SEMIGROUPS WITH INVERSE SKELETONS AND ZAPPA-SZÉP PRODUCTS

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ABSTRACT. The aim of this paper is to study semigroups possessing E-regular elements, where an element a of a semigroup S is *E-regular* if a has an inverse a° such that $aa^\circ, a^\circ a$ lie in $E \subseteq E(S)$. Where S possesses 'enough' (in a precisely defined way) E-regular elements, analogues of Green's lemmas and even of Green's theorem hold, where Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{D}$ are replaced by $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E, \widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$. Note that S itself need not be regular. We also obtain results concerning the extension of (one-sided) congruences, which we apply to (one-sided) congruences on maximal subgroups of regular semigroups.

If S has an inverse subsemigroup U of E-regular elements, such that $E \subseteq U$ and U intersects every $\widetilde{\mathcal{H}}_E$-class exactly once, then we say that U is an *inverse skeleton* of S. We give some natural examples of semigroups possessing inverse skeletons and examine a situation where we can build an inverse skeleton in a $\widetilde{\mathcal{D}}_E$-simple monoid. Using these techniques, we show that a reasonably wide class of $\widetilde{\mathcal{D}}_E$-simple monoids can be decomposed as Zappa-Szép products. Our approach can be immediately applied to obtain corresponding results for bisimple inverse monoids.

1. INTRODUCTION

Decomposing semigroups using Green's relations is the classical approach to semigroup structure. Regular $\mathcal{D}$-classes are particularly well understood, given that the left and right translations afforded by Green's lemmas result in Green's theorem, which states that the $\mathcal{H}$-class of an element a is a subgroup if and only if $a\,\mathcal{H}\,a^2$. For non-regular $\mathcal{D}$-classes, indeed for non-regular semigroups, an approach using Green's relations is not always the most appropriate. As an alternative, one can make use of the extensions $\mathcal{K}^*$ of Green's relations $\mathcal{K}$, where $K \in \{R, L, H, D\}$ or the yet wider relations $\widetilde{\mathcal{K}}_E$, where E is a set of idempotents. The aim of this current paper is to take an approach that is something of a synthesis: we study semigroups possessing E-regular elements, where an element a of a semigroup S is *E-regular* if a has an inverse a° such that $aa^\circ, a^\circ a$ lie in $E \subseteq E(S)$.

After recalling the definitions of $\widetilde{\mathcal{K}}_E$ in Section 2, we show that where E-regular elements exist in particular places, then analogues of Green's lemmas hold where $\mathcal{K}$ is replaced by $\widetilde{\mathcal{K}}_E$. With some extra conditions on our semigroup we also have an analogue of Green's

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theorem. Namely, we show that under these conditions, if $a\,\widetilde{\mathcal{H}}_E\,a^2$, then $\widetilde{H}_E^a$, the $\widetilde{\mathcal{H}}_E$-class of a, is a monoid with identity from E. In Section 3 we show that if $\widetilde{\mathcal{H}}_E$ is a congruence on a semigroup S, then any right congruence on the submonoid $\widetilde{H}_E^e$, where $e \in E$, can be extended to a congruence on S. We also have a result for two sided congruences, with some further restrictions on S. We stress that for regular semigroups with $E = E(S)$ we have $\widetilde{\mathcal{K}}_E = \mathcal{K}^* = \mathcal{K}$, so our results can be immediately applied to maximal subgroups of regular semigroups.

In Section 4 we introduce the idea of an *inverse skeleton* U of a semigroup S. Here U is an inverse subsemigroup of E-regular elements, such that $E \subseteq U$ and U intersects every $\widetilde{\mathcal{H}}_E$-class exactly once (it follows that $E = E(U)$). We examine some conditions under which we obtain skeletons from monoids having a particular submonoid L of the $\widetilde{\mathcal{L}}_E$-class of the identity. A monoid with such a submonoid L is called *special*. Our most complete results are for restriction monoids, which for convenience we briefly define in Section 2.

Finally, in Section 5, we investigate the decomposition of some special $\widetilde{\mathcal{D}}_E$-simple monoids as what we refer to as *Zappa-Szép products*, also known as *general products*. The concept of Zappa-Szép product was first studied for groups by Neumann [15] and subsequently by Zappa [19] and Casadio [1]. The Zappa-Szép product of two groups is a natural generalisation of the notion of semidirect product, which itself extends that of direct product. Szép initiated the study of Zappa-Szép products in settings other than groups in [17, 18]. Zappa-Szép products for monoids have been further investigated by, for example, Kunze [10, 11, 12] and Lavers [13]. In particular, Kunze gave applications of Zappa-Szép products to translational hulls, Bruck-Reilly extensions and Rees matrix semigroups. In this paper we focus on a result of Kunze [10] for the Bruck-Reilly extension $\mathrm{BR}(M,\theta)$ of a monoid M, showing that $\mathrm{BR}(M,\theta)$ is a Zappa-Szép product of $\mathbb{N}^0$ under addition and a semidirect product $M \rtimes \mathbb{N}^0$. Certainly $\mathrm{BR}(M,\theta)$ is special, with L isomorphic to $\mathbb{N}^0$. We put Kunze's result in more general framework and prove in particular that a special $\widetilde{\mathcal{D}}_E$-simple restriction monoid can be decomposed in an analogous way. Again, our results apply immediately to inverse monoids.

A few words on notation. Given a semigroup S, we denote by $E(S)$ its set of idempotents and by E a subset of $E(S)$. We assume that the reader is familiar with Green's relations and their associated preorders and the starred versions thereof. Details of the latter and of the relations $\widetilde{\mathcal{K}}_E$, which we define below, can be found in the notes [6].

2. The relations $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E$ and analogues of Green's lemmas

We recall that the relation $\leq_{\widetilde{\mathcal{R}}_E}$ on S is defined by the rule that for all $a, b \in S$ we have $a \leq_{\widetilde{\mathcal{R}}_E} b$ if and only if

$$\{e \in E : eb = b\} \subseteq \{e \in E : ea = a\}.$$

It is clear that $\leq_{\widetilde{\mathcal{R}}_E}$ is a pre-order on S, that is, a relation that is reflexive and transitive. The associated equivalence relation is denoted by $\widetilde{\mathcal{R}}_E$. Thus for any $a, b \in S$ we have $a\,\widetilde{\mathcal{R}}_E\,b$ if and only if a and b have same set of left identities in E. It is easy to see that $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$. The relations $\leq_{\widetilde{\mathcal{L}}_E}$ and $\widetilde{\mathcal{L}}_E$ are defined dually so that clearly $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_E$.

Note that any $e \in E$ is a left (right) identity for its $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class). If S is regular and $E = E(S)$, then the foregoing inclusions are replaced by equalities. More generally, if $e, f \in E$ then $e \,\widetilde{\mathcal{R}}_E\, f$ if and only if $e \,\mathcal{R}\, f$ and $e \,\widetilde{\mathcal{L}}_E\, f$ if and only if $e \,\mathcal{L}\, f$. In general, however, the inclusions are strict.

We will show that, under certain circumstances, $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ behave like $\mathcal{R}$ and $\mathcal{L}$. In general, however, they do not. The first thing to observe is that, unlike $\mathcal{R}$ and $\mathcal{R}^*$, the relation $\widetilde{\mathcal{R}}_E$ need not be a left congruence; of course the dual remark is also true. We say that S *satisfies the Congruence Condition* (C) *with respect to* E (or, more simply, S *satisfies* (C)) if $\widetilde{\mathcal{R}}_E$ is a left congruence and $\widetilde{\mathcal{L}}_E$ is a right congruence. A second observation is that, as is the case with $\mathcal{R}^*$ and $\mathcal{L}^*$, the relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ need not commute. We denote by $\widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$ the intersection and join of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ respectively. Note that from the previous remark, it is not usually the case that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Deviating slightly from standard terminology, we will denote the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class, $\widetilde{\mathcal{H}}_E$-class, $\widetilde{\mathcal{D}}_E$-class) of any $a \in S$ by $\widetilde{R}^a_E$ ($\widetilde{L}^a_E$, $\widetilde{H}^a_E, \widetilde{D}^a_E$).

One class of semigroups having the congruence condition is the class of restriction semigroups. Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by ${}^+$. The identities that define a left restriction semigroup S are:

$$a^+a = a,\ a^+b^+ = b^+a^+,\ (a^+b)^+ = a^+b^+ \text{ and } ab^+ = (ab)^+a.$$

Putting $E = \{a^+ : a \in S\}$, it is easy to see that E is a semilattice, the *semilattice of projections* of S. Dually, right restriction semigroups form a variety of unary semigroups, where in this case the unary operation is denoted by *. A bi-unary semigroup S (that is, a semigroup with two unary operations) which is both left restriction and right restriction and which also satisfies the linking identities

$$(a^+)^* = a^+ \text{ and } (a^*)^+ = a^*$$

is called a *restriction semigroup*. We remark that an inverse semigroup is restriction, where we define $a^+ = aa^{-1}$ and $a^* = a^{-1}a$. If a restriction semigroup S has an identity element 1, then it is easy to see that $1^+ = 1^* = 1$. Such a restriction semigroup is naturally called a *restriction monoid*.

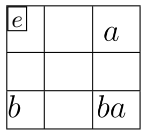
A restriction semigroup satisfies (C) (with respect to E) and is such that the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class) of an element a contains a unique element of E, namely a^+ (a^*). Restriction semigroups and their one sided versions have been studied from various point of view and under different names since the 1960s. They were formerly called *weakly E-ample semigroups*, to emphasize that the class naturally extends the class of *ample semigroups*. For detailed studies of the basic properties of these structures and a historical overview, the reader is referred to [5] and [6].

The next remark is folklore, but worth stating as a lemma.

Lemma 2.1. *If S satisfies (C), then $\widetilde{H}^e_E$ is a monoid with identity e, for any $e \in E$.*

Lemma 2.2. *Let S be a semigroup satisfying (C). Then if $a, b \in S$ and $a \,\widetilde{\mathcal{R}}_E\, e \,\widetilde{\mathcal{L}}_E\, b$, for some $e \in E$, we have that $a \,\widetilde{\mathcal{L}}_E\, ba \,\widetilde{\mathcal{R}}_E\, b$.*

Proof. As $a \,\widetilde{\mathcal{R}}_E\, e$ and $\widetilde{\mathcal{R}}_E$ is left congruence, we have $ba \,\widetilde{\mathcal{R}}_E\, be = b$. Dually, $ba \,\widetilde{\mathcal{L}}_E\, a$.



□

Definition 2.3. An element $c \in S$ is *E-regular* if c has an inverse c° such that $cc^\circ, c^\circ c \in E$.

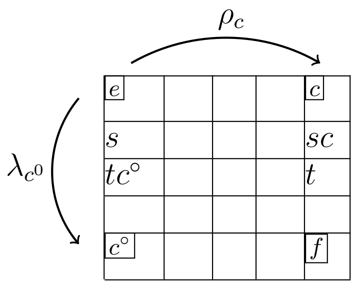
We emphasise that the notation c° will always be used with this meaning. Of course, if c is E-regular, then so is c°. Observe that if $c \in S$ is E-regular and $g, h \in E$ with $g \,\widetilde{\mathcal{R}}_E\, c \,\widetilde{\mathcal{L}}_E\, h$, then $cc^\circ \,\mathcal{R}\, c \,\widetilde{\mathcal{R}}_E\, g$ and $c^\circ c \,\mathcal{L}\, c \,\widetilde{\mathcal{L}}_E\, h$, so that by an earlier remark, $cc^\circ \,\mathcal{R}\, g$ and $c^\circ c \,\mathcal{L}\, h$. It follows from standard results for regular elements that c has an inverse c' such that $cc' = g$ and $c'c = h$. It is also easy to see (in view of earlier remarks concerning idempotents), that if $h, k \in S$ are E-regular, then $h \,\widetilde{\mathcal{K}}_E\, k$ if and only if $h \,\mathcal{K}\, k$, where K is R, L or H.

We first show that analogues of Green's Lemmas hold with $\mathcal{R}$, $\mathcal{L}$ replaced by $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ where there is a suitable E-regular element.

Lemma 2.4. *Suppose that $\widetilde{\mathcal{L}}_E$ is a right congruence and S has an E-regular element c such that $e = cc^\circ$ and $f = c^\circ c$. Then the right translations*

$$\rho_c : \widetilde{L}^e_E \to \widetilde{L}^f_E \quad \textit{and} \quad \rho_{c^\circ} : \widetilde{L}^f_E \to \widetilde{L}^e_E$$

are mutually inverse $\widetilde{\mathcal{R}}_E$-class preserving bijections.



Proof. Notice that $e \,\mathcal{R}\, c \,\mathcal{L}\, f$. Let $s \in \widetilde{L}^e_E$. Since $\widetilde{\mathcal{L}}_E$ is a right congruence, $sc \,\widetilde{\mathcal{L}}_E\, ec = c$ so there is a map $\rho_c : \widetilde{L}^e_E \to \widetilde{L}^f_E$ defined by $s\rho_c = sc$. Now $s = se = scc^\circ \,\mathcal{R}\, sc$, so that certainly ρ_c is $\widetilde{\mathcal{R}}_E$-class preserving. Dually, $\rho_{c^\circ} : \widetilde{L}^f_E \to L^e_E$ is $\widetilde{\mathcal{R}}_E$-class preserving.

For any $s \in \widetilde{L}^e_E$ and $t \in \widetilde{L}^f_E$ we have $s = se = s(cc^\circ) = s\rho_c\rho_{c^\circ}$ and similarly, $t = t\rho_{c^\circ}\rho_c$, so that ρ_c and ρ_{c° are mutually inverse on the specified domains. □

Note that we are not assuming that the $\widetilde{\mathcal{D}}_E$-class depicted above is an "egg-box", since as $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ need not commute, some of the cells may be empty.

For convenience we now state the dual of Lemma 2.4.

Lemma 2.5. *Suppose that $\widetilde{\mathcal{R}}_E$ is a left congruence and S has an E-regular element c such that $e = cc^\circ$ and $f = c^\circ c$. Then the left translations*

$$\lambda_{c^\circ} : \widetilde{R}^e_E \to \widetilde{R}^f_E \quad \textit{and} \quad \lambda_c : \widetilde{R}^f_E \to \widetilde{R}^e_E$$

are mutually inverse $\widetilde{\mathcal{L}}_E$-class preserving bijections.

Corollary 2.6. *Let S be a semigroup with (C). Let c be an E-regular element of S such that $e = cc^\circ$ and $f = c^\circ c$. Then $\widetilde{H}^e_E \cong \widetilde{H}^f_E$.*

Proof. By Lemmas 2.4 and 2.5, $\rho_c : \widetilde{H}^e_E \to \widetilde{H}^c_E$ and $\lambda_{c^\circ} : \widetilde{H}^c_E \to \widetilde{H}^f_E$ are bijections. Now For any $x, y \in \widetilde{H}^e_E$ we have

$$\begin{aligned} (xy)\rho_c\lambda_{c^\circ} &= c^\circ xyc \\ &= c^\circ xcc^\circ yc \qquad \text{as } cc^\circ = e \\ &= (x\rho_c\lambda_{c^\circ})(y\rho_c\lambda_{c^\circ}). \end{aligned}$$

Thus $\rho_c\lambda_{c^\circ}$ is an isomorphism and hence $\widetilde{H}^e_E \cong \widetilde{H}^f_E$. □

If we have enough E-regular elements, then we can say much more than in Corollary 2.6. First, we recall that S is *weakly E-abundant* if every $\widetilde{\mathcal{R}}_E$- and every $\widetilde{\mathcal{L}}_E$-class of S contains an idempotent of E. Clearly a regular semigroup S is weakly $E(S)$-abundant; on the other hand, any monoid is weakly $\{1\}$-abundant. A less extreme example is $M_n(R)$, the monoid of $n \times n$ matrices over a principal ideal domain, under matrix multiplication [4]. In such a monoid we have $\widetilde{\mathcal{R}}_E = \mathcal{R}^*$ and $\widetilde{\mathcal{L}}_E = \mathcal{L}^*$, where $E = E\big(M_n(R)\big)$, and further, every $\mathcal{H}^*$-class contains a regular element. The reader will see other natural examples as the article progresses.

Lemma 2.7. *If every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, then S is weakly E-abundant. Moreover if S has (C), then $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (so that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E$) and if $a, b \in S$ with $a \,\widetilde{\mathcal{D}}_E\, b$, then $|\widetilde{H}^a_E| = |\widetilde{H}^b_E|$.*

Proof. The first statement is clear. Suppose that $a, c \in S$ with $a \,\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E\, c$.

				$b^\circ b$
a		bb°		b
$cb^\circ a$				c

There exists an E-regular $b \in S$ such that $a \,\widetilde{\mathcal{R}}_E\, b \,\widetilde{\mathcal{L}}_E\, c$. Choose an inverse b° of b such that $bb^\circ, b^\circ b \in E$. Notice that $c \,\widetilde{\mathcal{L}}_E\, b^\circ b$ and $a \,\widetilde{\mathcal{R}}_E\, bb^\circ$. Using (C), $cb^\circ a \,\widetilde{\mathcal{R}}_E\, cb^\circ b = c$ and $cb^\circ a \,\widetilde{\mathcal{L}}_E\, bb^\circ a = a$. Then $a \,\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E\, c$. Together with the dual argument we have that $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. In view of the remarks following Definition 2.3, the proof of the final statement follows easily from Lemmas 2.4 and 2.5. □

Green's theorem, a pivot of classical semigroup theory, states that if $k \in S$ and $k\,\mathcal{H}\,k^2$, then H_k is a group. We now consider semigroups with (C) such that the analogue of Green's theorem holds, by which we mean, if $k\,\widetilde{\mathcal{H}}_E\,k^2$, then $\widetilde{H}_E^k$ is a monoid with identity an element of E: in view of Lemma 2.1, this is equivalent to containing an element of E.

The set of idempotents $E(T)$ of any semigroup T may be endowed with the two pre-orders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$, under which it has the structure of a *biordered set*; if T is regular, then $E(T)$ is a *regular biordered set*. Conversely, any biordered set is the biordered set of idempotents of a semigroup, which is regular if E is regular [14, 3]. Suppose now that S is our semigroup with $E \subseteq E(S)$; [14, Theorem 1.3] gives necessary and sufficient conditions such that E generates a regular subsemigroup $S' = \langle E \rangle$ of S such that $E(S') = E$. Clearly, if these conditions hold, and if $h \in S'$ with $h\,\widetilde{\mathcal{H}}_E\,h^2$ in S, then as $E \subseteq S'$ we have $h\,\widetilde{\mathcal{H}}_E\,h^2$ in S'. It follows that $h\,\mathcal{H}\,h^2$ in S' so that $h\,\mathcal{H}\,u$ in S' for some $u \in E(S') = E$. Certainly then $\widetilde{H}_E^h$ (in either S or S') contains u.

To obtain a more general result, we need to introduce the following concept.

Definition 2.8. We say that $E \subseteq E(S)$ is *closed under E-conjugation* if for any $e \in E$ and E-regular $c \in S$ (with $cc^\circ, c^\circ c \in E$), if $cec^\circ \in E(S)$, then $cec^\circ \in E$.

Notice that the above definition is symmetric, since $(c^\circ)^\circ = c$.

Lemma 2.9. *Let S be a restriction semigroup, let $c \in S$ be E-regular and let $e \in E$. Then cec° (and hence also $c^\circ ec$) lie in E.*

Proof. Let c, e be as above. Then

$$cec^\circ = (ce)^+ cc^\circ \in E$$

as E is a semilattice. □

The next lemma follows the pattern for regular semigroups, as stated in [7, Result 2]. However, we need a little care as E need not consist of all idempotents of S.

Lemma 2.10. *The E-regular elements of S form a subsemigroup T with $E = E(T)$ if and only if ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation.*

Proof. Let T denote the set of E-regular elements of S. The direct statement is clear.

Conversely, suppose that ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation. Let $h, k \in T$ and choose inverses h°, k° of h and k respectively, such that $hh^\circ, f = h^\circ h, e = kk^\circ, k^\circ k \in E$. Let u be an inverse of fe such that $ufe, feu \in E$. It is easy to check that $k^\circ uh^\circ$ is an inverse of hk. We then have $(hk)(k^\circ uh^\circ) \in E(S)$ and

$$(hk)(k^\circ uh^\circ) = hf(kk^\circ)uh^\circ = h(feu)h^\circ,$$

so that $(hk)(k^\circ uh^\circ) \in E$ as $feu \in E$ and E is closed under E-conjugation. Similarly, $(k^\circ uh^\circ)hk \in E$. Thus $hk \in T$ as required. □

Corollary 2.11. *Suppose that ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation. If $h \in S$ is E-regular and $h\,\widetilde{\mathcal{H}}_E\,h^2$, then $\widetilde{H}_E^h$ contains an idempotent of E; hence if S satisfies (C), then $\widetilde{H}_E^h$ is a monoid with identity from E.*

Proof. From Lemma 2.10 we have that the E-regular elements of S form a subsemigroup T with $E = E(T)$. Certainly $h, h^2 \in T$ with $h \,\widetilde{\mathcal{H}}_E\, h^2$ in T. Then $h \,\mathcal{H}\, h^2$ in T so that as $E = E(T)$ we have $\widetilde{H}_E^h$ (in either T or S) contains an idempotent of E. $\square$

Whereas the previous result uses Green's theorem, the next does not, but has rather restrictive hypotheses.

Lemma 2.12. *Suppose that $E \subseteq E(S)$ is a band, every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, $\widetilde{\mathcal{H}}_E$ is a congruence and S satisfies (C). Then for $k \in S$ with $k \,\widetilde{\mathcal{H}}_E\, k^2$, we have $E \cap \widetilde{H}_E^k \neq \emptyset$.*

Proof. Notice that as $\widetilde{\mathcal{H}}_E$ is a congruence and $k \,\widetilde{\mathcal{H}}_E\, k^2$, we have that $\widetilde{H}_E^k$ is a subsemigroup.

$\boxed{h}$ $k,\, ef$				$\boxed{hh^\circ = e}$
$\boxed{h^\circ h = f}$				$\boxed{h^\circ}$ fe

By hypothesis there exists an E-regular element $h \in \widetilde{H}_E^k$ such that $hh^\circ = e, h^\circ h = f \in E$. Then

$$h^\circ = h^\circ h h^\circ \,\widetilde{\mathcal{H}}_E\, h^\circ h h h^\circ = fe \in E.$$

By Lemma 2.2, $ef \in \widetilde{H}_E^k$ and $ef \in E$ as E is a band. Hence $E \cap \widetilde{H}_E^k \neq \emptyset$. $\square$

3. Extending congruences

Let M be a subsemigroup of a semigroup S and let ρ be a congruence (respectively, right congruence) on M. We denote by $\widetilde{\rho}$ (respectively, $\bar{\rho}$) the congruence (respectively, right congruence) on S generated by ρ. We briefly review the circumstances under which $\rho = \widetilde{\rho} \cap (M \times M)$ or $\rho = \bar{\rho} \cap (M \times M)$, where $M = \widetilde{H}_E^e$ for some $e \in E$, in the context of the conditions discussed in this article.

Definition 3.1. A subsemigroup M of a semigroup S has the *(right) congruence extension property* in S if for any (right) congruence ρ on M we have

$$\rho = \widetilde{\rho} \cap (M \times M) \text{ (respectively, } \rho = \bar{\rho} \cap (M \times M)).$$

Lemma 3.2. *Let S be a weakly E-abundant semigroup with (C). Suppose that $\widetilde{\mathcal{H}}_E$ is a congruence. Let $e \in E$. Then $M = \widetilde{H}_E^e$ has the right congruence extension property in S.*

Proof. Let ρ be a right congruence on M. Clearly $\rho \subseteq \bar{\rho} \cap (M \times M)$. Let $a \in M, b \in S$ and suppose $a \,\bar{\rho}\, b$. Then either $a = b$ (so that clearly $a \,\rho\, b$) or there exists a sequence

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ \cdots,\ d_n t_n = b$$

for some $n \in \mathbb{N}$, where $(c_i, d_i) \in \rho$, $t_i \in S$, $1 \leq i \leq n$ (see, for example, [9, Chapter 1]). As $a, c_1, d_1, \cdots, c_n, d_n \in M$, which has identity e, we have

$$a = c_1 t_1',\ d_1 t_1' = c_2 t_2',\ \cdots,\ d_n t_n' = b \quad \text{where } t_i' = e t_i.$$

Since $\widetilde{\mathcal{H}}_E$ is a congruence we have

$$a = c_1 t_1' \, \widetilde{\mathcal{H}}_E \, e t_1' = t_1' \, \widetilde{\mathcal{H}}_E \, d_1 t_1' = c_2 t_2' \, \widetilde{\mathcal{H}}_E \, e t_2' = t_2' \, \widetilde{\mathcal{H}}_E \, \cdots \, \widetilde{\mathcal{H}}_E \, e t_n' = t_n'.$$

We conclude that $t_1', \cdots, t_n' \in M$ and so $b \in M$ and $a \, \rho \, b$. Hence M has the right congruence extension property. $\square$

Note that what we have shown above is something a little stronger than claimed, namely that $\bar{\rho}$ saturates M.

Corollary 3.3. *Let S be a regular semigroup such that $\mathcal{H}$ is a congruence. Then for any $e \in E(S)$, the maximal subgroup H_e has the right congruence extension property.*

Let M be a subsemigroup of S and let ρ be a congruence on M. We say that ρ is *closed under E-conjugation* if for $u, v \in M$ with $u \, \rho \, v$ and for any E-regular $c \in S$ with $cuc^\circ, cvc^\circ \in M$, we have $cuc^\circ \, \rho \, cvc^\circ$; if $E = E(S)$, we simply say that ρ is *closed under conjugation.*

Proposition 3.4. *Let S be a semigroup with (C) such that every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, $\widetilde{\mathcal{H}}_E$ is a congruence and if $k \, \widetilde{\mathcal{H}}_E \, k^2$, then $\widetilde{H}_E^k$ contains an idempotent of E. Let $e \in E$ and $M = \widetilde{H}_E^e$ and let ρ be a congruence on M. Then*

$$\rho = \widetilde{\rho} \cap (M \times M),$$

if and only if ρ is closed under E-conjugation.

Proof. It is clear that if $\rho = \widetilde{\rho} \cap (M \times M)$, then ρ is closed under E-conjugation.

Conversely, suppose that ρ is closed under E-conjugation. Let $a \in M, b \in S$ and suppose that

$$a = cpd, \; cqd = b,$$

where $(p, q) \in \rho$ and $c, d \in S^1$. As $p \, \widetilde{\mathcal{H}}_E^e \, q$ and $\widetilde{\mathcal{H}}_E$ is a congruence, we see that $b \in M$. It follows that

$$a = c'pd', \; c'qd' = b,$$

where $c' = ece$ and $d' = ede$. Then

$$a \leq_{\widetilde{\mathcal{R}}_E} c' \leq_{\widetilde{\mathcal{R}}_E} e \, \widetilde{\mathcal{R}}_E \, a,$$

so that $a \, \widetilde{\mathcal{R}}_E \, c'$. Dually, $a \, \widetilde{\mathcal{L}}_E \, d'$.

$\boxed{e}$ a	v°			c' u
u°				f
d' v u^*	g			$d'c'$ w

From the comments following Definition 2.3, there exist E-regular elements $u \in \widetilde{H}_E^{c'}$ and $v \in \widetilde{H}_E^{d'}$ such that $uu^\circ = e, u^\circ u = f \in E$ and $v^\circ v = e, vv^\circ = g \in E$. Now $vu \in \widetilde{R}_E^v \cap \widetilde{L}_E^u$ by Lemma 2.2 and $vu \,\widetilde{\mathcal{H}}_E\, d'c'$. Since

$$uv \,\widetilde{\mathcal{H}}_E\, c'd' = c'ed' \,\widetilde{\mathcal{H}}_E\, c'pd' = a \,\widetilde{\mathcal{H}}_E\, e$$

we have

$$vuvu \,\widetilde{\mathcal{H}}_E\, veu \,\widetilde{\mathcal{H}}_E\, vu.$$

By assumption, there exists an idempotent $w \in E \cap \widetilde{H}_E^{d'c'}$. Let $u^* \in \widetilde{H}_E^{d'}$ be an inverse of u such that $uu^* = e$ and $u^*u = w$. Then

$$a = c'wpwd' = (c'u^*)(upu^*)(ud') \text{ and } b = c'wqwd' = (c'u^*)(uqu^*)(ud').$$

Now $u^* \,\widetilde{\mathcal{H}}_E\, d'$ gives that $c'u^* \,\widetilde{\mathcal{H}}_E\, c'd' \,\widetilde{\mathcal{H}}_E\, e$, so $c'u^* \in M$ and similarly $u \,\widetilde{\mathcal{H}}_E\, c'$ gives that $ud' \,\widetilde{\mathcal{H}}_E\, c'd' \,\widetilde{\mathcal{H}}_E\, e$, so that $ud' \in M$. Further,

$$upu^* = e(upu^*)e \,\widetilde{\mathcal{H}}_E\, (c'u^*)(upu^*)(ud') = a \in M,$$

and similarly, $uqu^* \in M$. Since ρ is closed under E-conjugation it follows that $upu^* \,\rho\, uqu^*$ and so $a \,\rho\, b$.

Now consider $h \in M, k \in S$ with $h \,\widetilde{\rho}\, k$. Either $h = k$ (so that certainly $h \,\rho\, k$), or h is connected to k via a ρ-sequence

$$h = c_1p_1d_1, c_1q_1d_1 = c_2p_2d_2, \cdots, c_nq_nd_n = k,$$

for some $n \in \mathbb{N}$, where $(p_i, q_i) \in \rho$, $c_i, d_i \in S^1$, $1 \leq i \leq n$ (see, for example, [8, Chapter 1]). It follows from the above that $c_iq_id_i \in M$ and $h \,\rho\, c_iq_id_i$ for $1 \leq i \leq n$. Hence $h \,\rho\, k$ and

$$\rho = \widetilde{\rho} \cap (M \times M).$$

□

Corollary 3.5. *Let S be a regular semigroup such that $\mathcal{H}$ is a congruence. Let $G = H_e$ be the maximal subgroup with identity $e \in E(S)$. Then for any right congruence ρ on G we have $\rho = \widetilde{\rho} \cap (G \times G)$ if and only if ρ is closed under conjugation.*

Note that if E is a band, then from Lemma 2.12, the remaining hypotheses of Proposition 3.4 will guarantee that $\widetilde{H}_E^k$ contains an idempotent of E.

In the following, M is a monoid with identity e.

Example 3.6. Let B be a band. With $E = \{e\} \times B$, the direct product $M \times B$ satisfies the hypotheses of Proposition 3.4.

The next three examples are essentially folklore, but they can all be found in [2].

Example 3.7. Let $S = \mathcal{B}^\circ(M, I)$ be a 'Brandt' semigroup. That is,

$$S = (I \times M \times I) \cup \{0\}$$

with multiplication given by

$$(i, m, j)(j, n, k) = (i, mn, k),$$

all other products being 0. Then with

$$E = \{(i, 1, i) : i \in I\} \cup \{0\}$$

we have that for any $(i, m, j), (k, n, l) \in M$

$$(i, m, j) \, \widetilde{\mathcal{R}}_E \, (k, n, l) \text{ if and only if } i = k$$

and

$$(i, m, j) \, \widetilde{\mathcal{L}}_E \, (k, n, l) \text{ if and only if } j = l.$$

It follows that S is restriction with distinguished semilattice E, $\widetilde{\mathcal{H}}_E$ is a congruence on S and with

$$U = \{(i, e, j) : i, j \in I\} \cup \{0\}$$

we have that U is an inverse subsemigroup of E-regular elements, intersecting every $\widetilde{\mathcal{H}}_E$-class exactly once. In particular, S satisfies the hypotheses of Proposition 3.4.

Example 3.8. Let $S = \mathrm{BR}(M, \theta)$, where $\theta : M \to M$ is a monoid morphism. That is,

$$S = \mathbb{N}^0 \times M \times \mathbb{N}^0$$

and multiplication is given by

$$(m, a, n)(h, b, k) = (m - n + u, a\theta^{u-n} b\theta^{u-h}, k - h + u) \text{ where } u = \max\,(n, h).$$

With

$$E = \{(m, e, m) : m \in \mathbb{N}^0\}$$

we have that for any $(m, a, n), (h, b, k) \in S$,

$$(m, a, n) \, \widetilde{\mathcal{R}}_E \, (h, b, k) \text{ if and only if } m = h$$

and

$$(m, a, n) \, \widetilde{\mathcal{L}}_E \, (h, b, k) \text{ if and only if } n = k.$$

It is then easy to see that $\widetilde{\mathcal{H}}_E$ is a congruence on S and S is restriction. Moreover, with

$$U = \{(m, e, n) : m, n \in \mathbb{N}^0\}$$

we have that U is an inverse subsemigroup of E-regular elements of S intersecting every $\widetilde{\mathcal{H}}_E$-class exactly once. In particular, S satisfies the hypotheses of Proposition 3.4. Note that S is a monoid with identity $(0, e, 0)$.

Note that the assumption in [2] that the image of θ is contained in $\widetilde{H}^1_E$, is not needed for the above.

Example 3.9. Let $S = \mathrm{BR}(M, \mathbb{Z}, \theta)$ be the extended Bruck-Reilly extension of a monoid M. The underlying set is

$$S = \mathbb{Z} \times M \times \mathbb{Z}$$

and the semigroup operation on S is defined as in Example 3.8. The semigroup S has the same properties as in that example, with the exception of being a monoid.

Example 3.10. Let $S = [Y; S_\alpha; \chi_{\alpha,\beta}]$ be a strong semilattice Y of monoids $S_\alpha, \alpha \in Y$, with connecting morphims $\chi_{\alpha,\beta}$ for $\alpha \geqslant \beta$. Denoting the identity of S_α by e_α we have that S is restriction with

$$E = \{e_\alpha : \alpha \in Y\} \cong Y,$$

and the S_αs are the $\widetilde{\mathcal{H}}_E$-classes. Certainly then $\widetilde{\mathcal{H}}_E$ is a congruence on S and S satisfies the hypotheses of Proposition 3.4.

4. Semigroups with skeletons

We continue to examine semigroups with 'enough' E-regular elements, now moving towards decompositions of such semigroups. It is clear from Lemma 2.7 that if every $\widetilde{\mathcal{H}}_E$-class of a semigroup S with (C) contains an E-regular element, and $e\,\widetilde{\mathcal{D}}_E\,a$ where $e \in E$, then every element of $\widetilde{H}_E^a$ has a unique decomposition as upv, where u, v are fixed E-regular elements and $p \in \widetilde{H}_E^e$. For results leading further to structure theorems, we will concentrate in this section on the case where E is a semilattice.

Definition 4.1. Let $V \subseteq W$ be subsets of a semigroup S such that W is a union of $\widetilde{\mathcal{H}}_E$-classes. We say that V is an $\widetilde{\mathcal{H}}_E$-*transversal* of W if

$$|V \cap \widetilde{H}_E^a| = 1 \quad \text{for all } a \in W.$$

Lemma 4.2. *Let E be a semilattice and let $c \in S$ be E-regular. Then there is only one choice of c°. Moreover, if $d \in S$ is E-regular and $c\,\widetilde{\mathcal{H}}_E\,d$, then $c^\circ\,\widetilde{\mathcal{H}}_E\,d^\circ$.*

Proof. If c°, c' are both inverses of c with $cc^\circ, cc', c^\circ c, c'c \in E$, then we have

$$c\,\widetilde{\mathcal{L}}_E\,c^\circ c\,\widetilde{\mathcal{L}}_E\,c'c \text{ and } cc^\circ\,\widetilde{\mathcal{R}}_E\,c\,\widetilde{\mathcal{R}}_E\,cc'.$$

Since E is a semilattice, any $\widetilde{\mathcal{R}}_E$-class or $\widetilde{\mathcal{L}}_E$-class contains at most one idempotent of E, so that $c^\circ c = c'c = e$ and $cc^\circ = cc' = f$ say. Thus $c^\circ, c' \in R_e \cap L_f$ so that (as any $\mathcal{H}$-class contains at most one inverse of c) we have $c^\circ = c'$.

The proof of the second statement is similar. □

Clearly the above shows that if E is a semilattice and $c \in S$ is E-regular, then $(c^\circ)^\circ = c$. We recall that S is said to be *weakly E-adequate* if S is weakly E-abundant and E is a semilattice. In this case there is a unique idempotent in the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class) of $a \in S$, which we denote by a^+ (a^*, respectively).

Note 4.3. Let S be a weakly E-adequate semigroup and let $c \in S$ be E-regular. Then

$$c\,\widetilde{\mathcal{R}}_E\,c^+\,\widetilde{\mathcal{R}}_E\,cc^\circ,$$

so that we must have $c^+ = cc^\circ$ and similarly $c^* = c^\circ c$. Hence also $(c^\circ)^+ = c^\circ c$ and $(c^\circ)^* = cc^\circ$.

Proposition 4.4. *Let S be weakly E-adequate with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, and let $e \in E$. Suppose there is an $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}_E^e$ such that every $c \in L$ is E-regular, and $e \in L$. Then:*

(1) $R = \{c^\circ : c \in L\}$ *is an* $\widetilde{\mathcal{H}}_E$*-transversal of* $\widetilde{R}^e_E$*;*
(2) $D = LR$ *is an* $\widetilde{\mathcal{H}}_E$*-transversal of* $\widetilde{D}^e_E$*;*
(3) *if* S *has* (C)*, then every element of* $\widetilde{D}^e_E$ *has a unique decompostion as* cpd°*, where* $c, d \in L$ *and* $p \in \widetilde{H}^e_E$*.*

Proof. (1) Let $c \in L$. As E is a semilattice and $c\,\widetilde{\mathcal{L}}_E\,e$, we must have that $e = c^\circ c$ so that $e\,\widetilde{\mathcal{R}}_E\,c^\circ$. From Lemma 4.2, clearly R intersects any $\widetilde{\mathcal{H}}_E$-class at most once. On the other hand, let $a \in \widetilde{R}^e_E$. Then $a\,\widetilde{\mathcal{L}}_E\,f \in E$ and as $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, we have that $f\,\widetilde{\mathcal{R}}_E\,c$ for some $c \in L$. It follows that $a\,\widetilde{\mathcal{H}}_E\,c^\circ$, so that R is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^e_E$.
(2) It is clear from Lemma 2.2 that for any $c, d \in L$ we have $cd^\circ \in \widetilde{R}^c_E \cap \widetilde{L}^{d^\circ}_E$. Since $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, it follows that D is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{D}^e_E$, as required.
(3) This follows from Lemmas 2.4 and 2.5.

□

We anticipate that Proposition 4.4 can be used to develop structure theorems for classes of weakly E-adequate semigroups analogous to those for inverse semigroups.

Definition 4.5. Let U be an inverse subsemigroup of S consisting of E-regular elements such that $E \subseteq U$. If U is an $\widetilde{\mathcal{H}}_E$-transversal of S, then U is an *inverse skeleton* of S.

Example 4.6. The semigroups of Examples 3.7, 3.8 and 3.10 all have inverse skeletons, with E being the skeleton in Example 3.10.

Lemma 4.7. *Let* S *be a semigroup containing an inverse skeleton* U*. Then* $E = E(U)$ *is a semilattice,* S *is weakly* E*-adequate and if in addition* S *has* (C)*, we have* $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$*.*

Proof. We are given that $E \subseteq E(U)$. If $u \in E(U)$, then as u is E-regular, $u\,\mathcal{R}\,uu^\circ \in E$. We are given that $E(U)$ is a semilattice and so $u = uu^\circ \in E$. The remainder of the lemma is immediate from Lemma 2.7. □

Naturally, we say that S is $\widetilde{\mathcal{D}}_E$*-simple* if it is a single $\widetilde{\mathcal{D}}_E$-class.

Theorem 4.8. *Let* S *be a* $\widetilde{\mathcal{D}}_E$*-simple weakly* E*-adequate monoid with* $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$*. Suppose there is a submonoid* $\widetilde{\mathcal{H}}_E$*-transversal* L *of* $\widetilde{L}^1_E$ *such that every* $c \in L$ *is* E*-regular and for all* $c \in L$*,* $e \in E$ *we have* $cec^\circ, c^\circ ec \in E$*. Let*

$$R = \{c^\circ : c \in L\}.$$

(1) R *is a submonoid* $\widetilde{\mathcal{H}}_E$*-transversal of* $\widetilde{R}^1_E$*;*
(2) $RL \subseteq \widetilde{R}^1_E \cup \widetilde{L}^1_E$ *if and only if* E *is a chain;*
(3) *if* S *is restriction then* $U = \langle R \cup L\rangle$ *is an inverse subsemigroup of* S *with* $E(U) = E$*;*
(4) *if* S *is restriction and* $RL \subseteq R \cup L$*, then* $U = LR$ *and* U *is an inverse skeleton for* S*.*

Proof. From the condition that $cec^\circ, c^\circ ec \in E$ for all $c \in L$, and the fact that E is a semilattice, it is easy to see that for any $u, v \in R \cup L$ we have uv is E-regular with suitable inverse $v^\circ u^\circ$.

(1) From Proposition 4.4, we know that R is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^1_E$. Let $c, d \in L$ so that $c^\circ, d^\circ \in R$. From the above, cd is E-regular with $(cd)^\circ = d^\circ c^\circ$. As $cd \in L$ we have $d^\circ c^\circ \in R$. Clearly, $1 = 1^\circ \in R$, so that R is a submonoid.

(2) Let $e, f \in E$ and let $c, d \in L$ be such that $cc^\circ = e, dd^\circ = f$. As above, $c^\circ d$ is E-regular with $(c^\circ d)^\circ = d^\circ c$. We have $c^\circ d \in \widetilde{R}^1_E$ if and only if $1 = c^\circ dd^\circ c$, which implies (multiplying on the front by c and the back by c°) that $e = efe$ so that $e \leq f$. On the other hand, if $e \leq f$, then $c^\circ d \, \widetilde{\mathcal{R}}_E \, c^\circ ef = c^\circ e = c^\circ \, \widetilde{\mathcal{R}}_E \, 1$. Similarly, we see that $c^\circ d \in \widetilde{L}^1_E$ if and only if $f \leq e$. Statement (2) follows.

(3) Let $u = x_1 x_2 \ldots x_k \in U$, where $x_i \in L \cup R$ for $1 \leq i \leq n$. We show by induction on k that u is E-regular with $u^\circ = x_k^\circ \ldots x_1^\circ$. Clearly this is true for $k = 1$ and we commented above that this is true for $k = 2$.

Suppose now that $k \geqslant 3$ and the result is true for words in U of shorter length. Our inductive hypothesis gives that $x_1 \ldots x_{k-1}$ is E-regular with inverse $x_{k-1}^\circ \ldots x_1^\circ$. Then

$$\begin{aligned}(x_1 \cdots x_k)(x_k^\circ \cdots x_1^\circ)(x_1 \cdots x_k) &= (x_1 \cdots x_{k-1})(x_k x_k^\circ)[(x_{k-1}^\circ \cdots x_1^\circ)(x_1 \cdots x_{k-1})]x_k \\ &= (x_1 \cdots x_{k-1})[(x_{k-1}^\circ \cdots x_1^\circ)(x_1 \cdots x_{k-1})](x_k x_k^\circ)x_k \\ &= x_1 \cdots x_{k-1} x_k\end{aligned}$$

and

$$(x_1 \cdots x_k)(x_k^\circ \cdots x_1^\circ) = x_1(x_2 \cdots x_k x_k^\circ \cdots x_2^\circ)x_1^\circ \in E$$

by induction and hypothesis. Together with the dual argument, we obtain that $u = x_1 \cdots x_k$ is E-regular with $u^\circ = x_k^\circ \cdots x_1^\circ$.

Certainly $E \subseteq E(U)$. To show that U is inverse, we use the fact that S is restriction. Let $e \in E(U)$. Then

$$e^+ = ee^\circ = eee^\circ = ee^+$$

so that using the identity $xy^+ = (xy)^+x$ we have

$$e^+ = ee^+ = (ee)^+e = e^+e = e,$$

so that $E(U) = E$. Hence $E(U)$ is a semilattice and U is inverse.

(4) To see that $U = LR$, let $u \in U$. Since R and L are submonoids, we can write $u = l_1 r_1 l_2 r_2 \cdots l_m r_m$ where $l_1, \ldots, l_m \in L$ and $r_1, \ldots, r_m \in R$ and m is least with respect to such a decomposition of u. If $m \geqslant 2$, then either $r_1 l_2 \in R$ or $r_1 l_2 \in L$, so that as

$$u = l_1(r_1 l_2 r_2) \cdots l_m r_m = (l_1 r_1 l_2) r_2 \cdots l_m r_m$$

we have violated the minimality of m. Hence $m = 1$ and $U = LR$. From Proposition 4.4, U is an $\widetilde{\mathcal{H}}_E$-transversal of S, so that U is an inverse skeleton of S.

□

Example 4.9. Let $S = \mathrm{BR}(M, \theta)$ and put
$$L = \{(m, e, 0) : m \in \mathbb{N}^0\}.$$
We have that L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}_E^1$ consisting of E-regular elements and $S \times S = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. With
$$R = \{(0, e, m) : m \in \mathbb{N}^0\} = \{(m, e, 0)^\circ : m \in \mathbb{N}^0\}$$
we see that $RL \subseteq R \cup L$. Then U defined as in Theorem 4.8 coincides with U as given in Example 3.8.

5. $\widetilde{\mathcal{D}}_E$-simple monoids and Zappa-Szép products

We build on the results of previous sections to show how certain $\widetilde{\mathcal{D}}_E$-simple restriction monoids decompose as Zappa-Szép products of submonoids. In particular, we show how Kunze's [10] result for the Bruck-Reilly extension of a monoid may be put into a general framework.

For the convenience of the reader we begin by recalling the basic definitions relating to Zappa-Szép products.

Definition 5.1. Let U and V be monoids and suppose that we have maps
$$V \times U \to U,\ (t, s) \mapsto t \cdot s \text{ and } V \times U \to V,\ (t, s) \mapsto t^s$$
such that for all $s, s' \in U, t, t' \in V$:

(ZS1) $tt' \cdot s = t \cdot (t' \cdot s)$;	(ZS5) $t \cdot 1_U = 1_U$;
(ZS2) $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$;	(ZS6) $t^{1_U} = t$;
(ZS3) $(t^s)^{s'} = t^{ss'}$;	(ZS7) $1_V \cdot s = s$;
(ZS4) $(tt')^s = t^{t' \cdot s} t'^s$;	(ZS8) $1_V^s = 1_V$.

Define a binary operation on $U \times V$ by
$$(s, t)(s', t') = (s(t \cdot s'), t^{s'} t').$$
Then $U \times V$ is a monoid, most recently referred to as the (external) *Zappa-Szép product of U and V* and denoted by $U \bowtie V$.

It is clear that $U \bowtie V$ contains submonoids $U' = U \times \{1_V\}$ and $V' = \{1_U\} \times V$ such that every element of $U \bowtie V$ has a unique expresssion as $u'v'$ where $u \in U', v \in V'$. Thus $U \bowtie V$ is the internal Zappa-Szép product of U' and V', where we say that a monoid S is the *internal Zappa-Szép product* of submonoids U and V if $S = UV$ and every element of S has a *unique* expression as $uv, u \in U, v \in V$. In this case, writing
$$vu = (v \cdot u)(v^u)$$
we have that U and V act on each other satisfying (ZS1)–(ZS8) and $S \cong U \bowtie V$ under the isomorphism $uv \mapsto (u, v)$ [13].

Note that if one of the above actions is trivial (that is, by identity maps), then the second action is by morphisms, and we obtain the semidirect product $U \rtimes V$ (if U acts trivially) or $U \ltimes V$ (if V acts trivially).

Definition 5.2. Let S be a monoid. We say that S is *special* if there is a submonoid $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}^1_E$ such that every $c \in L$ is E-regular.

Example 5.3. We have observed in Example 4.9 that $S = \mathrm{BR}(M, \theta)$ is special with

$$L = \{(m, e, 0) : n \in \mathbb{N}^0\}$$

being a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$. Moreover, $\widetilde{\mathcal{H}}_E$ is a congruence on S.

Theorem 5.4. *Let S be a weakly E-adequate monoid with (C). Then S is $\widetilde{\mathcal{D}}_E$-simple with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and special if and only if S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$, where L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$.*

Proof. Suppose that S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$, where L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$.

Let $a, b \in S$ and write $a = lr$, $b = l'r'$ where $l, l' \in L$ and $r, r' \in \widetilde{R}^1_E$. Then $lr', l'r \in S$,

$$a = lr \, \widetilde{\mathcal{R}}_E \, lr' \, \widetilde{\mathcal{L}}_E \, l'r' = b$$

and

$$a = lr \, \widetilde{\mathcal{L}}_E \, l'r \, \widetilde{\mathcal{R}}_E \, l'r' = b.$$

Thus $\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = S \times S$. Finally we need to show that L consists of E-regular elements. For this let $l \in L$ and write $l^+ = uv$ where $u \in L$ and $v \in \widetilde{R}^1_E$. Then $u \, \widetilde{\mathcal{R}}_E \, l$ so that $u = l$, since $|L \cap \widetilde{H}^a_E| = 1$ for all $a \in L$.

$\boxed{1}$		$v = l^\circ$	
$\boxed{l = u}$		$l^+ = uv$	

Therefore $l^+ = lv$ and $l = l\,1 = l^+ l = l(vl)$ and $vl \in \widetilde{H}^1_E$ by Lemma 2.2. By uniqueness of factorisation, $vl = 1$. Thus $v = vlv$ and $lv, vl \in E$, so that l is E-regular as required. Thus S is special.

Conversely, suppose that $\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = S \times S$ and S is special. Let $s \in S$. Then $1 \, \widetilde{\mathcal{L}}_E \, l \, \widetilde{\mathcal{R}}_E s$ for some $l \in L$ and as l is E-regular we have $s = l^+ s = ll^\circ s$. Now observe that $l^\circ s \, \widetilde{\mathcal{R}}_E \, l^\circ l = 1$ so that $l^\circ s \in \widetilde{R}^1_E$. To see that this factorisation is unique, suppose that $s = lr = kt$ where $l, k \in L$ and $r, t \in \widetilde{R}^1_E$. Now $\widetilde{\mathcal{R}}_E$ is a left congruence, so that $l \, \widetilde{\mathcal{R}}_E \, k$, giving $l = k$. As l is E-regular, we have $1 = l^\circ l$ and we deduce that $r = t$. Thus S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$. $\square$

We now examine the actions in the situation where the hypotheses of Theorem 5.4 hold. For $r \in \widetilde{R}^1_E$ and $l \in L$ we have

$$rl = (rl)^+ rl = dd^\circ rl$$

where $d \in L$. Observe now that $d^\circ rl \, \widetilde{\mathcal{R}}_E \, d^\circ (rl)^+ = d^\circ dd^\circ = d^\circ \widetilde{\mathcal{R}}_E \, 1$. It follows that

$$r \cdot l = d \text{ and } r^l = d^\circ rl \text{ where } rl \, \widetilde{\mathcal{R}}_E \, d \in L.$$

We explain these actions with the help of an egg-box picture.

$\boxed{1}$		r	$r^l = d^\circ rl$
$\boxed{l}$			
$\boxed{r \cdot l = d}$			rl

We can proceed further in Theorem 5.4 to decompose $\widetilde{R}^1_E$ as a Zappa-Szép product, under the additional hypothesis that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Recall from Theorem 4.8 that this guarantees that $R = \{c^\circ : c \in L\}$ is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^1_E$.

Theorem 5.5. *Let S be a weakly E-adequate monoid with (C) such that S is $\widetilde{\mathcal{D}}_E$-simple with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and special. Suppose in addition that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Then $\widetilde{R}^1_E$ is the internal Zappa-Szép product of $\widetilde{H}^1_E$ and R.*

It follows that $\widetilde{R}^1_E \cong \widetilde{H}^1_E \bowtie R$. Further, if $\widetilde{\mathcal{H}}_E$ is a congruence on S, then the action of $\widetilde{H}^1_E$ on R is trivial and $\widetilde{R}^1_E \cong \widetilde{H}^1_E \rtimes R$.

Proof. Let $t \in \widetilde{R}^1_E$. For $r \in R$ with $r \,\widetilde{\mathcal{H}}_E\, t$, we have $rr^\circ = 1$ and $r^\circ r = f \in E$ and certainly $f \,\widetilde{\mathcal{L}}_E\, r$. From Lemma 2.4, $\rho_r : \widetilde{H}^1_E \to \widetilde{H}^r_E$ is a bijection. Thus every element of $\widetilde{R}^1_E$ has a unique decomposition as hr for some $h \in \widetilde{H}^1_E$ and $r \in R$, that is, $\widetilde{R}^1_E = \widetilde{H}^1_E R$ is the internal Zappa-Szép product of $\widetilde{H}^1_E$ and R.

It follows that $\widetilde{R}^1_E \cong \widetilde{H}^1_E \bowtie R$. We now examine the mutual actions of $\widetilde{H}^1_E$ and R. Let $h \in \widetilde{H}^1_E, r \in R$ and let $t \in R$ be such that $rh \,\widetilde{\mathcal{H}}_E\, t$, so that $rh \,\widetilde{\mathcal{L}}_E\, f = t^\circ t$. Then $rh = (rh)f = (rh)(t^\circ t)$ and $rht^\circ \in \widetilde{H}^1_E$, again by Lemma 2.4. Hence $r \cdot h = rht^\circ$ and $r^h = t$:

$\boxed{1}$ h $r \cdot h = rht^\circ$		$\boxed{r}$	$\boxed{t = r^h}$ rh	
$\boxed{t^\circ}$			$\boxed{t^\circ t}$	

Finally, if $\widetilde{\mathcal{H}}_E$ is congruence, then $rh \,\widetilde{\mathcal{H}}_E\, r1 = r$, so that $t = r$ and $r^h = r$. $\square$

6. Some applications and examples

If S is such that every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element and S has (C), then we have noted in Lemma 2.7 that $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Moreover, if S is special and restriction, then we immediately see from Lemma 2.9 that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. In particular, if S is an inverse monoid, then certainly with $E = E(S)$, S is restriction, every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element and $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (since $\widetilde{\mathcal{K}}_E = \mathcal{K}$, for all relevant K). We thus immediately deduce from Theorems 5.4 and 5.5 the following: notice that we have reverted to the more usual notation of K_a for the $\mathcal{K}$-class of $a \in S$.

Theorem 6.1. *Let S be an inverse monoid. Then S is bisimple and special if and only if S is the internal Zappa-Szép product of L and R_1, where L is a submonoid $\mathcal{H}$-transversal of L_1. Moreover, in this case, R_1 is the internal Zappa-Szép product of H_1 and R where $R = \{r^{-1} : r \in L\}$, and is a semidirect product if $\mathcal{H}$ is a congruence.*

Now we deduce [10, Section 5.4].

Corollary 6.2. *Let $S = BR(M, \theta)$. Then with*

$$L = \{(n, e, 0) : n \in \mathbb{N}^0\} \text{ and } R = \{(0, e, n) : n \in \mathbb{N}^0\}$$

we have that

$$S \cong \mathbb{N}^0 \bowtie (M \rtimes \mathbb{N}^0).$$

Proof. We have observed that S is restriction and special with L and R as given. Moreover, $S \times S = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{H}}_E$ is a congruence. From Theorems 5.4 and 5.5 we have $S \cong L \bowtie (\widetilde{H}^1_E \rtimes R)$ and then as $L \cong \mathbb{N}^0$, $\widetilde{H}^1_E \cong M$ and $L \cong \mathbb{N}^0$, we deduce the result.

We now consider the relevant actions. For $(n, e, 0) \in L$ and $(0, a, m) \in \widetilde{R}^1_E$, with $k = \max(m, n)$ we have

$$(0, a, m)(n, e, 0) = (k - m, a\theta^{k-m}, k - n)$$

so that from the recipe in Theorem 5.4 we have

$$(0, a, m) \cdot (n, e, 0) = (k - m, e, 0) \text{ and } (0, a, m)^{(n,e,0)} = (0, a\theta^{k-m}, k - n).$$

Considering now the action of R on $\widetilde{H}^1_E$ we have

$$(0, e, m) \cdot (0, a, 0) = (0, e, m)(0, a, 0)(m, e, 0) = (0, a\theta^m, 0).$$

Using the natural isomorphisms $(n, e, 0) \mapsto n$, $(0, e, m) \mapsto m$ and $(0, a, 0) \mapsto a$ we have that $\mathbb{N}^0$ acts on S by

$$m \cdot a = a\theta^m$$

giving the semidirect product $S \rtimes \mathbb{N}^0$ and then $S \rtimes \mathbb{N}^0$ and $\mathbb{N}^0$ act on eachother mutually by

$$(a, m) \cdot n = k - m \text{ and } (a, m)^n = (a\theta^{k-m}, k - n).$$

□

Of course, the above can be applied to the bicyclic monoid (with M trivial) or to bisimple inverse ω-semigroups (with M a group).

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