Free idempotent generated Semigroups II

Dandan Yang
The University of York

Outline

- Summary of results
- A Reidemeister-Schreier type presentation for maximal subgroups
- Singular squares and a presentation for the maximal subgroup of $IG(E)$
- Maximal subgroups of $IG(E)$ arising from full transformation semigroup T_n

Summary of results

Let S be a semigroup, S' the idempotent generated subsemigroup of S and $IG(E)$ the free idempotent generated semigroup over $E(S)$. Then we have the following results:

Summary of results

Let S be a semigroup, S' the idempotent generated subsemigroup of S and $IG(E)$ the free idempotent generated semigroup over $E(S)$. Then we have the following results:

(i) There exists a natural homomorphism $\phi : IG(E) \longrightarrow S'$.

Summary of results

Let S be a semigroup, S' the idempotent generated subsemigroup of S and $IG(E)$ the free idempotent generated semigroup over $E(S)$. Then we have the following results:

(i) There exists a natural homomorphism $\phi : IG(E) \longrightarrow S'$.

(ii)$E(IG(E)) \cong E$.

Summary of results

Let S be a semigroup, S' the idempotent generated subsemigroup of S and $IG(E)$ the free idempotent generated semigroup over $E(S)$. Then we have the following results:

(i) There exists a natural homomorphism $\phi : IG(E) \longrightarrow S'$.

(ii)$E(IG(E)) \cong E$.

(iii)ϕ maps the $\mathcal{R}$-class(resp. $\mathcal{L}$-class)of $e \in E$ onto the corresponding class of e in S'; this induces a bijection between the set of all $\mathcal{R}$-classes(resp. $\mathcal{L}$-classes) in the $\mathcal{D}$-class of e in $IG(E)$ and the corresponding set in S'.

Summary of results

Let S be a semigroup, S' the idempotent generated subsemigroup of S and $IG(E)$ the free idempotent generated semigroup over $E(S)$. Then we have the following results:

(i) There exists a natural homomorphism $\phi : IG(E) \longrightarrow S'$.

(ii)$E(IG(E)) \cong E$.

(iii)ϕ maps the $\mathcal{R}$-class(resp. $\mathcal{L}$-class)of $e \in E$ onto the corresponding class of e in S'; this induces a bijection between the set of all $\mathcal{R}$-classes(resp. $\mathcal{L}$-classes) in the $\mathcal{D}$-class of e in $IG(E)$ and the corresponding set in S'.

(iv) The restriction of ϕ to the maximal subgroup of $IG(E)$ containing $e \in E$ (i.e. to the $\mathcal{H}$-class of e in $IG(E)$) is a homomorphism onto the maximal subgroup of S' containing e.

A Reidemeister-Schreier type presentation for maximal subgroups

Let S be a semigroup, H the maximal subgroup of S with identity e, and let $R = \{H_j : j \in J\}$ be the $\mathcal{R}$-class of e. Here we suppose that $1 \in J$ and use H_1 to denote the $\mathcal{H}$-class of e.

A Reidemeister-Schreier type presentation for maximal subgroups

Let S be a semigroup, H the maximal subgroup of S with identity e, and let $R = \{H_j : j \in J\}$ be the $\mathcal{R}$-class of e. Here we suppose that $1 \in J$ and use H_1 to denote the $\mathcal{H}$-class of e.

Recall that for any elements $s, t \in S$ such that $st\mathcal{R}s$, the mapping $\rho_t : x \mapsto xt$ is an $\mathcal{H}$-class preserving bijection between the $\mathcal{L}$-class L_s and L_{st}. Furthermore, if $stu = s$, then the mapping ρ_t and $\rho_u : L_{st} \to L_s$, $x \mapsto xu$, are mutually inverse bijections.

A Reidemeister-Schreier type presentation for maximal subgroups

Let S be a semigroup, H the maximal subgroup of S with identity e, and let $R = \{H_j : j \in J\}$ be the $\mathcal{R}$-class of e. Here we suppose that $1 \in J$ and use H_1 to denote the $\mathcal{H}$-class of e.

Recall that for any elements $s, t \in S$ such that $st\mathcal{R}s$, the mapping $\rho_t : x \mapsto xt$ is an $\mathcal{H}$-class preserving bijection between the $\mathcal{L}$-class L_s and L_{st}. Furthermore, if $stu = s$, then the mapping ρ_t and $\rho_u : L_{st} \to L_s$, $x \mapsto xu$, are mutually inverse bijections.

Now we define an action of S on $J \cup \{0\}$ by $(j, s) \mapsto j.s = l$, if $l, j \in J$ and $H_j s = H_l$; otherwise, $j.s = 0$.

A Reidemeister-Schreier type presentation for maximal subgroups

Suppose $S = \langle A \rangle$. Let $r_j (j \in J)$ be the elements of S^1 such that $H_1 r_j = H_j$ (or $1.r_j = j$) for all $j \in J$. Then $\exists r'_j$ such that $h r_j r'_j = h$ and $h' r'_j r_j = h'$, for all $h \in H_1$ and $h' \in H_j$.

A Reidemeister-Schreier type presentation for maximal subgroups

Suppose $S = \langle A \rangle$. Let $r_j (j \in J)$ be the elements of S^1 such that $H_1 r_j = H_j$ (or $1.r_j = j$) for all $j \in J$. Then $\exists r_j'$ such that $hr_j r_j' = h$ and $h' r_j' r_j = h'$, for all $h \in H_1$ and $h' \in H_j$.

It was proved that a generating set for H is given by

$$\{ e r_j a r_{j.a}' : j \in J, a \in A, j.a \neq 0 \}.$$

A Reidemeister-Schreier type presentation for maximal subgroups

Suppose $S = \langle A \rangle$. Let $r_j (j \in J)$ be the elements of S^1 such that $H_1 r_j = H_j$ (or $1.r_j = j$) for all $j \in J$. Then $\exists r'_j$ such that $hr_j r'_j = h$ and $h' r'_j r_j = h'$, for all $h \in H_1$ and $h' \in H_j$.

It was proved that a generating set for H is given by

$$\{er_j a r'_{j.a} : j \in J, a \in A, j.a \neq 0\}.$$

Next, we let S be a presentation given by $S = \langle A | R \rangle$, and $e, r_j, r'_j \in A^*$. For convenience, we introduce a new alphabet

$$B = \{[j, a] : j \in J, a \in A, j.a \neq 0\}$$

representing the generators above.

A Reidemeister-Schreier type presentation for maximal subgroups

Define a rewriting mapping:

$$\phi : \{(j, w) : j \in J, w \in A^*, j.w \neq 0\} \longrightarrow B^*$$

inductively by

$$\phi(j, \varepsilon) = \varepsilon, \phi(j, aw) = [j, a]\phi(j.a, w).$$

A Reidemeister-Schreier type presentation for maximal subgroups

Define a rewriting mapping:

$$\phi : \{(j, w) : j \in J, w \in A^*, j.w \neq 0\} \longrightarrow B^*$$

inductively by

$$\phi(j, \varepsilon) = \varepsilon, \phi(j, aw) = [j, a]\phi(j.a, w).$$

Suppose that the words $r_j (j \in J)$ form a Schreier system, i.e. every prefix of every r_j is equal to some other r_k. Then H is defined by the presentation with generators:

$$B = \{[j, a] : j \in J, a \in A, j.a \neq 0\}$$

and the defining relations:

$$[j, a] = 1 \qquad (j \in J, a \in A, j.a \neq 0, r_{j.a} = r_j a),$$

$$\phi(j, u) = \phi(j, v) \qquad (j \in J, (u = v) \in R, j.u \neq 0)$$

Singular squares and a presentation for the maximal subgroup of $IG(E)$

Let $S = \langle E(S) \rangle$, $e_{11} \in E$, and $H = H(IG(E), e_{11})$.

Remark: The action of any generator $e \in E$ on the $\mathcal{H}$-class of $R_{e_{11}}$ in $IG(E)$ is equivalent to the action of e on the $\mathcal{H}$-class of $R_{e_{11}}$ in S.

Singular squares and a presentation for the maximal subgroup of $IG(E)$

Let $S = \langle E(S) \rangle$, $e_{11} \in E$, and $H = H(IG(E), e_{11})$.

Remark: The action of any generator $e \in E$ on the $\mathcal{H}$-class of $R_{e_{11}}$ in $IG(E)$ is equivalent to the action of e on the $\mathcal{H}$-class of $R_{e_{11}}$ in S.

Let D be the $\mathcal{D}$-class of e_{11}, $R_i (i \in I)$ be the $\mathcal{R}$-classes contained in D, $L_j (j \in J)$ be the $\mathcal{L}$-classes contained in D, and $H_{i,j} = R_i \cap L_j$, $j \in J, i \in I$.

Singular squares and a presentation for the maximal subgroup of $IG(E)$

Let $S = \langle E(S) \rangle$, $e_{11} \in E$, and $H = H(IG(E), e_{11})$.

Remark: The action of any generator $e \in E$ on the $\mathcal{H}$-class of $R_{e_{11}}$ in $IG(E)$ is equivalent to the action of e on the $\mathcal{H}$-class of $R_{e_{11}}$ in S.

Let D be the $\mathcal{D}$-class of e_{11}, $R_i (i \in I)$ be the $\mathcal{R}$-classes contained in D, $L_j (j \in J)$ be the $\mathcal{L}$-classes contained in D, and $H_{i,j} = R_i \cap L_j$, $j \in J, i \in I$.

Let $K = \{(i,j) \in I \times J : H_{ij} \text{ is a group}\}$, and $e_{ij} \in H_{ij}$, for $(i,j) \in K$.

Let R_1 be the $\mathcal{R}$-class of e_{11} contained in D and $H_j = H_{1j}$. Let $r_j \in E^*$ $(j \in J)$ be a Schreier system. It was proved that every element of D can be expressed as a product of idempotents from D of the form $e_{i_1 j_1} e_{i_2 j_2} \ldots e_{i_n j_n}$ such that $(i_{q+1}, j_q) \in K$, $(q = 1, \ldots, n-1)$. So we can choose r_j entirely of such products.

Singular squares and a presentation for the maximal subgroup of $IG(E)$

Recall that $IG(E) = \langle E : e.f = ef, (e, f) \text{ is a basic pair}\rangle$. Hence, the maximal subgroup H with identity e_{11} of $IG(E)$ is given by the presentation with generators:

$$B = \{[j, e] : j \in J, e \in E, j.e \neq 0\}$$

and the defining relations:

$$[j, e] = 1 \qquad (j \in J, e \in E, j.a \neq 0, r_{j.e} = r_j e),$$

$$[j, ef] = [j, e][j.e, f] \qquad (j \in J, (e, f) \text{ is a basic pair}, [j, ef] \neq 0).$$

Singular squares and a presentation for the maximal subgroup of $IG(E)$

Finally, the author introduced the concept of singular squares.

Singular squares and a presentation for the maximal subgroup of $IG(E)$

Finally, the author introduced the concept of singular squares.

A quadruple $(i, k; j, l) \in I \times I \times J \times J$ is a square if $(i, j), (i, l), (k, j), (k, l) \in K$. It is a singular square if there exists $e \in E$ s.t. one of the following dual conditions holds:

$$ee_{ij} = e_{ij}, ee_{kj} = e_{kj}, e_{ij}e = e_{il}, e_{kj}e = e_{kl}$$

or

$$e_{ij}e = e_{ij}, e_{il}e = e_{il}, ee_{ij} = e_{kj}, ee_{il} = e_{kl}.$$

Singular squares and a presentation for the maximal subgroup of $IG(E)$

For every $i \in I$, fix $j(i) \in J$ such that $(i, j(i)) \in K$. Then by using the properties of singular squares, the author obtained an equivalent presentation of H_{11} to the presentation above.

$$B = \{f_{ij} : (i,j) \in K\}$$

and the defining relations:

$$f_{ij} = f_{il} \ ((i,j),(i,l) \in K, r_j e_{il} = r_{j.e_{il}}),$$

$$f_{i,j(i)} = 1 \qquad (i \in I).$$

$$f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl} \quad ((i,k;j,l) \in \Sigma)$$

Where $f_{ij} = [j(i), e_{ij}]$.