Zappa-Szép products of groups and semigroups

Rida-E-Zenab
University of York

Contents

Contents

Contents

- Introduction
- History and definitions
- Our progress

Introduction

Zappa-Szép products (also known as *knit products*) is a natural generalization of a semidirect product, whereas, a semidirect product is a natural generalization of a direct product.

Zappa-Szép product tells us how to construct a group from its two subgroups.

Examples

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Examples

- Hall's Theorem is an important example of Zappa-Szép product which shows that every soluble group is a Zappa-Szép product of a Hall p-subgroup and a Sylow p-subgroup.
- A nilpotent group G of class at most 2 can form the Zappa-Szép product $P = G \bowtie G$ with the left and right conjugation actions of G on itself.
- A general linear group $G = GL(n, C)$ of invertible $n \times n$ matrices over the field of complex numbers is the Zappa-Szép product of unitary group $U(n)$ and the group of upper triangular matrices with positive diagonal entries.

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- Zappa-Szép products in semigroup theory were introduced by *M. Kunze* in 1983.

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- In 2007 *M. Lawson* studied Zappa-Szép product of a free monoid and a group from the view of self similar group action and completely determined their structure.
- Recently *Suha Wazzan* studied Zappa-Szép products from the view of regular and inverse semigroups. She generalized some results of Lawson and found necessary and sufficient conditions for the Zappa-Szép products of regular and inverse semigroups to be regular and inverse.

Semidirect product

Definition

Let S and T be semigroups. T is said to act on S by endomorphisms if for every $t \in T$, there is a map $s \to t \cdot s$ from S to itself satisfying the following two axioms for all $t, t' \in T$ and for all $s, s' \in S$:

(1) $t \cdot (ss') = (t \cdot s)(t \cdot s')$;

(2) $tt' \cdot s = t \cdot (t' \cdot s)$.

If T is a monoid having identity 1, then the following condition also holds:

(3) $1 \cdot s = s$ for all $s \in S$.

These three axioms are equivalent to the existence of a homomorphism from T to the monoid of endomorphisms of S. Thus

$$S \rtimes T = \{(s, t) : s \in S,\ t \in T\}$$

is the *semidirect product* with multiplication

$$(s, t)(s', t') = (s(t \cdot s'), tt').$$

Reverse semidirect product

Dually we have *reverse semidirect product* when S acts on the right on T by endomorphisms; that is for every $s \in S$, there is a map $t \to t^s$ from T to itself satisfying above three axioms. Thus

$$S \ltimes T = \{(s, t) : s \in S,\ t \in T\}$$

is reverse semidirect product with multiplication

$$(s, t)(s', t') = (ss', t^{s'} t').$$

Zappa-Szép product of semigroups

The construction of *Zappa-Szép product* involves both semidirect and reverse semidirect product.
Let S and T be semigroups and suppose that we have maps

$$T \times S \to S,\ (t,s) \mapsto t \cdot s$$
$$T \times S \to T,\ (t,s) \mapsto t^s$$

such that for all $s, s' \in S, t, t' \in T$, the following hold:
(ZS1) $tt' \cdot s = t \cdot (t' \cdot s)$;
(ZS2) $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$;
(ZS3) $(t^s)^{s'} = t^{ss'}$;
(ZS4) $(tt')^s = t^{t' \cdot s} t'^s$.
Define a binary operation on $S \times T$ by

$$(s,t)(s',t') = (s(t \cdot s'), t^{s'} t').$$

Then $S \times T$ is a semigroup, known as the Zappa-Szép product of S and T and denoted by $S \bowtie T$.

Zappa-Szép product of monoids

If S and T are monoids then we insist that the following four axioms also hold:

(ZS5) $t \cdot 1_S = 1_S$;
(ZS6) $t^{1_S} = t$;
(ZS7) $1_T \cdot s = s$;
(ZS8) $1_T^s = 1_T$.

Then $S \bowtie T$ is monoid with identity $(1_S, 1_T)$.

M. Kunze has recorded following properties of Zappa-Szép product of monoids.

Theorem

Let $M = S \bowtie T$ be a Zappa-Szép product of S and T. Then for $s_1, s_2 \in S, t_1, t_2 \in T$

- $(s_1, t_1) \mathcal{R} (s_2, t_2) \Rightarrow s_1 \mathcal{R} s_2$ in S.

Suha has proved that if $Z = M \bowtie G$ is a Zappa-Szép product of a monoid M and a group G, then

$$(a, b) \mathcal{R} (c, d) \text{ in } Z \Leftrightarrow a \mathcal{R} c \text{ in } M.$$

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- $(s_1, t_1) \leq_{\mathcal{R}} (s_2, t_2) \Rightarrow s_1 \leq_{\mathcal{R}} s_2$ in S.

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Definition

Let S be a semigroup and $a, b \in S$. The relation $\mathcal{R}^*$ is defined by the rule that $a \mathcal{R}^* b$ if and only if

$$xa = ya \Leftrightarrow xb = yb$$

for all $x, y \in S^1$.

The relation $\mathcal{L}^*$ is defined dually.

Proposition

Let S, T be monoids and $Z = S \bowtie T$ be Zappa-Szép product of S and T. Then

- $(a, b) \mathcal{R}^* (c, d)$ in Z implies $a \mathcal{R}^* c$ in S.

The Green's* Relations

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Let S be a semigroup and $a, b \in S$. The relation $\mathcal{R}^*$ is defined by the rule that $a \, \mathcal{R}^* \, b$ if and only if

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- $(a, b) \, \mathcal{R}^* \, (c, d)$ in Z implies $a \, \mathcal{R}^* \, c$ in S.
- $(a, b) \, \mathcal{L}^* \, (c, d)$ in Z implies $b \, \mathcal{L}^* \, d$ in T.

The Green's* Relations

Our question was that if S and T are semigroups and $a\,\mathcal{R}^*\,c$ in S, then is it true that $(a,b)\,\mathcal{R}^*\,(c,d)$ in $S \bowtie T$.

Theorem

Let $Z = S \bowtie T$ be Zappa-Szép product of semigroups S and T where T is right cancellative. Suppose S acts faithfully on the right of T. Suppose also that if $a\,\mathcal{R}^*\,c$ in S, then $\ker a = \ker c$. Then $a\,\mathcal{R}^*\,c$ in S implies that $(a,b)\,\mathcal{R}^*\,(c,d)$ in Z.

The relations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$

Definition

Let S be a semigroup and E be set of idempotents. For $a \in S$ and all $e \in E$, the relation $\widetilde{\mathcal{R}}_E$ is defined by $a \widetilde{\mathcal{R}}_E b$ if and only if

$$ea = a \Leftrightarrow eb = b.$$

The relation $\widetilde{\mathcal{L}}_E$ is dual.

$\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are equivalence relations.

Note that $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$.

The relations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$

Proposition

Let $Z = S \bowtie T$ be Zappa-Szép product of S and T where S, T are monoids. Then

- $(a, b)\,\widetilde{\mathcal{R}}_{F_1}\,(c, d)$ in Z if and only if $a\,\widetilde{\mathcal{R}}_E\,c$ in S for $E \subseteq E(S)$;
- $(a, b)\,\widetilde{\mathcal{L}}_{F_2}\,(c, d)$ in Z if and only if $b\,\widetilde{\mathcal{L}}_E\,d$ in T for $E \subseteq E(T)$,

where $F_1 = \{(e, 1) : e \in E \subseteq E(S)\}$ and $F_2 = \{(1, e) : e \in E \subseteq E(T)\}$ are set of idempotents in Z.

Definition

A semigroup S with distinguished semilattice E is called *left restriction* if the following hold:

- E is a semilattice;

Right restriction semigroups are defined dually.

A semigroup is *restriction* if it is left and right restriction with respect to the same distinguished samilattice.

Zappa-Szép product and restriction semigroups

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- every $\widetilde{\mathcal{R}}_E$ class contains an idempotent of E,
- the relation $\widetilde{\mathcal{R}}_E$ is a left congruence and
- the left ample condition holds, that is, for all $a \in S$ and $e \in E$,

$$ae = (ae)^+ a.$$

Right restriction semigroups are defined dually.

A semigroup is *restriction* if it is left and right restriction with respect to the same distinguished samilattice.

Theorem A

Let S be a left restriction semigroup and $E = \{a^+ : a \in S\}$, the distinguished set of idempotents. Define an action of S on E by $s \cdot e = (se)^+$ and an action of E on S by $s^e = se$, Then $Z = E \bowtie S$ is Zappa-Szép product.

We wanted to know that what are idempotents of this Zappa-Szép product. So we have the following result.

Theorem

Suppose $Z = E \bowtie S$ is a Zappa-Szép product of a restriction semigroup S and distinguished set of idempotents E under the actions defined in Theorem A. Then

$$E(Z) = \{(e, s) : e \leq s^+, s = ses\}.$$

Also $\overline{E} = \{(e, e) : e \in E\}$ is a semilattice isomorphic to E and if $E(S) = E$, then $\overline{E} = E(Z)$.

Zappa-Szép product and restriction semigroups

Theorem B

Suppose $Z = E \bowtie S$ is a Zappa-Szép product of a restriction semigroup S and distinguished set of idempotents E under the action defined in Theorem A. Then:

- $\overline{E} = \{(e, e) : e \in E\}$ is a semilattice isomorphic to $E(S)$;

Zappa-Szép product and restriction semigroups

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- the map $\alpha : Z \to S$ separates the idempotents of $\overline{E}$;

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- $\overline{E} = \{(e, e) : e \in E\}$ is a semilattice isomorphic to $E(S)$;
- the map $\alpha : Z \to S$ separates the idempotents of $\overline{E}$;
- $(g, g)(e, s) = (e, s)$ for some $(g, g) \in \overline{E}$ if and only if $ge = e$ and $es = s$; in this case $(e, s)\, \widetilde{\mathcal{R}}_{\overline{E}}\, (e, e)$;

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- $(e,s)(f,f) = (e,s)$ for some (f,f) if and only if $e \leq s^+, s = sf$ for some $f \in E(S)$, and then

$$(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f) \Leftrightarrow s\,\widetilde{\mathcal{L}}_E\,f;$$

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- $(e,s)(f,f) = (e,s)$ for some (f,f) if and only if $e \leq s^+, s = sf$ for some $f \in E(S)$, and then

$$(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f) \Leftrightarrow s\,\widetilde{\mathcal{L}}_{E}\,f;$$

- $(e,e)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f)$ for some $e,\ f \in E$ implies $(e,s) = (s^+, s)$.

Zappa-Szép product and restriction semigroups

Further, $U = \{(s^+, s) : s \in S\} \cong S$.

Now our question was that is this Zappa-Szép product left restriction? Unfortunately it is not. But we found a subsemigroup of this Zappa-Szép product which is left restriction.

Theorem

Suppose $Z = E \bowtie S$ is a Zappa-Szép product of a restriction semigroup S and distinguished set of idempotents E under the actions defined in Theorem A and let $T = \{(e, s) : s^+ \leq e\} = \{(e, s) : es = s\}$. Then T is left restriction subsemigroup of Z with $(e, s)^+ = (e, e)$.

The Bruck-Reilly extension of a monoid

Kunze discovered that the Bruck-Reilly extension $BR(S,\theta)$ is the Zappa-Szép product of $(\mathbb{N},+)$ and semidirect product, $\mathbb{N} \rtimes S$, where multiplication in $\mathbb{N} \rtimes S$ is defined by the following rule:

$$(k,s)\cdot(l,t) = (k+l,(s\theta^{l})t).$$

Define for $m \in \mathbb{N}$ and $(l,s) \in \mathbb{N} \rtimes S$

$$(l,s)\cdot m = (g-m, s\theta^{g-l}) \text{ and } m^{(l,s)} = g-l$$

where g is greater of m and l. Then $(\mathbb{N} \rtimes S) \times \mathbb{N}$ is Zappa-Szép product with composition rule

$$[(k,s),m]\circ[(l,t),n] = [(k-m+g, s\theta^{g-m}t\theta^{g-l}), n-l+g],$$

where again g is greater of m and l.

Applications to semigroups

We would like to understand this result in terms of Green's relations, in order to generalize it to arbitrary *bisimple inverse monoids*. Here is the result specialized to *bicyclic semigroup*.

Theorem

The *bicyclic semigroup B* can be seen as Zappa-Szép product of $\mathcal{L}$-classes and $\mathcal{R}$-classes, where

$$L = \{(m, 0) : m \in \mathbb{N}^0\} \cong \mathbb{N}^0$$

$$R = \{(0, n) : n \in \mathbb{N}^0\} \cong \mathbb{N}^0$$

The actions of R on L and L on R are defined respectively as:

$$(0, m) \cdot (n, 0) = (max(m, n) - m, 0)$$

and

$$(0, m)^{(n,o)} = (0, max(m, n) - n)$$

Applications to semigroups

More generally we have the following nice result in which we have seen a combinatorial bisimple inverse monoid as Zappa-Szép product of an $\mathcal{L}$-class and an $\mathcal{R}$-class.

Theorem

Suppose S is combinatorial bisimple inverse monoid. Let L be $\mathcal{L}$-class of identity and R be $\mathcal{R}$-class of identity. Then $Z = L \bowtie R$ is Zappa-Szép product of L and R under the actions defined by:

$$r \cdot l = c \text{ where } c^+ = (rl)^+$$

and

$$r^l = d \text{ where } d^* = (rl)^*$$

for $l \in L$ and $r \in R$.

Also $Z \cong S$.