A common framework for restriction semigroups and regular $*$-semigroups.

Peter Jones
Marquette University

At The University of York
November 22, 2010

Background on left restriction semigroups, *aka* **weakly left E-ample semigroups.**

One view: consider the semigroup of partial transformations $\mathcal{PT}_X$ on a set as a unary semigroup under the additional unary operation $^+$, where $\alpha^+ = 1_{\mathsf{dom}\alpha}$. The left restriction semigroups are the abstractions of the (unary) semigroups of partial transformations. Notice that the set E of partial identity maps is a semilattice that is a proper subset of the set of idempotents of $\mathcal{PT}_X$.

An alternative view is that S is a semigroup with a designated subsemilattice E of idempotents, every element a is $\widetilde{\mathcal{R}}_E$ - related to a unique member a^+ of E (S is weakly left E-adequate), $\widetilde{\mathcal{R}}_E$ is a left congruence and the left ample condition $ae = (ae)^+a$ is satisfied for all $e \in E$.

Either way, as unary semigroups they are defined by the identities [Cockett and Lack, 2002; Gould "notes" 2009]:

$$x^+x = x, \quad x^+x^+ = x^+, \quad (xy)^+ = (xy^+)^+,$$

$$x^+y^+ = y^+x^+, \quad xy^+ = (xy)^+x.$$

Within this variety is the quasivariety of left ample semigroups – where $\widetilde{\mathcal{L}}_E = \mathcal{L}^*$ – which correspond to the (unary) semigroups of one-one transformations.

Every inverse semigroup $(S, \cdot, ^{-1})$ induces a left restriction semigroup by setting $a^+ = aa^{-1}$; and dually, it induces a right restriction semigroup by setting $a^* = a^{-1}a$.

A restriction semigroup is a 'bi-unary' semigroup $(S, \cdot, ^+, ^*)$, the operations being attached to a common subsemilattice E. So every inverse semigroup induces a restriction semigroup.

At the opposite extreme, every monoid $(S, \cdot, 1)$ induces a restriction semigroup by setting $a^+ = 1 = a^*$.

Generalizing restriction semigroups.

First of all, we want to retain ‘adequacy’. In the past, this was approached by allowing E to be a band instead of a semilattice.

Rather than using E itself as the focus, we consider semigroups obtained by inducing one or both of the operations $a^{+} = aa^{-1}$ and $a^{*} = a^{-1}a$ from a ‘nice’ class of semigroups endowed with an inversion operation.

Now E is just the set of ‘projections’, so we prefer to denote it P_S.

A *regular $*$-semigroup* [Nordahl and Scheiblich, 1978] is a semigroup $(S, \cdot, ^{-1})$ with a regular involution:

$$xx^{-1}x = x, \quad x^{-1}xx^{-1} = x^{-1}$$

$$(x^{-1})^{-1} = x, \quad (xy)^{-1} = y^{-1}x^{-1}.$$

Induced operations are obtained as above. Now $P_S = \{a^+ : a \in S\} = \{a^* : a \in S\}$ is the usual set of projections, in the standard terminology.

The induced unary semigroup $(S, \cdot, ^+)$ satisfies:

$x^+x = x, \quad x^+x^+ = x^+, \quad (xy)^+ = (xy^+)^+,$
$(x^+y)^+ = x^+y^+x^+.$

The last identity is purely a consequence of the involutory property.

I call such unary semigroups *left P-Ehresmann semigroups*, modelled on the term for left restriction semigroups 'minus the left ample condition'. The right P-Ehresmann semigroups are the duals $(S, \cdot, ^*)$.

P-Ehresmann semigroups are the obvious bi-unary semigroups $(S, \cdot, ^+, ^*)$.

The bi-unary semigroup $(S, \cdot, ^+, ^*)$ induced from a regular $*$-semigroup $(S, \cdot, ^{-1})$ also satisfies the 'generalized left and right ample' identities

$$(xy)^+x = xy^+x^*, \quad x(yx)^* = x^+y^*x.$$

Again, these are consequences of the involutory property only.

A *P-restriction semigroup* is a P-Ehresmann semigroup that, in addition, satisfies the generalized ample identities.

Projection algebras and the generalized Munn representation

There are useful one-sided versions of the Munn representation for P-Ehresmann semigroups, but the best outcome is for P-restriction semigroups.

The Munn representation for an *inverse* semigroup $(S, \cdot, ^{-1})$ maps each $a \in S$ to the isomorphism

$$\theta_a : (aa^{-1})\!\downarrow \longrightarrow (a^{-1}a)\!\downarrow$$

between principal ideals of the semilattice E_S that is defined by conjugation:

$$e\theta_a = a^{-1}ea = (ea)^{-1}ea.$$

This definition has a natural interpretation in the induced restriction semigroup $(S\cdot,^+,^*)$, which also makes sense for P-restriction semigroups: define

$$\theta_a : a^+\!\downarrow \longrightarrow\ a^*\!\downarrow$$

by

$$e\theta_a = (ea)^*.$$

Then θ_a is an order isomorphism between principal ideals of the poset of projections, with inverse ϕ_a, given by

$$f\phi_a = (af)^+$$

.

The projection algebra of a P-restriction semigroup.

In the case of inverse semigroups (and of restriction semigroups), the Munn semigroup comprises isomorphisms between principal ideals, and it is a subsemigroup of the symmetric inverse semigroup on the semilattice E.

- In the case of P-restriction semigroups, the operation cannot be composition, because this would force projections to commute.

- How can θ_a and ϕ_a be regarded as isomorphisms? How can the set of projections be regarded as an algebra?

The set P_S of projections on any left P-Ehresmann semigroup $(S, \cdot, ^+)$ is turned into a 'left projection algebra' $(P_S, \times)$ by virtue of the last defining identity: put

$$e \times f = efe.$$

Dually, on any right P-Ehresmann semigroup $(S, \cdot, ^*)$, $(P_S, \star)$ becomes a right projection algebra under:

$$e \star f = fef.$$

And on a (two-sided) P-Ehresmann semigroup, $(P_S, \times, \star)$ becomes a 'projection algebra'.

Here are the axioms for a right projection algebra:

1. $e \star e = e$;

2. $(f \star e) \star e = e \star (f \star e) = f \star e$;

3. $g \star (f \star e) = ((g \star e) \star f) \star e$;

4. $(g \star f) \star e = ((g \star f) \star e) \star (f \star e)$.

Let $(P, \times, \star)$ be a projection algebra.

The 'generalized Munn semigroup', T_P, consists of the isomorphisms between the principal ideals of P.

Its binary operation is a 'sandwich' product in the symmetric inverse semigroup on P.*

*If α has range $f\!\downarrow$ and β has domain $g\!\downarrow$, then $\alpha \star \beta$ is the product $\alpha\pi_{g,f}\beta$ in the symmetric inverse semigroup, where

$$\pi_{g,f} : (g \star f)\!\downarrow \to (f \star g)\!\downarrow,$$

$$e\pi_{g,f} = e \star f.$$

Theorem. If P is a projection algebra, then T_P is a fundamental, regular $*$-semigroup whose projection algebra is isomorphic with P.

Corollary. The projection algebras of P-Ehresmann semigroups, P-restriction semigroups and regular $*$-semigroups form the same class of algebras.

Theorem. For any P-restriction semigroup $(S, \cdot, {}^+, {}^*)$, the map

$$\theta : a \to \theta_a$$

defines a $({}^+, {}^*)$-preserving representation onto a full subsemigroup of T_{P_S}, which induces an isomorphism between their respective projection algebras.

If S is the P-restriction semigroup induced from a regular $*$-semigroup, then θ preserves the operation of inversion. (Cf [Imaoka, 1981], [Nambooripad and Pastijn, 1981])

Projection sets and adequacy.

Projection algebras provide an abstract characterization of the sets of projections of left, right, or two-sided P-Ehresmann semigroups.

To describe left P-Ehresmann semigroups in terms of 'adequacy', an internal characterization of the sets of projections is needed.

Theorem. A semigroup S is a left P-Ehresmann semigroup if and only if it is weakly left P-adequate with respect to a set P of idempotents that satisfies

$$(ef)^2 = ef \text{ and } efe \in P, \text{ for all } e, f \in P,$$

and the relation $\widetilde{\mathcal{R}}_P$ is a left congruence.

If S is a left P-Ehresmann semigroup and P_S is a band, then it is a right regular band and the operation on the projection algebra is just the product in the band. This occurs if, for example, $P_S = E_S$, or if S satisfies the left ample identity.

So an E-Ehresmann semigroup is just a P-Ehresmann semigroup with P_S a subsemilattice; and a restriction semigroup is a P-restriction semigroup with P_S a subsemilattice.

References:

A common framework for regular $*$-semigroups and restriction semigroups, submitted.

Weak P-abundancy and P-adequacy: a general analysis in varietal terms, manuscript.