B$, $B$ is closed under weak conjugation. For a detailed discussion of Ash's theorem the reader is referred to Almeida [2]. We remark that the results of [9] are phrased for monoids; here we use the corresponding semigroup versions.

THEOREM 3.5. *[9] Let $S$ be a finite semigroup in which the idempotents form a band; then $\{s,t\} \subseteq S$ is pointlike if and only if there are factorisations of $s$ and $t$:*

$$s = e_0c_1e_1c_2 \dots e_{k-1}c_ke_k \text{ and } t = f_0d_1f_1d_2 \dots f_{k-1}d_kf_k$$

*where $e_i, f_i \in E(S^1), i \in \{0, \dots, k\}$, and for each $i \in \{1, \dots, k\}$, either $c_i = d_i$ or $c_i, d_i \in W(b_i)$ for some $b_i \in S$.*

PROPOSITION 3.6. *Let $S$ be a finite wlqa semigroup. If $\{s,t\}$ is pointlike and $xs\,\widetilde{\mathcal{R}}\,x\,\widetilde{\mathcal{R}}\,xt$, then $xs = xt$.*

PROOF. By Ash's theorem above, there are factorizations

$$s = e_0c_1e_1c_2 \dots e_{k-1}c_ke_k \text{ and } t = f_0d_1f_1d_2 \dots f_{k-1}d_kf_k$$

where $e_i, f_i \in E(S^1), i \in \{0, \dots, k\}$, and for each $i \in \{1, \dots, k\}$, either $c_i = d_i$ or $c_i, d_i \in W(b_i)$ for some $b_i \in S$.

Let

$$s_1 = e_0, s_2 = e_0c_1, s_3 = e_0c_1e_1, \dots, s_{2k+1} = s$$

and

$$t_1 = f_0, t_2 = f_0d_1, t_3 = f_0d_1f_1, \dots, t_{2k+1} = t.$$

It follows from Lemma 3.1 that

$$xs_i\,\widetilde{\mathcal{R}}\,x\,\widetilde{\mathcal{R}}\,xt_i$$

for each $i \in \{1, \dots, 2k+1\}$. Repeated applications of Lemmas 3.2 and 3.4 give the required result. ∎

In the next result, which is crucial, we show that each $\widetilde{\mathcal{R}}$-class of $S$ intersects with each pointlike set in at most one element.

LEMMA 3.7. *Let $S$ be a finite wlqa semigroup. Then for any $s, t$ with $\{s, t\}$ pointlike and $s^+ = t^+$, we have that $s = t$.*

PROOF. Let $\{s, t\} \subseteq S$ be pointlike such that $s^+ = t^+$. We have

$$s^+s\,\widetilde{\mathcal{R}}\,s^+\,\widetilde{\mathcal{R}}\,s^+t$$

so that $s^+s = s^+t$ by Proposition 3.6. Hence $s = t$. ∎

We thus are able to prove our main result.

THEOREM 3.8. *Each finite weakly left quasi-ample semigroup $S$ has a finite proper weakly left quasi-ample cover.*

PROOF. If $\widetilde{\mathcal{R}}$ is the identity relation then $S$ is proper. Otherwise, let $\{a_1, b_1\}, \ldots, \{a_k, b_k\}$ be a list of all pairs of distinct $\widetilde{\mathcal{R}}$-related elements of $S$. By Lemma 3.7, none of these pairs is pointlike. Hence for each $i \in \{1, \ldots, k\}$ there is a finite group $G_i$ and a relational morphism $\phi_i : S \to G_i$ such that $a_i\phi_i \cap b_i\phi_i = \emptyset$. Now define

$$\phi : S \to G_1 \times \ldots \times G_k = G,\ s\phi = s\phi_1 \times \ldots \times s\phi_k.$$

It is easy to check that $\phi$ is a relational morphism.

Let $R$ and $\theta$ constitute the "standard" cover obtained from $\phi$ as in the Introduction. As commented there, $E(R) = E(S) \times \{1\}$. Now if $(s, g) \in R$, then in the direct product $S \times G$,

$$(s, g)^+ = (s^+, g^+) = (s^+, 1) \in R$$

so that $R$ is a $(2, 1)$-subalgebra of $S \times G$. By virtue of the fact that wlqa semigroups form a quasivariety, $R$ is wlqa. It remains to show that $R$ is proper.

Let $(s, g), (t, h) \in R$ with $(s, g)\ \widetilde{\mathcal{R}} \cap \sigma\ (t, h)$. Then

$$(s, g)^+ = (t, h)^+ \text{ and } (e, 1)(s, g) = (e, 1)(t, h)$$

for some $e \in E(S)$. It follows that $s^+ = t^+$ and $g = h \in s\phi \cap t\phi$. Since $g = (g_1, \ldots, g_k)$ where $g_i \in s\phi_i \cap t\phi_i$, the set $\{s, t\}$ must be pointlike. By Lemma 3.7, $s = t$. Hence $R$ is proper. ∎

Since the cover $\widehat{S}$ of $S$ constructed in Theorem 3.8 is a $(2, 1)$-subalgebra of $S \times G$, where $G$ is a group, it follows that if $S$ is wla, then so is $\widehat{S}$.

Moreover, if $S$ is left ample, that is, $\mathcal{R}^* = \widetilde{\mathcal{R}}$ (see [6]), or equivalently it satisfies the quasi-identity

$$(s^+ = t^+ \wedge xs = ys) \Rightarrow xt = yt,$$

then so does $\widehat{S}$. Indeed $\widehat{S}$ is always left ample if $S$ is wla [6].

Finally, we remark that Theorem 3.8 holds equally well for monoids, in view of the fact that in a relational morphism $\phi$ between monoids, $1 \in 1\phi$ so that in Theorem 3.8, $(1,1) \in R$ and $(1,1)\theta = 1$.

## 4. Representations

Enlarging on a comment in the Introduction, we have the following result, which stems from a number of sources. We denote the identity map on a set $Y$ by $I_Y$ and for a set $X$, we put

$$E_X = \{I_Y : Y \subseteq X\}.$$

PROPOSITION 4.1. *[7, 10, 11] Let $\mathcal{PT}_X$ denote the monoid of partial transformations on a set $X$ and let $\alpha \mapsto \alpha^+$ be the unary operation on $\mathcal{PT}_X$ which takes $\alpha$ to $I_{dom\ \alpha}$. Then any $(2,1)$-subalgebra $S$ of $\mathcal{PT}_X$ such that $E(S) \subseteq E_X$ is a wla semigroup.*

*Conversely, if $S$ is a wla semigroup, then there is a $(2,1)$-embedding $\phi$ from $S$ into $\mathcal{PT}_X$ such that $E(S\phi) \subseteq E_X$.*

The aim of this section is to represent wlqa semigroups by partial transformations, or by direct products of such, in such a way that if $S$ is wla then the representation is faithful. In the proof of Proposition 4.1, $X$ may be taken to be $S$. For wlqa semigroups we need to develop further the alternative approach of generalised Schützenberger representations, already utilised in the previous section. We adopt standard convention in denoting the $\widetilde{\mathcal{R}}$-class of an element $a$ of a semigroup $S$ by $\widetilde{R}_a$; we put

$$\mathcal{X} = \{\widetilde{R}_a : a \in S\}.$$

LEMMA 4.2. *Let $S$ be a wlqa semigroup and let $X \in \mathcal{X}$. Then $S$ acts on $X$ on the right by partial functions if we define*

$$x \cdot s = \begin{cases} xs & \textit{if } xs\,\widetilde{\mathcal{R}}\,x \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

PROOF. We denote the action of $s$ by $\rho_s^X$. We need only show that dom $\rho_s^X \rho_t^X =$ dom $\rho_{st}^X$. But this is a consequence of Lemma 3.1. ∎

COROLLARY 4.3. *Let $S$ be wlqa and let $X \in \mathcal{X}$. Then $\phi^X : S \to \mathcal{PT}_X$ given by $s\phi^X = \rho_s^X$ is a semigroup morphism. For any $e \in E(S)$, $e\phi^X \in E_X$. Consequently, if $\widetilde{\mathcal{R}}$ is a left congruence, $\phi^X$ preserves $^+$.*

PROOF. If $e \in E(S)$ and $x \in \mathrm{dom}\ \rho_e^X$, then $xe \,\widetilde{\mathcal{R}}\, x$ so that by Lemma 3.2,

$$x\rho_e^X = xe = x.$$

If $\widetilde{\mathcal{R}}$ is a left congruence, then for any $s \in S$, $\mathrm{dom}\ \rho_s^X = \mathrm{dom}\ \rho_{s^+}^X$ so that

$$(s\phi^X)^+ = (\rho_s^X)^+ = I_{\mathrm{dom}\ \rho_s^X} = I_{\mathrm{dom}\ \rho_{s^+}^X} = \rho_{s^+}^X = s^+\phi^X.$$

■

In any semigroup $S$ containing a zero 0, $\{0\}$ is clearly an $\widetilde{\mathcal{R}}$-class. If $S$ is wlqa then choosing $X$ to be $\widetilde{R}_0$ in the above result, the representation of $S$ in $\mathcal{PT}_X$ is trivial. We wish our representation to be as 'faithful as possible'; more precisely, that it identifies only $\mathcal{L}$-related idempotents, and is faithful if $S$ is wla. To achieve the result we require we take the direct product of all the representations $\phi^X$.

PROPOSITION 4.4. *Let $S$ be a wlqa semigroup and let*

$$\phi : S \to \mathcal{P} = \Pi_{X \in \mathcal{X}} \mathcal{PT}_X$$

*be given by $(s\phi)_X = s\phi^X$. Then $\phi$ is a semigroup morphism such that*

*(i) if $s\phi \in E(\mathcal{P})$, then $s \in E(S)$;*

*(ii) $E(S\phi) \subseteq \Pi_{X \in \mathcal{X}} E_X$;*

*(iii) for any $e \in E(S)$, $(e\phi)\phi^{-1}$ is the $\mathcal{L}$-class of $e$.*

PROOF. (i) Suppose that $s \in S$ and $s\phi \in E(\mathcal{P})$, so that $\rho_s^X = \rho_{s^2}^X$ for all $X \in \mathcal{X}$. Let $X = \widetilde{R}_{s^+}$. Since $s^+s \,\widetilde{\mathcal{R}}\, s^+$ we have $s^+ \in \mathrm{dom}\ \rho_s^X = \mathrm{dom}\ \rho_{s^2}^X$ and $s^+s = s^+s^2$, so that $s = s^2$.

(ii) This is an immediate consequence of (i) and Corollary 4.3.

(iii) Let $e \in E(S)$ so that by (i), $(e\phi)\phi^{-1} \subseteq E(S)$. Suppose that $f, g \in (e\phi)\phi^{-1}$. Then $\rho_f^X = \rho_g^X$ for all $X \in \mathcal{X}$; putting $X = \widetilde{R}_f$ we have $fg \,\widetilde{\mathcal{R}}\, f$ since $f \in \mathrm{dom}\ \rho_f^X$. Thus $f = fg$ by Lemma 2.3; we deduce that $(e\phi)\phi^{-1} \subseteq L_e$.

Conversely, suppose that $h \in E(S)$ and $h \,\mathcal{L}\, e$. For any $X \in \mathcal{X}$,

$$\{x \in X : x \,\widetilde{\mathcal{R}}\, xe\} = \{x \in X : x = xe\} =$$

$$\{x \in X : x = xh\} = \{x \in X : x \,\widetilde{\mathcal{R}}\, xh\}$$

so that $\rho_e^X = \rho_h^X$ and $e\phi = h\phi$. ■

We would like to deduce from the previous result that $S\phi$ is wla . If $\mathcal{Y}$ is any collection of sets and for each $Y \in \mathcal{Y}$, $T_Y$ is a $(2,1)$-subalgebra of $\mathcal{PT}_Y$ with $E(T_Y) \subseteq E_Y$, then by Lemma 2.1, $\Pi_{Y\in\mathcal{Y}}T_Y$ is wla. In Proposition 4.4 we certainly have that the idempotents of $S\phi$ are all of the correct form. However, consequent upon the following result, $S\phi$ may not be closed under $^+$.

THEOREM 4.5. *Let $S$ be a weakly left quasi-ample semigroup and let $\phi$ be defined as in Proposition 4.4. Then the following are equivalent:*

*(i) $\phi$ preserves $^+$;*

*(ii) for each $a \in S$, $(a\phi)^+ = e\phi$ for some $e \in E(S)$;*

*(iii) $S\phi$ is closed under $^+$;*

*(iv) $\widetilde{\mathcal{R}}$ is a left congruence.*

*If any of these conditions holds, $S\phi$ is weakly left ample.*

PROOF. That (i) implies (ii) is clear and that (iii) implies (ii) follows from Proposition 4.4. Suppose now that (ii) holds. Let $a \in S$ and suppose that $(a\phi)^+ = e\phi$ where $e \in E(S)$; we show that $(a\phi)^+ = a^+\phi$, so that (i) and (iii) hold. For any $X \in \mathcal{X}$, $(a\phi^X)^+ = e\phi^X$ so that $I_{\text{dom }\rho_a^X} = \rho_e^X$ and dom $\rho_a^X =$ dom $\rho_e^X$. This gives from Lemma 3.2 that

$$\{x \in X : x\,\widetilde{\mathcal{R}}\,xa\} = \{x \in X : x\,\widetilde{\mathcal{R}}\,xe\} = \{x \in X : x = xe\}.$$

Since $a^+\,\widetilde{\mathcal{R}}\,a^+a$, if we put $X = \widetilde{R}_{a^+}$ we obtain $a^+ = a^+e$. On the other hand, as $e = ee$ putting $X = \widetilde{R}_e$ we have $e\,\widetilde{\mathcal{R}}\,ea$ and so

$$e\,\widetilde{\mathcal{R}}\,ea = ea^+a \leq_{\widetilde{\mathcal{R}}} ea^+ \leq_{\widetilde{\mathcal{R}}} e$$

so that $e\,\widetilde{\mathcal{R}}\,ea^+$ and so $e = ea^+$, as $E(S)$ is a left regular band. Thus $e\,\mathcal{L}\,a^+$ and it follows from Proposition 4.4 that $(a\phi)^+ = e\phi = a^+\phi$.

Corollary 4.3 gives that (iv) implies (i). Finally, we assume that (i) holds, so that for any $X \in \mathcal{X}$ and $a \in S$, $(a\phi^X)^+ = a^+\phi^X$ so that dom $\rho_a^X =$ dom $\rho_{a^+}^X$. Let $x \in S$ and put $X = \widetilde{R}_{xa^+}$. Then $xa^+a^+\,\widetilde{\mathcal{R}}\,xa^+$ so that $xa^+ \in$ dom $\rho_{a^+}^X$ and hence $xa^+\,\widetilde{\mathcal{R}}\,xa^+a = xa$. It follows that if $u, v, w \in S$ and $v\,\widetilde{\mathcal{R}}\,w$, then

$$uv\,\widetilde{\mathcal{R}}\,uv^+ = uw^+\,\widetilde{\mathcal{R}}\,uw$$

and $\widetilde{\mathcal{R}}$ is a left congruence.

The final statement of the theorem follows from the comments immediately preceding its statement. ∎

COROLLARY 4.6. *Let $S$ be a wla semigroup. Then $\phi$ as defined in Proposition 4.4 is an embedding.*

PROOF. From Theorem 4.5, $\phi$ preserves $^+$. Suppose that $a, b \in S$ with $a\phi = b\phi$. Then $a^+\phi = b^+\phi$ so that by Proposition 4.4, $a^+ = b^+$. Putting $X = \widetilde{R}_{a^+}$ and using the fact that $\phi_a^X = \phi_b^X$ we obtain $a^+a = a^+b$, whence $a = b$. ∎

If $S$ is a two element null semigroup with an identity then it is easy to see that $S$ is wlqa but $\widetilde{\mathcal{R}}$ is not a left congruence.

The results of this section are immediately adaptable to monoids; in each case the representation preserves the identity.

## References

[1] ASH, C.J., 'Inevitable graphs: a proof of the type II conjecture and some related decision procedures', *Internat. J. Algebra Comput.* 1:127–146, 1991.

[2] ALMEIDA, J., 'On hyperdecidable pseudovarieties of simple semigroups', *Internat. J. Algebra Comput.* 10:261–284, 2000.

[3] EDWARDS, C.C., 'The minimum group congruence on an $\mathcal{L}$-unipotent semigroup', *Semigroup Forum* 18:9–14, 1979.

[4] GOMES, G.M.S., and V.A.R. Gould, 'Proper weakly left ample semigroups', *Internat. J. Algebra Comput.* 9:721–739, 1999.

[5] GOMES G.M.S., and V.A.R. Gould, 'Graph expansions of unipotent monoids', *Comm. Algebra* 28:447–463, 2000.

[6] GOMES, G.M.S., and V.A.R. Gould, 'Finite proper covers in a class of finite semigroups with commuting idempotents', *Semigroup Forum* 66:433–454, 2003.

[7] GOULD, V.A.R., '(Weakly) left $E$-ample semigroups', *http://www-users.york.ac.uk/˜varg1/*.

[8] HOWIE, J.M., *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford 1995.

[9] HENCKELL, K., S.W. MARGOLIS, J.-E. PIN, and J. RHODES, 'Ash's type II theorem, profinite topology and Malcev products. I', *Internat. J. Algebra Comput.* 1:411–436, 1991.

[10] JACKSON M., and T. STOKES, 'An invitation to $C$-semigroups', *Semigroup Forum* 62:279–310, 2001.

[11] LAWSON, M.V., 'Semigroups and ordered categories I. The reduced case', *J. Algebra* 141:422-462, 1991.

[12] McALISTER, D.B., 'Groups, semilattices and inverse semigroups', *Trans. Amer. Math. Soc.* 192:227-244, 1974.

[13] McALISTER, D.B., , 'Groups, semilattices and inverse semigroups II', *Trans. Amer. Math. Soc.* 192:351-370, 1974.

[14] PETRICH M., and N.R. REILLY, *Completely Regular Semigroups*, Canadian Mathematical Society Series of Monographs and Advanced Texts 23, Wiley, New York, 1999.

[15] PIN, J.-E., *Varieties of Formal Languages*, North Oxford Academic, 1986.

[16] TAKIZAWA, K., '$E$-unitary and $\mathcal{R}$-unipotent semigroups', *Bull. Tokyo Gakugei Univ. Ser. IV* 3:21–33, 1978.

Karl Auinger
Institut für Mathematik
Universität Wien
Strudlhofgasse 4
A-1090 Wien
AUSTRIA
`karl.auinger@univie.ac.at`

Gracinda M.S. Gomes
Centro de Álgebra da Universidade de Lisboa
Avenida Prof Gama Pinto, 2
1649-003 Lisboa
PORTUGAL
*and* Departamento de Matemática
Faculdade de Ciências
Universidade de Lisboa
1746–016 Lisboa
PORTUGAL
`ggomes@cii.fc.ul.pt`

Victoria Gould
Department of Mathematics
University of York
Heslington
York YO10 5DD
UK
`varg1@york.ac.uk`

Benjamin Steinberg
School of Mathematics and Statistics
Carleton University
Herzberg Laboratories
1125 Colonel By Drive
Ottawa, Ontario K1S 5B6
CANADA
`bsteinbg@math.carleton.ca`