A RANK FOR RIGHT CONGRUENCES ON INVERSE SEMIGROUPS

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Abstract. The *sandwich rank*, or S-rank, of a right congruence ρ on a semigroup S is the Cantor-Bendixson rank of ρ in the lattice of right congruences $\mathcal{RC}$ of S with respect to a topology we call the *finite type topology*. If every $\rho \in \mathcal{RC}$ possesses S-rank, then S is *ranked*. It is known that every right noetherian semigroup is ranked and every ranked inverse semigroup is weakly right noetherian. Moreover, if S is ranked, then so is every maximal subgroup of S. We show that a Brandt semigroup $\mathcal{B}^0(G,I)$ is ranked if and only if G is ranked and I is finite.

The bicyclic monoid B is certainly inverse, weakly right noetherian, and possesses trivial (hence ranked) subgroups. We show that nevertheless B is not ranked. Consequently, a ranked semigroup cannot contain a bicyclic $\mathcal{J}$-class.

Our notion of rank arose from considering stability properties of the theory T_S of existentially closed (right) S-sets over a right coherent monoid S. The property of right coherence guarantees that the existentially closed S-sets form an axiomatisable class. We argue that B is right coherent. Consequently, T_B is a theory of B-sets that is superstable but not totally transcendental.

1. Introduction

Let S be a semigroup; we denote by $\mathcal{RC} = \mathcal{RC}_S$ the lattice of right congruences of S. The notion of S-rank for $\rho \in \mathcal{RC}$ was introduced in [7], in an attempt to find a property intermediate between that of being right noetherian and of being weakly right noetherian. Recall that a semigroup is *right noetherian* if every right congruence is finitely generated, and *weakly right noetherian* if every right ideal is finitely generated. By considering Rees right congruences, it is easy to see that a right noetherian semigroup is weakly right noetherian. The converse is, however, far from true; to see this the reader need only consider a group a having a subgroup lattice that does not have the ascending chain condition.

The S-rank of $\rho \in \mathcal{RC}$ is the Cantor-Bendixson rank of ρ with respect to the finite type topology. The topology and the calculation of rank is given explicitly

Date: November 8, 2006.

1991 *Mathematics Subject Classification.* 20M30.

Key words and phrases. semigroup, monoid, right congruence, chain conditions, S-rank, superstable, totally transcendental.

in Section 2. For the purposes of this introduction, it is enough to say that the S-rank of $\rho \in \mathcal{RC}$ is an ordinal, or ∞ (considered to be greater than every ordinal), that measures how far ρ is from being finitely generated and finitely covered in $\mathcal{RC}$. A semigroup is *ranked* if every $\rho \in \mathcal{RC}$ has ordinal S-rank. We showed in [7] that a right noetherian semigroup is ranked, and a ranked inverse semigroup is weakly right noetherian.

The aim of this paper is to consider ranked inverse semigroups. From [7], if a Brandt semigroup $\mathcal{B}^0(G, I)$ over a group G is ranked, then I is finite and G is ranked. In Section 3 we prove the converse. Right congruences on Brandt semigroups may be described using inverse subsemigroups containing the 0 element [12]; for the purposes or our arguments it is easier to take a direct approach.

In Section 4 we concentrate on an E-unitary inverse semigroup S having a chain $E(S)$ of idempotents. By analysing right congruences contained in the least group congruence σ, we can show that if S is ranked then $E(S)$ is finite. Consequently, the Bicyclic monoid B, while being weakly right noetherian and having trivial (hence ranked) subgroups, is not ranked. It follows that a ranked semigroup cannot contain a bicyclic $\mathcal{J}$-class. Theorem 1.3 of [8] gives that the 0-simple principal factors of a right noetherian semigroup are completely 0-simple; we conjecture the same is true for ranked semigroups.

Our approach to rank was inspired by the model-theoretic Morley rank. From [7], every (complete 1)-type of the theory T_S of existentially closed (right) S-sets over a right coherent monoid S has Morley rank, that is, T_S is totally transcendental, if and only if S is weakly right noetherian and ranked. From [11, 4] T_S is superstable if and only if S is weakly right noetherian. In Section 5 we show that B is right coherent. Thus T_B exists, is superstable, but not totally transcendental. For background in model theory we recommend [2] and [3]. The reader interested in stability theory can find further details in the books [1, 9, 13].

We remark that S-rank was introduced in [7] for pairs (I, ρ) where $\rho \in \mathcal{RC}$ and I is a ρ-saturated right ideal of S. Our aim in doing so was to get a close bound on the Morley rank of types over the theory T_S, where S is a right coherent monoid. However, if all pairs of the form $(\emptyset, \rho)$ possess S-rank then so do all pairs (I, ρ). For the purposes of this paper, which makes no explicit model-theoretic considerations, it is enough to consider pairs $(\emptyset, \rho)$. Thus we define S-rank simply for elements of $\mathcal{RC}$.

2. S-rank for right congruences

Sandwich rank, or S-rank, for right congruences on a semigroup S is defined as follows.

Let ν denote a finitely generated right congruence on S and let $K \subseteq S \times S$ be finite. Then

$$[\nu, K] = \{\rho \in \mathcal{RC} : \nu \subseteq \rho \subseteq S \times S \setminus K\};$$

we say that $[\nu, K]$ is a *subset of finite type*. Clearly subsets of finite type are convex and $[\mu, H] \cap [\nu, K] = [\langle \mu \cup \nu \rangle, H \cup K]$, so that finite intersections of subsets of finite type are of finite type. Hence the set of subsets of finite type are a basis for a topology on $\mathcal{RC}$; we call this the *finite type topology*.

Let $\rho \in \mathcal{RC}$. The S-rank of ρ, $\mathrm{S}(\rho)$, is the Cantor-Bendixson rank of ρ in $\mathcal{RC}$ equipped with the finite type topology. We make this explicit by defining subsets $\mathcal{RC}^\alpha$ of $\mathcal{RC}$ for each ordinal α, as follows:

(I) $\mathcal{RC}^0 = \mathcal{RC}$;

(II) if α is a limit ordinal, then

$$\mathcal{RC}^\alpha = \bigcap \{\mathcal{RC}^\beta : \beta < \alpha\};$$

(III) $\rho \in \mathcal{RC}^{\alpha+1}$ if and only if $\rho \in \mathcal{RC}^\alpha$ and for all subsets of finite type $[\nu, K]$ with $\rho \in [\nu, K]$, there exists $\theta \in \mathcal{RC}$ with

$$\theta \in [\nu, K], \theta \neq \rho \text{ and } \theta \in \mathcal{RC}^\alpha.$$

The *S-rank* of $\rho \in \mathcal{RC}$ is $\mathrm{S}(\rho)$, where if $\rho \in \mathcal{RC}^\alpha$ for all α then $\mathrm{S}(\rho) = \infty$, and otherwise $\mathrm{S}(\rho) = \alpha$ where $\rho \in \mathcal{RC}^\alpha \setminus \mathcal{RC}^{\alpha+1}$. If $\mathrm{S}(\rho) < \infty$, then we say that ρ *has S-rank*. Notice that for any $\rho \in \mathcal{RC}$ and ordinal α, $\mathrm{S}(\rho) \geq \alpha$ if and only if $\rho \in \mathcal{RC}^\alpha$. We remark that in [7], $\mathrm{S}(\rho)$ is denoted by $\mathrm{S}(\emptyset, \rho)$.

We say that a semigroup S is *ranked* if every element of $\mathcal{RC}$ has S-rank. Notice that if ρ is finitely generated and has finitely many covers $\{\rho_i : i \in I\}$, then choosing $(u_i, v_i) \in \rho_i \setminus \rho$, and putting

$$K = \{(u_i, v_i) : i \in I\},$$

it is clear that $[\rho, K]$ isolates ρ. Thus $\mathrm{S}(\rho) = 0$. Hence any finite semigroup is ranked, and if the universal relation ω on any semigroup is finitely generated, then $\mathrm{S}(\omega) = 0$.

We recall the following results from [7], which will be needed in later sections.

Proposition 2.1. *Let S, T be semigroups and let $\psi : S \to T$ be an onto morphism. If S is ranked, then so is T.*

Proposition 2.2. *Let S be a semigroup and let T be a monoid principal factor, a monoid $\mathcal{J}$-class, or a maximal subgroup. If S is ranked, then so is T.*

Proposition 2.3. *Let S be a ranked inverse semigroup. Then S is weakly right noetherian.*

3. Ranked Brandt semigroups

Let $\mathcal{B}^0 = \mathcal{B}^0(G, I)$ be a Brandt semigroup. If $\mathcal{B}^0$ is ranked, then from Propositions 2.2 and 2.3 we know that G is ranked and I is finite. The aim of this section is to prove the converse.

To this end, let $\mathcal{B}^0 = \mathcal{B}^0(G, I)$ be a Brandt semigroup. For each $i \in I$ we put

$$G_i = \{(i, a, i) : a \in G\},$$

so that G_i is a subgroup of $\mathcal{B}^0$ isomorphic to G. For any $\rho \in \mathcal{RC}$ we put

$$\rho_i = \rho \cap (G_i \times G_i),$$

so that ρ_i is a right congruence on G_i. We also denote the ρ-class of 0 by U_ρ, so that in particular U_ρ is a right ideal.

We make the observation that

$$\omega = \langle \{(0, (i, 1, i)) : i \in I\} \rangle.$$

Theorem 3.1. *Let $\mathcal{B}^0 = \mathcal{B}^0(G, I)$ be a Brandt semigroup over a group G. Then $\mathcal{B}^0$ is ranked if and only if G is ranked and I is finite.*

Proof. We assume that G is ranked and I is finite. From the observation preceding the theorem, ω is finitely generated. As remarked in Section 2, it follows that $\mathrm{S}(\omega) = 0$.

Our aim then is to show that if $\rho \in \mathcal{RC}$ with

$$\{i \in I : (i, 1, i) \notin U_\rho\} \neq \emptyset,$$

then we have that $\mathrm{S}(\rho) < \infty$.

Choose and fix a non-empty subset J of I. Without loss of generality we can assume that

$$J = \{1, 2, \ldots, m\}$$

for some $m \in \mathbb{N}$. Put $J' = I \setminus J$ and

$$U = \{(i, a, j) : i \in J'\} \cup \{0\},$$

so that U is a right ideal. Let

$$\mathcal{RC}(J) = \{\rho \in \mathcal{RC} : U_\rho = U\}.$$

We proceed via a series of lemmas to show that every $\rho \in \mathcal{RC}(J)$ has S-rank.

Lemma 3.2. *Let $\rho \in \mathcal{RC}(J)$. Then for any $i, j \in J$,*

$$(i, a, k)\, \rho\, (j, b, l) \text{ implies that } k = l.$$

Further,

$$(i, a, k)\, \rho\, (j, b, k) \text{ if and only if } (i, a, m)\, \rho\, (j, b, m)$$

for any $m \in I$.

Proof. If $(i, a, k)\, \rho\, (j, b, l)$, then as ρ is a right congruence,

$$(i, a, k) = (i, a, k)(k, 1, k)\, \rho\, (j, b, l)(k, 1, k)$$

so that $k = l$ since $(i, a, k) \notin U$. Suppose now that

$$(i, a, k)\, \rho\, (j, b, k)$$

and $m \in I$. Then

$$(i, a, k)(k, 1, m)\, \rho\, (j, b, k)(k, 1, m),$$

so that

$$(i, a, m)\, \rho\, (j, b, m)$$

as required. ∎

For any $\rho \in \mathcal{RC}(J)$ we define a relation $\tilde{\rho}$ on J by the rule that $i\,\tilde{\rho}\,j$ if and only if there exist $(i,a,l),(j,b,l)$ with $(i,a,l)\,\rho\,(j,b,l)$. Clearly $\tilde{\rho}$ is reflexive and symmetric. If $i\,\tilde{\rho}\,j\,\tilde{\rho}\,k$ then, calling upon Lemma 3.2, for some $(i,a,l),(j,b,l),(j,c,l),(k,d,l)$ we have that

$$(i,a,l)\,\rho\,(j,b,l) \text{ and } (j,c,l)\,\rho\,(k,d,l)$$

so that

$$(i,a,l)\,\rho\,(j,b,l) = (j,c,l)(l,c^{-1}b,l)\,\rho(k,d,l)(l,c^{-1}b,l) = (k,dc^{-1}b,l),$$

whence $i\,\tilde{\rho}\,k$ and $\tilde{\rho}$ is an equivalence on J.

We denote by $|\rho|$ the number of $\tilde{\rho}$-classes, so that $1 \leq |\rho| \leq m$. If $\theta,\rho \in \mathcal{RC}(J)$ and $\rho \subseteq \theta$, then clearly $\tilde{\rho} \subseteq \tilde{\theta}$ so that $|\rho| \geq |\theta|$ and $|\rho| = |\theta|$ if and only if $\tilde{\rho} = \tilde{\theta}$.

For the meanwhile we fix $\rho \in \mathcal{RC}(J)$ and show how to construct a subset of finite type (of $\mathcal{RC}_{\mathcal{B}^0}$) containing ρ from subsets of finite type (of $\mathcal{RC}_{G_i}$) containing $\rho_i, i \in J$. For distinct $i,j \in J$ with $i\,\tilde{\rho}\,j$ we have that

$$(i,a,k)\,\rho\,(j,b,k)$$

say, so that

$$(i,1,i)\,\rho\,(j,ba^{-1},i).$$

We choose and fix $g_{ij} \in G$ with $(i,1,i)\,\rho\,(j,g_{ij},i)$. Suppose that for each $i \in J$, $\rho_i \in [\nu_i, K_i]$ where $\nu_i = \langle H_i \rangle$. We define H,ν and K by

$$H = \bigcup_{i\in J} H_i \cup \{((i,1,i),(j,g_{ij},i)) : i \neq j \in J, i\,\tilde{\rho}\,j\} \cup \{(k,1,k),0) : k \in J'\},$$

$$\nu = \langle H \rangle$$

and

$$K = \bigcup_{i\in J} K_i \cup \{(i,1,i),0) : i \in J\}.$$

It is easy to see that $\rho \in [\nu, K]$. Moreover, if $\theta \in [\nu, K]$, then $\theta \in \mathcal{RC}(J)$ and for each $j \in J$, $\theta_i \in [\nu_i, K_i]$.

Lemma 3.3. *Let ρ, H, ν and K be as above. If $\theta \in [\nu, K]$ and $\theta_i = \rho_i$ for each $i \in J$, then $\rho \subseteq \theta$.*

Proof. We know that $\theta \in \mathcal{RC}(J)$ so that in particular U is the θ-class of 0 and

$$\theta \cap (U \times U) = U \times U = \rho \cap (U \times U).$$

Suppose now that $(i,a,k)\,\rho\,(j,b,k)$ for some $i,j \in J$. Then $(i,1,i)\,\rho\,(j,ba^{-1},i)$. If $i = j$ then $(i,1,i)\,\rho_i\,(j,ba^{-1},i)$; but $\rho_i = \theta_i$ so that $(i,1,i)\,\theta\,(j,ba^{-1},i)$, whence $(i,a,k)\,\theta\,(j,b,k)$.

Consider now the case where $i \neq j$. As $i\,\tilde{\rho}\,j$ we have a pair $((i,1,i),(j,g_{ij},i)) \in H$. Hence

$$(j,ba^{-1},i)\,\rho\,(j,g_{ij},i);$$

Lemma 3.2 and the fact that $\theta_j = \rho_j$ now yield

$$(j, ba^{-1}, i)\,\theta\,(j, g_{ij}, i).$$

Since $H \subseteq \theta$ we have that

$$(i, 1, i)\,\theta\,(j, ba^{-1}, i)$$

so that

$$(i, a, k)\,\theta\,(j, b, k)$$

and $\rho \subseteq \theta$ as required. □

Lemma 3.4. *Let ρ, H, ν and K be as above. If $\theta \in [\nu, K]$, $\theta_i = \rho_i$ for each $i \in J$ and $\tilde{\theta} = \tilde{\rho}$, then $\theta = \rho$.*

Proof. By Lemma 3.3, we know that $\rho \subseteq \theta$. For the opposite inclusion, suppose that $(i, a, k)\,\theta\,(j, b, k)$ for some $i, j \in J$. If $i \neq j$ then as $i\,\tilde{\theta}\,j$ we must have that $i\,\tilde{\rho}\,j$ so that there is a pair $((i, 1, i), (j, g_{ij}, i)) \in H$. Arguing as in the proof of Lemma 3.3 yields the result. □

We can now proceed to complete our proof that if $\rho \in \mathcal{RC}(J)$, then ρ has S-rank. We use induction on

$$(\mathrm{S}(\rho_1), \mathrm{S}(\rho_2), \dots, \mathrm{S}(\rho_m), |\rho|)$$

where the product set has the lexicographic ordering.

Suppose first that $\mathrm{S}(\rho_i) = 0$ for each $i \in J$ and $|\rho| = 1$. Then each ρ_i is isolated, say by $[\nu_i, K_i]$. Let $[\nu, K]$ be constructed as above, so that $\rho \in [\nu, K]$. If $\theta \in [\nu, K]$, then $\theta_i \in [\nu_i, K_i]$ so that as ρ_i is isolated by this set, $\theta_i = \rho_i$ for each $i \in J$. By Lemma 3.3 we have that $\rho \subseteq \theta$. Hence

$$1 = |\rho| \geq |\theta|$$

forces $|\rho| = |\theta|$ so that $\tilde{\rho} = \tilde{\theta}$. By Lemma 3.4 we have that $\rho = \theta$. Thus $[\nu, K]$ isolates ρ and $\mathrm{S}(\rho) = 0$.

Proceeding by induction, we suppose that for all $\psi \in \mathcal{RC}(J)$ with

$$(\mathrm{S}(\psi_1), \mathrm{S}(\psi_2), \dots, \mathrm{S}(\psi_m), |\psi|) < (\mathrm{S}(\rho_1), \mathrm{S}(\rho_2), \dots, \mathrm{S}(\rho_m), |\rho|)$$

we have that $\mathrm{S}(\psi) < \infty$.

For each $i \in J$ let $\alpha_i = \mathrm{S}(\rho_i)$ and let $[\nu_i, K_i]$ isolate ρ_i amongst right congruences of G_i with S-rank greater than or equal to α_i. Construct $[\nu, K]$ as above. If $\theta \in [\nu, K]$, then $\theta \in \mathcal{RC}(J)$ and $\theta_i \in [\nu_i, K_i]$. Consequently, $\mathrm{S}(\theta_i) \leq \mathrm{S}(\rho_i)$ for each i. If $\mathrm{S}(\theta_i) < \mathrm{S}(\rho_i)$ for some i, then

$$(\mathrm{S}(\theta_1), \mathrm{S}(\theta_2), \dots, \mathrm{S}(\theta_m), |\theta|) < (\mathrm{S}(\rho_1), \mathrm{S}(\rho_2), \dots, \mathrm{S}(\rho_m), |\rho|)$$

so that $\mathrm{S}(\theta)$ exists by our inductive assumption.

Otherwise, $\mathrm{S}(\theta_i) = \mathrm{S}(\rho_i) = \alpha_i$ for each i. But then $\theta_i = \rho_i$ for each i, so that by Lemma 3.3, $\rho \subseteq \theta$, giving that $|\rho| \geq |\theta|$. If this inequality is strict, then again we have that

$$(\mathrm{S}(\theta_1), \mathrm{S}(\theta_2), \dots, \mathrm{S}(\theta_m), |\theta|) < (\mathrm{S}(\rho_1), \mathrm{S}(\rho_2), \dots, \mathrm{S}(\rho_m), |\rho|)$$

and so $\mathrm{S}(\theta)$ exists. On the other hand, if $|\rho| = |\theta|$, then $\tilde{\rho} = \tilde{\theta}$ so that $\rho = \theta$ by Lemma 3.4.

Our conclusion then is that for $\theta \in [\nu, K]$ we have that $\theta = \rho$ or $\mathrm{S}(\theta) < \infty$. Now if α is an ordinal larger than $\mathrm{S}(\theta)$ for any $\theta \in [\nu, K]$ with $\theta \neq \rho$ (and α certainly exists since $[\nu, K]$ is a *set*), we cannot have that $\mathrm{S}(\rho) \geq \alpha + 1$. Hence $\mathrm{S}(\rho) < \infty$ as required.

□

4. S-rank for inverse semigroups having a chain of idempotents

We recall that a semigroup S is *E-unitary* if for any $e \in E(S)$ and $a \in S$, $ea \in E(S)$ implies that $a \in E(S)$. We say that S is a *chain semigroup* if $E(S)$ forms a chain under the natural partial order. We prove in this section that if S is a ranked E-unitary inverse chain semigroup, then $E(S)$ is finite. In view of Proposition 2.2, any ranked semigroup S cannot contain an E-unitary inverse ω-semigroup as a $\mathcal{J}$-class. In particular, it cannot contain a bicyclic $\mathcal{J}$-class.

Let S be an E-unitary inverse semigroup with chain of idempotents $E = E(S)$. We aim to utilise the results of [7] for ranked chains by describing certain elements of $\mathcal{RC}_S$ by their restriction in $\mathcal{RC}_E$. We say that a partition

$$\mathcal{E} = \{E_i : i \in I\}$$

of E is a *convex* partition if each E_i is convex. That is, if $e, f, g \in E$ with $e > f > g$ and $e, g \in E_i$, then $f \in E_i$. Certainly such a partition inherits a partial ordering from E, so that $i < j$ if $x < y$ for all (any) $x \in E_i, y \in E_j$. It is easy to see that if $\rho \in \mathcal{RC}_E$, then the ρ-classes form a convex partition. Conversely, the equivalence relation associated with a convex partition is a congruence on E.

We recall that the least group congruence on an inverse semigroup S is given by the rule

$$a \,\sigma\, b \text{ if and only if } ea = eb \text{ for some } e \in E.$$

Clearly, if S is E-unitary, then E forms a σ-class.

Let S be an inverse chain semigroup and let $\mathcal{E}$ be a convex partition. Let $\rho_{\mathcal{E}}$ be the relation on S given by

$$\rho_{\mathcal{E}} = \{(a, b) : \exists i \in I \text{ such that } aa', bb' \in E_i \text{ and } ea = eb \text{ for some } e \in E_i\}.$$

Lemma 4.1. *Let S be an inverse chain semigroup, and let $\mathcal{E}, \rho_{\mathcal{E}}$ be as above. Then $\rho_{\mathcal{E}}$ is a right congruence on S, contained in σ, inducing the partition $\mathcal{E}$ of E.*

Proof. It is clear that a convex partition satisfies condition (N) of [10]. The left-right dual of Theorem 3.1 of [10] gives that $\rho_{\mathcal{E}}$ is a right congruence, indeed the minimum right congruence inducing the partition $\mathcal{E}$ of E. Clearly $\rho_{\mathcal{E}} \subseteq \sigma$.

□

The following technical result, which makes implicit use of the natural partial order on an inverse semigroup, will be used repeatedly.

Lemma 4.2. *Let S be an E-unitary inverse chain semigroup, and let $a, b \in S$ with $a\,\sigma\,b$. Then $ba' = ab' \in E$. If $b \leq_{\mathcal{L}} a$ then $ba' = bb'$.*

Proof. Since $a\,\sigma\,b$ we have that $ea = eb$ for some $e \in E$. Now

$$eaa' = eba'$$

so that $ba' \in E$ as S is E-unitary. If $b \leq_{\mathcal{L}} a$ then $b = ba'a$ so that

$$b \leq_{\mathcal{R}} ba' \leq_{\mathcal{R}} b$$

and consequently, $ba' = bb'$ since S is inverse.

□

For an inverse semigroup S we let $\mathcal{G}$ denote the set of right congruences contained in σ.

Lemma 4.3. *Let S be an E-unitary inverse chain semigroup. Then for $\rho, \tau \in \mathcal{G}$,*

$$\rho \subseteq \tau \text{ if and only if } \rho \cap (E \times E) \subseteq \tau \cap (E \times E).$$

Proof. We suppose that $\rho \cap (E \times E) \subseteq \tau \cap (E \times E)$ and show that $\rho \subseteq \tau$.

Let $a, b \in S$ with $a\,\rho\,b$. Without loss of generality we assume that $b \leq_{\mathcal{L}} a$; Lemma 4.2 gives that $ba' = bb'$. Since $aa'\,\rho\,ba'$ and $\rho \cap (E \times E) \subseteq \tau \cap (E \times E)$ we must have that $aa'\,\tau\,ba'$ so that as τ is a right congruence,

$$a = aa'a\,\tau\,ba'a = b.$$

□

Corollary 4.4. *Let S be an E-unitary inverse chain semigroup. Then there is a bijection between $\mathcal{G}$ and the set of convex partitions of E.*

Proof. If $\tau \in \mathcal{G}$ and $\mathcal{E}$ is the partition of E induced by τ, then as commented earlier, $\mathcal{E}$ must be convex. On the other hand for any convex partition $\mathcal{F}$ of E, $\rho_{\mathcal{F}}$ induces $\mathcal{F}$. Invoking Lemma 4.3 gives our result. □

Since the set of convex partitions of E is in bijective correspondence with $\mathcal{RC}_E$, it follows that there is a bijection between $\mathcal{G}$ and $\mathcal{RC}_E$.

We proceed to examine sets of generators for right congruences in $\mathcal{G}$.

Lemma 4.5. *Let S be an E-unitary inverse chain semigroup.*

For any $\rho \in \mathcal{G}$, if

$$\rho = \langle \{(a_j, b_j) : j \in J\} \rangle,$$

then

$$\rho = \langle \{(a_j a_j', b_j b_j') : j \in J\} \rangle.$$

Proof. Let $\rho \in \mathcal{G}$ with

$$\rho = \langle \{(a_j, b_j) : j \in J\} \rangle.$$

Let $j \in J$ and assume that $b_j \leq_{\mathcal{L}} a_j$. By Lemma 4.2 we have that $b_j b_j' = b_j a_j'$ and

$$a_j a_j'\,\rho\,b_j a_j' = b_j b_j'.$$

Hence

$$\tau = \langle \{(a_j a'_j, b_j b'_j) : j \in J\} \rangle \subseteq \rho.$$

Conversely,

$$b_j = b_j a'_j a_j = b_j b'_j a_j \,\tau\, a_j a'_j a_j = a_j,$$

so that $\rho \subseteq \tau$ and we have that $\rho = \tau$. □

For the purposes of the next result, we denote the right congruence on a subsemigroup T of S generated by $H \subseteq T \times T$ as $\langle H \rangle_T$.

Lemma 4.6. *Let S be an E-unitary inverse chain semigroup, let $H \subseteq E \times E$ and let $\rho = \langle H \rangle_S$. Then $\rho \in \mathcal{G}$ and $\rho \cap (E \times E) = \langle H \rangle_E$.*

Proof. Let $H \subseteq E \times E$. If $a \,\rho\, b$, then $a = b$ (so that $a \,\sigma\, b$), or there is a sequence

$$a = c_1 t_1, d_1 t_1 = c_2 t_2, \ldots, d_l t_l = b,$$

where for each k, (c_k, d_k) or $(d_k, c_k) \in H$ and $t_k \in S^1$. Since E is a chain we can find a least element e in

$$\{c_1, d_1, \ldots, c_l, d_l\}$$

and then

$$ea = ec_1 t_1 = et_1 = ed_1 t_1 = ec_2 t_2 = \ldots = ed_l t_l = eb$$

so that $a \,\sigma\, b$ and $\rho \in \mathcal{G}$.

Clearly

$$\langle H \rangle_E \subseteq \rho \cap (E \times E).$$

On the other hand, if $e, f \in E$ and $e \,\rho\, f$, then $e = f$ or there is a sequence

$$e = u_1 s_1, v_1 s_1 = u_2 s_2, \ldots, v_l s_l = f,$$

where for each k, (u_k, v_k) or (v_k, u_k) is in H and $s_k \in S^1$. Since S is inverse we have a sequence

$$e = u_1 s_1 s'_1, v_1 s_1 s'_1 = u_2 s_2 s'_2, \ldots, v_l s_l s'_l = f,$$

so that $e \,\langle H \rangle_E\, f$. Thus

$$\rho \cap (E \times E) = \langle H \rangle_E.$$

□

Proposition 4.7. *Let S be an E-unitary inverse chain semigroup and let $\rho \in \mathcal{G}$. Then ρ is finitely generated (in $\mathcal{RC}_S$) if and only if $\rho \cap (E \times E)$ is finitely generated (in $\mathcal{RC}_E$).*

Proof. If ρ is finitely generated, then by Lemma 4.5, ρ has a finite set of generators H where $H \subseteq E \times E$. By Lemma 4.6,

$$\rho \cap (E \times E) = \langle H \rangle_E$$

and thus is finitely generated.

Conversely, suppose that $\rho \cap (E \times E)$ is finitely generated by F. If $a \,\rho\, b$, then by Lemma 4.5, $aa' \,\rho\, bb'$ so that $aa' \,\langle F \rangle_E\, bb'$. Hence $aa' = bb'$ or there is a sequence

$$aa' = c_1 t_1, d_1 t_1 = c_2 t_2, \ldots, d_l t_l = bb',$$

where for each k, (c_k, d_k) or $(d_k, c_k) \in F$ and $t_k \in E^1$. Without loss of generality, assume that $b \leq_{\mathcal{L}} a$ so that $bb' = ba'$. If $aa' = bb'$ then

$$a = aa'a = bb'a = ba'a = b.$$

Otherwise,

$$a = aa'a = c_1 t_1 a, d_1 t_1 a = c_2 t_2 a, \ldots, d_l t_l a = bb'a = ba'a = b,$$

so that $a \,\langle F \rangle_S\, b$. Consequently, $\rho = \langle F \rangle_S$ and hence is finitely generated. □

We are finally in a position to prove the main result of this section. To do so we call upon the corresponding result in [7] for the case of S a chain.

Theorem 4.8. *Let S be an E-unitary inverse chain semigroup. If S is ranked, then E is finite.*

Proof. We proceed by transfinite induction to show that for any $\rho \in \mathcal{G}$ that is not finitely generated, and for any ordinal α, $\mathrm{S}(\rho) \geq \alpha$. As usual, the step at limit ordinals is trivial. Suppose now that $\mathrm{S}(\mu) \geq \alpha$ for all $\mu \in \mathcal{G}$ that are not finitely generated. Let $\rho \in \mathcal{G}$ where ρ is not finitely generated and suppose that

$$\rho \in [\nu, K]$$

where $[\nu, K]$ is an interval of finite type. Clearly

$$\nu \subset \rho \subseteq \sigma.$$

By Proposition 4.7, $\nu \cap (E \times E)$ is finitely generated but $\rho \cap (E \times E)$ is not. From Lemma 5.6 of [7], we can find a non-finitely generated congruence θ on E such that

$$\nu \cap (E \times E) \subset \theta \subset \rho \cap (E \times E).$$

Let $\psi = \langle \theta \rangle_S$ so that $\psi \in \mathcal{G}$ from Lemma 4.6 and

$$\psi \cap (E \times E) = \langle \theta \rangle_E = \theta.$$

From Proposition 4.7 we have that ψ is not finitely generated. Lemma 4.3 gives that

$$\nu \subset \psi \subset \rho.$$

Thus $\psi \in [\nu, K]$ and $\psi \neq \rho$. Our inductive hypothesis tells us that $\mathrm{S}(\psi) \geq \alpha$, so that $\mathrm{S}(\rho) \geq \alpha + 1$.

It follows that for any non-finitely generated $\rho \in \mathcal{G}$, $\mathrm{S}(\rho) = \infty$. If E is infinite, then it contains an unbounded subset, so that as in Proposition 5.7 of [7], $\mathcal{RC}_E$ contains a non-finitely generated congruence θ. Then $\langle \theta \rangle_S$ fails to be finitely generated, so that $\mathrm{S}(\langle \theta \rangle_S) = \infty$ and S is not ranked. □

Corollary 4.9. *Let S be a ranked semigroup. Then S cannot contain a monoid $\mathcal{J}$-class that is an inverse ω-semigroup. In particular, S cannot contain a bicyclic $\mathcal{J}$-class.*

Proof. Let S be ranked. If S contains an inverse ω-semigroup T as a $\mathcal{J}$-class, then T is perforce ranked by Proposition 2.2. As T is a $\mathcal{J}$-class of S, T is a simple inverse ω-semigroup; it need not be E-unitary, but from the characterisation of T as a Bruck-Reilly semigroup, it is clear that there is an onto morphism from T to the bicyclic monoid B. Thus by Proposition 2.1, B is ranked. But B is an E-unitary inverse ω-semigroup, so cannot be ranked in view of Theorem 4.8. $\square$

5. The bicyclic monoid is right coherent

A monoid S is *right coherent* if every finitely generated S-subset of any finitely presented (right) S-set is finitely presented. It is known that S is right coherent if and only if the theory of S-sets has a model companion; the model companion is then a set of sentences which axiomatises the existentially closed S-sets. Further details can be found in [5, 14]. In [6] we show that an inverse monoid with central idempotents is right coherent. The aim of this section is to extend this result to show that an inverse monoid with an inversely well-ordered chain of idempotents is right coherent.

Lemma 5.1. *Let I be a right ideal of a monoid S, let $\rho \in \mathcal{RC}$ and let $x, y \in S$.*

(i) The set

$$I\rho = \{s \in S : s\,\rho\,a \text{ for some } a \in I\} = \bigcup\{[a] : a \in I\}$$

is a right ideal of S containing I.

(ii) The set

$$(I, x) = \{t \in S : xt \in I\}$$

is empty or is a right ideal.

(iii) If $x\,\rho\,y$ then

$$(I\rho, x) = (I\rho, y).$$

Proof. It is easy to see that (i) and (ii) hold. For (iii), observe that $xt \in I\rho$ if and only if $yt \in I\rho$, since $I\rho$ is a union of ρ-classes.

$\square$

Theorem 5.2. *Let S be a regular monoid such that every right ideal is principal. Then S is right coherent.*

Proof. Let ρ be a finitely generated right congruence on S. Since every right ideal of S is principal, the set of right ideals of S is linearly ordered by inclusion. Clearly then for $[a], [b] \in S/\rho$, we have that $[a]S \cap [b]S = [a]S$ or $[b]S$. From Corollary 3.4 of [6], it remains to show that the right congruence

$$r([a]) = \{(u, v) \in S \times S : au\,\rho\,av\}$$

is finitely generated.

Let $\{(x_i, y_i)\}$ be a finite set of generators for ρ and let $[a] \in S/\rho$. Then $(aS)\rho$ is a right ideal of S, so by assumption

$$(aS)\rho = eS \text{ for some } e \in E.$$

Clearly $ea = a$. Now $e \in (aS)\rho$ gives that $e \,\rho\, at$ for some $t \in S$. Consequently,

$$a = ea \,\rho\, ata$$

so that

$$(1, ta) \in r([a]).$$

For any i,

$$(eS, x_i) = (eS, y_i) = \emptyset$$

or

$$(eS, x_i) = (eS, y_i) = g_i S \text{ for some } g_i \in E(S),$$

using Lemma 5.1. In the latter case,

$$atx_i g_i \,\rho\, ex_i g_i = x_i g_i \,\rho\, y_i g_i = ey_i g_i \,\rho\, aty_i g_i$$

so that

$$(tx_i g_i, ty_i g_i) \in r([a]).$$

Putting

$$\tau = \langle \{(1, ta), (tx_i g_i, ty_i g_i)\} \rangle$$

we have shown that

$$\tau \subseteq r([a]).$$

Conversely, suppose that $(u, v) \in r([a])$, so that $au \,\rho\, av$. If $au = av$, then

$$u \,\tau\, tau = tav \,\tau\, v.$$

Otherwise, we have a sequence

$$au = c_1 t_1, d_1 t_1 = c_2 t_2, \ldots, d_l t_l = av,$$

where for each $j \in \{1, \ldots, l\}$, $t_j \in S$ and

$$(c_j, d_j) = (x_{i_j}, y_{i_j}) \text{ or } (c_j, d_j) = (y_{i_j}, x_{i_j}).$$

Observe that for each j,

$$t_j \in (eS, x_{i_j}) = (eS, y_{i_j}) = g_{i_j} S,$$

so that

$$t_j = g_{i_j} t_j.$$

We rewrite our sequence as

$$au = c_1 g_{i_1} t_1, d_1 g_{i_1} t_1 = c_2 g_{i_2} t_2, \ldots, d_l g_{i_l} t_l = av,$$

and multiply every equation by t to obtain

$$tau = tc_1 g_{i_1} t_1, td_1 g_{i_1} t_1 = tc_2 g_{i_2} t_2, \ldots, td_l g_{i_l} t_l = tav.$$

Since for each j we have that

$$(tc_j g_{i_j}, td_j g_{i_j}) \text{ or } (td_j g_{i_j}, ct_j g_{i_j})$$

is a generator of τ, we obtain

$$u \,\tau\, tau \,\tau\, tav \,\tau\, v,$$

so that

$$r([a]) \subseteq \tau$$

and we deduce that $r([a]) = \tau$ is finitely generated as required.

□

Corollary 5.3. *The bicyclic monoid is right coherent.*

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