Inverse monoids and immersions of cell complexes

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Immersions

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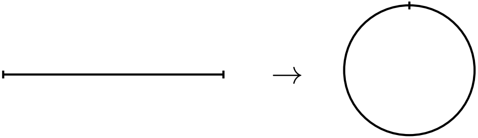

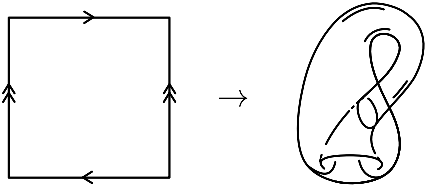

Further examples: coverings

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A covering is a continous map $f\colon Y \to X$ for which there exists an open cover U_α of X such that for each α, $f^{-1}(U_\alpha)$ is a disjoint union of open sets in Y, each of which is mapped homeomorphically onto U_α by f.

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Fact: connected covers of a topological space $\longleftrightarrow$ conjugacy classes of subgroups of its fundamental group.

Fundamental group: homotopy classes of closed paths around a given point, equipped with concatenation

Why this works: for any path p in X and any y point in $f^{-1}(\alpha(p))$, there exists a unique lift of p starting at y.
To characterize f, it suffices to keep track of which **closed** paths lift to **closed** paths, these correspond to a subgroup of the fundamental group.

Immersions — the idea

If $f\colon Y \to X$ is an immersion and p is a path in X, then **if** p lifts at some point in $f^{-1}(\alpha(p))$, **then** p is unique, however, it may be that p doesn't lift or lifts only partially.

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This algebraic structure will be an **inverse monoid** of paths.

Inverse monoids

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A monoid $(S, \cdot)$ is called an inverse monoid if for all $s \in S$ there exists an element $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$, furthermore idempotents commute.

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Natural partial order: $a \leq b$ iff there exists an idempotent e with $a = be$.

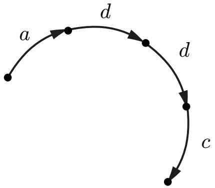
Free inverse monoids exist. (Notation: $\mathrm{FIM}(X)$)

Inverse monoid actions

Definition

An inverse monoid S acts on the set X if there is a homomorphism $S \to \mathrm{SIM}(X)$.

Example: let Γ be a graph edge-labeled in a deterministic and co-deterministic way over a set A, then $\mathrm{FIM}(A)$ acts on $V(\Gamma)$.



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Suppose $x.s = y$. Then

$$s^{-1} \operatorname{Stab}(x) s \subseteq \operatorname{Stab}(y),$$

$$s \operatorname{Stab}(y) s^{-1} \subseteq \operatorname{Stab}(x).$$

In this case, we say $\operatorname{Stab}(x)$ and $\operatorname{Stab}(y)$ are conjugate.

The loop monoid of graphs

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Definition (Margolis, Meakin)

The loop monoid $L(\Gamma, v)$ is the inverse monoid consisting of $\approx$-classes of closed paths around v, with respect to concatenation.

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Note: if Γ be a digraph edge-labeled over the set X in a deterministic and co-deterministic way, then Then **paths starting at v are words over $X \cup X^{-1}$**, hence $L(\Gamma, v) \leq \mathrm{FIM}(X)$, in fact $L(\Gamma, v) = \mathrm{Stab}(v)$ under the action of $FIM(X)$ on Γ.

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Remark: $L(\Gamma, v)$ and $L(\Gamma, v')$ are conjugate, but that doesn't imply isomorphic (unlike in the case of the fundamental group)

The theorem classifying graph immersions

An immersion between graphs: a topological immersion that respects the graph structure.

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Connected immersions over a connected graph $\Gamma \longleftrightarrow$ *conjugacy classes of closed inverse submonoid of* $L(\Gamma, v)$ *for any* $v \in \Gamma$.

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- for any $M \leq^{\omega} L(\Gamma_1, v_1)$, the ω-coset graph of M immerses into Γ_1
- $H, K \subseteq L(\Gamma_1, v_1)$ correspond to the same immersion iff they are conjugate

Immersions in higher dimensions

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1-dimensional CW-complexes = graphs

In a CW-complex $\mathcal{C}$, every cell has an attaching map $\varphi\colon S^n \to \mathcal{C}$ and a characteristic map $\sigma\colon B^n \to \mathcal{C}$.

Immersions in higher dimensions

Δ-complexes: CW-complexes with restricted attaching maps:

Each cell has a distinguished characteristic map $\sigma\colon \Delta^n \to \mathcal{C}$ such that the restriction to a face of Δ^n is also a characteristic map of some cell.

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Fix an ordering $v_0, \ldots, v_n$ on the vertices of Δ^n. (Notation: $\Delta^n = [v_0, \ldots, v_n]$.) We call the smallest vertex v_0 the **root** of the simplex, $\sigma(v_0)$ is called the **root** of the cell, denoted by $\alpha(C)$.

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Immersion between Δ-complexes: a topological immersion that commutes with the characteristic maps.

The loop monoid of a Δ-complex

The idea:
We need an algebraic structure to keep track of which paths and cells lift, and which closed paths lift to closed paths.
Let $\mathcal{C}$ be a Δ-complex.
A **generalized** path in $\mathcal{C}$ is a sequence of **cells** $s_1 \dots s_n$ such that $\omega(s_j) = \alpha(s_{j+1})$.

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We will introduce equivalence relations on generalized paths (in addition to the inverse monoid relations) which reflect the above properties.

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For any n-cell C ($n \geq 2$), we designate the following generalized path on the boundary of C.

- if $n = 2$, let $bw(C)$ be the image of the path (v_0, v_1, v_2, v_0) under σ;
- if $n > 2$, let $C_i = [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$, and let $bw(C)$ be the image of $C_n C_{n-1} \ldots C_1(v_0, v_1)C_0(v_1, v_0)$ under σ.

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Note: $bw(\rho) := \ell(bw(C))$, where $\ell(C) = \rho$, is well-defined.

The loop monoid

Take a Δ-complex labeled over $X \cup P$, and consider the inverse monoid $M_{X,P} = \langle X \cup P \rangle$, defined by the following relations: for any $\rho \in P$,

- $\rho^2 = \rho$, and
- $\rho \leq \ell(bw(\rho))$.

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$L(\mathcal{C}, v) :=$ generalized paths around v wrt the above relations
$=$ the stablizer of v under this action

Note:

- $L(\mathcal{C}, v) \leq^{\omega} M_{X,P}$;
- the greatest group homomorphic image of $L(\mathcal{C}, v)$ is $\pi_1(\mathcal{C})$.

The main theorem

Theorem (Meakin, Sz.)

Connected immersions over a connected Δ-complex $\mathcal{C}$ $\longleftrightarrow$ conjugacy classes of closed inverse submonoid of $L(\mathcal{C}, v)$ for any $v \in \mathcal{C}^0$.

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Remark: the above theorem was proven by Meakin and Sz. for CW-complexes in the 2-dimensional case.

Thank you for your attention!