STABILITY OF THE THEORY OF EXISTENTIALLY CLOSED S-ACTS OVER A RIGHT COHERENT MONOID S

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For Professor L.A. Bokut on the occasion of his 70th birthday

Abstract. Let L_S denote the language of (right) S-acts over a monoid S and let Σ_S be a set of sentences in L_S which axiomatises S-acts. A general result of model theory says that Σ_S has a model companion, denoted by T_S, precisely when the class $\mathcal{E}$ of existentially closed S-acts is axiomatisable and in this case, T_S axiomatises $\mathcal{E}$. It is known that T_S exists if and only if S is right coherent. Moreover, by a result of Ivanov, T_S has the model-theoretic property of being stable.

In the study of stable first order theories, superstable and totally transcendental theories are of particular interest. These concepts depend upon the notion of type: we describe types over T_S algebraically, thus reducing our examination of T_S to consideration of the lattice of right congruences of S. We indicate how to use our result to confirm that T_S is stable and to prove another result of Ivanov, namely that T_S is superstable if and only if S satisfies the maximal condition for right ideals. The situation for total transcendence is more complicated but again we can use our description of types to ascertain for which right coherent monoids S we have that T_S is totally transcendental and is such that the U-rank of any type coincides with its Morley rank.

1. Introduction

In this paper we are concerned with the investigation of stability properties of certain complete theories of S-acts. We emphasise that we are taking the algebraist's approach in the sense that our first aim is to associate stability properties of T_S with algebraic properties of the monoid S, our further investigations then focussing on the latter.

Stability properties (see Sections 2 and 5 for the relevant definitions) arose from the question of how many models a theory (a set of sentences of a first order language) has of any given cardinality. The seminal work of Shelah shows that

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an unstable theory, indeed a non-superstable theory, has 2^λ models of cardinality λ for any $\lambda > |T|$ [30]. The philosophy then is that, in these cases, there are too many models to attempt to classify by means of a sensible structure theorem. It is reasonable therefore for the algebraist to consider for a given class of algebras 'how stable' is the theory associated with it, before embarking on the search for structure or classification theorems.

For a monoid S, a *(right) S-act* is simply a set A upon which S acts on the right with the identity of S acting as the identity map on A. Associated with S is the first order language L_S for S-acts. We denote by Σ_S the set of sentences axiomatising S-acts, and refer to Σ_S as the *theory* of S-acts. Futher details are given in Section 2. We can think of an S-act as being analogous to a module over a ring; this observation inspires our characterisation of types and our approach to stability and superstability, after which more significant differences arise between the situation for modules and that for acts.

The model theory of modules has been and continues to be extensively investigated (see [28]), yielding both structure results for modules and giving concrete realisations of model theoretic concepts. In contrast, only a few studies have been made of the model theory of S-acts. Some results in the latter theory are close parallels of corresponding results for modules. As indicated above, there are, however, several major differences between the two theories. Essentially, these differences arise since right congruences on monoids cannot be determined by right ideals (as is the case for rings). For the model theorist, this means that atomic formulae without parameters cannot be replaced by formulae involving parameters.

Given any R-module M over a ring R, or any S-act A over a monoid S, we can consider the set of all sentences (in the appropriate language) that are true in M or in A. These theories are exactly the complete theories of R-modules or S-acts, where a theory T is *complete* if for any sentence ϕ of the language, $\phi \in T$ or $\neg\phi \in T$. A notable difference between the model theory of modules and that of S-acts is that, as demonstrated by Mustafin [23], for some monoids S, there are S-acts which have unstable theories whereas all complete theories of modules are stable. Mustafin goes on to describe all monoids S for which every S-act has a stable theory or superstable theory. The thrust of his later papers in this area is to move toward a description of those monoids S over which all S-acts are ω-stable [3, 24]. On the other hand Stepanova [31] has characterised monoids such that all *regular* S-acts have stable, superstable or ω-stable theories.

Rather than imposing conditions on the theories $\mathrm{Th}(A)$ for *all* S-acts A over a given S, we are concerned here with theories of existentially closed S-acts: we now explain our motivation. An important notion of model theory is that of *model companion*. For a theory T one defines in a natural way the notion of an existentially closed model of T and we denote the class of existentially closed models of T by $\mathcal{E}(T)$. If T is an inductive theory, such as Σ_S, then T has a model companion if and only if $\mathcal{E}(T)$ is axiomatisable. In this case, $\mathrm{Th}(\mathcal{E}(T))$ is a model companion of T. Following Wheeler [32] the notion of right coherence for monoids was introduced in [11] where it is shown that the theory of all S-acts (for fixed S) has a model companion T_S if and only if S is right coherent. It follows that the models of T_S are precisely the existentially closed S-acts, and further, that T_S is a complete theory so that $T_S = \mathrm{Th}(M)$ for any existentially closed S-act M. Ivanov [17] argues that T_S is a normal theory (see [25]) and hence stable.

Given that T_S is stable it is natural to investigate conditions under which it satisfies the stronger stability properties of being superstable, ω-stable or totally transcendental. In [4] the corresponding questions in module theory are posed and answered. This work both inspired and heavily influenced the present paper. For a right coherent ring R, the model companion of the theory of all R-modules is denoted by T_R. Properties such as stability are dependent upon the number of types (the details of which are given in Section 3). In [4] types are characterised by pairs consisting of a right ideal of R and an R-homomorphism. This is the key to a thorough analysis of complete types and so to finding for which rings R the theory T_R is superstable or totally transcendental.

In a ring R, a right congruence is determined by a right ideal, but as remarked above, this is not true for monoids in general. For this reason, in the case of right S-acts, complete types are characterised by triples consisting of a right ideal of S, a right congruence on S and an S-morphism. It is this result which allows us to translate model theoretic properties of T_S into algebraic properties of S and hence to apply the theory of semigroups. An immediate consequence is that we can easily find upper bounds for the number of types. This enables us to deduce Ivanov's result [17, Proposition 1.4] that the theory T_S is stable. Further, if every right ideal of S is finitely generated, then T_S is superstable, and if in addition S is countable and has at most $\aleph_0$ right congruences, then T_S is ω-stable.

To obtain the converse of these results we use the U-rank of types and the fact that a complete theory is superstable if and only if the U-rank of each type is defined (see [27]). Our approach is similar to but slightly more complicated than that of Bouscaren. The end results are that T_S is superstable if and only if every right ideal of S is finitely generated and that for a countable S, T_S is ω-stable if and only if S has at most $\aleph_0$ right congruences and every right ideal of S is finitely generated. The superstability result is also a straightforward consequence of [17, Theorem 2.4]. In these results there is of course the underlying assumption that S is right coherent, for this is needed for the theory T_S to exist. Right coherence does not follow from the property that every right ideal is finitely generated as shown by Example 3.1 in [12]. The equivalent results for modules are that superstability and total transcendence of T_R are both equivalent to R being right noetherian.

Another important rank of types is the Morley rank. This is used to define the concept of total transcendence, a complete theory T being totally transcendental if and only if every type has Morley rank. Morley rank is always greater than

U-rank, so that a totally transcendental theory is certainly superstable. In fact a countable theory T is totally transcendental if and only if it is ω-stable [27].

For a complete theory T of modules, the Morley rank of a type (when it exists) coincides with the U-rank of the type [28]. This is not the case for S-acts and we find necessary and sufficient conditions on S for the theory T_S to be totally transcendental with the Morley rank of any type being equal to its U-rank. The final section of the paper is devoted to a study of monoids which satisfy these conditions. If S is such a monoid and is weakly periodic, then S is finite. On the other hand, the infinite cyclic monoid satisfies the conditions.

The structure of this paper is as follows. In Section 2 we outline the basics of model theory we require; since the new work of this article is almost entirely semigroup theoretic, we keep these details to a minimum. We also give some details concerning S-acts over a monoid S. In Section 3 we discuss types and, crucially, show how a type of T_S over an S-act A is associated with what we call an A-triple. It is this result which allows us to translate arguments from model theory into algebra. We omit most proofs, since the ideas are rather straightforward and may be thought of as being inherent in the work of Ivanov [17]; full details appear in the notes [10]. The next section outlines how we may use our description of types to capture U-rank and the superstability result of [17]. Sections 3 and 4 may be regarded as a survey. The new material begins in Section 5 where we discuss Morley rank and find a criterion for a right coherent monoid S such that T_S is totally transcendental and the U-rank of any type of T_S coincides with its Morley rank. In our final section we investigate the monoids satisfying this criterion.

2. Preliminaries

This paper is intended to be accessible to algebraists with some familiarity with the basic ideas of first-order logic and, with the exception of the final section, only a very little semigroup theory. We recommend [6] and [9] for the former and [15] for the latter. Full accounts of the stability theory we use can be found in the books [1, 4, 19, 26, 27, 28]; we extract the key ideas and main results which we need. Any unreferenced results may be found in these texts.

We begin with some brief details concerning S-acts. Further details may be found in the comprehensive [18].

Let S be a monoid. A *(right) S-act* is a set A on which S acts on the right, that is, there is a map $\cdot$ from $A \times S$ to A satisfying :

$$(a \cdot s) \cdot t = a \cdot (st) \text{ and } a \cdot 1 = a$$

for all $s, t \in S, a \in A$, where $\cdot$ maps (a, s) to $a \cdot s$; we usually write as for $a \cdot s$. Clearly we can think of the elements of S as unary operation symbols and A as a unary algebra in the sense of universal algebra. We thus have all the standard concepts and results of universal algebra at our disposal (see, for example [21]). In particular, we have *S-subacts*, *S-morphisms*, *congruences on S-acts* and *quotient*

S-*acts* A/ρ where A is an S-act and ρ is a congruence on A. For an S-subact B of an S-act A, the relation ρ_B is defined by $a_1\rho_B a_2$ if and only if $a_1 = a_2$ or a_1, a_2 are both in B. It is easy to see that ρ_B is a congruence on A; the quotient S-act A/ρ_B is usually denoted by A/B and is called the *Rees quotient* of A by B. We differ from standard semigroup terminology in that we make the convention that the empty set $\emptyset$ is an S-subact of every S-act.

For any congruence ρ on an S-act A we denote the ρ-class of an element a of A by $a\rho$. For an S-morphism $f : A \to B$ we denote by $\mathrm{Ker}f$ the congruence on A determined by

$$(a, b) \in \ \mathrm{Ker}f \text{ if and only if } f(a) = f(b).$$

The multiplication in a monoid S makes S itself into a right S-act. The S-subacts of S are called *right ideals* of S and S-act congruences on S are called *right congruences* on S, to distinguish them from semigroup congruences on S.

The category of S-acts and S-morphisms has arbitrary products and coproducts. Another property enjoyed by this category which is useful for our purposes is the *strong amalgamation property.* This asserts that if A, B are S-acts with common S-subact U, then there is an S-act C and injective S-morphisms $f : A \to C, g : B \to C$ such that $f|U = g|U$ and $f(A) \cap g(B) = f(U)$.

Let I be a right ideal of a monoid S and ρ be a right congruence on S. The ρ-*closure* of I, denoted by $I\rho$, is defined by

$$I\rho = \{s \in S : s\,\rho\,t \text{ for some } t \in I\}.$$

It is easy to see that $I\rho$ is a right ideal of S containing I and that $(I\rho)\rho = I\rho$. We say that a right ideal J of S is ρ-*saturated* if $J\rho = J$; thus $I\rho$ is ρ-saturated for any right ideal I. If ν, ρ are right congruences on S and $\nu \subseteq \rho$, then any ρ-saturated right ideal is also ν-saturated.

When I is a ρ-saturated right ideal of S we say that the pair (I, ρ) is a *congruence pair*. We denote by $\mathcal{C}(S)$ or $\mathcal{C}$ the set of all congruence pairs of S.

This paper is concerned with one aspect of the model theory of S-acts. Let L be a first order language. A class $\mathcal{U}$ of L-structures is *axiomatisable* if there is a set of sentences Π of L such that an L-structure U lies in $\mathcal{U}$ if and only if every sentence of Π is true in U, that is, $U \models \Pi$. We use the standard notation that if $\phi(x_1, ..., x_n)$ is a formula of L, then the free variables of $\phi(x_1, ..., x_n)$ lie in $\{x_1, ..., x_n\}$. If T is a theory in L (that is, a set of sentences of L, which without loss of generality we may assume to be closed under deduction), then models of T will be denoted by letters M, N, P; we use the same notation for their universes. If M is an L-structure then $\mathrm{Th}(M)$ is the set of sentences true in M; if M is a model of a theory T then certainly $T \subseteq\mathrm{Th}(M)$. The letters $A, B, \ldots$ are used for subsets of models. For a set A, the language $L(A)$ is obtained from L by adding a new constant symbol to L for each element a of A. Again, we follow the usual practice and do not distinguish elements of A from the constants of $L(A)$ which they label. We may denote an $L(A)$-structure by $(B, a)_{a\in A}$, where $A \subseteq B$, so that if T is a theory in L and $A \subseteq M \models T$, $\mathrm{Th}(M, a)_{a \in A}$ is the set of sentences in $L(A)$ true in M.

The language L_S of the theory of S-acts consists of a unary function symbol f_s for each element s of S. We follow the usual convention and write as for $f_s(a)$. Clearly the class of S-acts is axiomatised by the set of sentences

$$\Sigma_S = \{(\forall x)(x1 = x)\} \cup \{(\forall x)((xs)t = x(st)) : s, t \in S\}.$$

An *equation* over an S-act A is an atomic formula of $L_S(A)$ and has one of the forms:

$$xs = xt, \; xs = yt, \; xs = a$$

where $s, t \in S$ and $a \in A$. An *inequation* over A is simply the negation of an equation over A.

A set Σ of equations and inequations over A is *consistent* if Σ has a solution in some S-act containing A. An S-act A is *existentially closed* if every consistent finite set of equations and inequations over A has a solution in A. Since the class of S-acts is inductive, that is, is closed under unions of chains, every S-act is contained in an existentially closed S-act.

To say that the theory of all S-acts has a *model companion* is equivalent to saying that the class of all existentially closed S-acts is axiomatisable by a theory T_S; then T_S is the required model companion. In [11] it is proved that T_S exists if and only if S is right coherent, where a monoid S is *right coherent* if for any finitely generated right congruence ρ on S, every finitely generated S-subact of S/ρ is finitely presented. This result was generalised to varieties of S-acts in [17]. A careful study of right coherence for S-acts is made in [12].

Given two existentially closed S-acts A, B it is certainly the case that A, B can be embedded in an S-act C (the coproduct of A and B for example) and C can be embedded in an existentially closed S-act. It follows from this and the model completeness of T_S that T_S (when it exists) is complete (Proposition 3.1.9 of [6]). That is, for any sentence ϕ of L_S either $\phi \in T_S$ or $\neg\phi \in T_S$; equivalently, $T_S = \mathrm{Th}(M)$ for any of its models M. Since the theory of all S-acts is universal and as T_S is actually the model completion of this theory [11], we have by Theorem 13.2 in [29] that T_S admits elimination of quantifiers. These properties, not all used explicitly here, ensure that T_S is precisely the kind of theory most amenable to the application of stability theory.

3. Types

The notion of a *type* is crucial to our investigations of stability properties of T_S. To define types, it is useful to employ the so-called monster model of a theory; justification of its existence (which uses the notion of saturation) and use can be found in [5]. Let T be a complete theory in L. The *monster model* of T is a model $\mathbf{M}$ of T such that all models of T are elementary substructures of $\mathbf{M}$ and all sets of parameters are subsets of $\mathbf{M}$.

Let A be a subset of $\mathbf{M}$ and let $c \in \mathbf{M}$. Then

$$\mathrm{tp}(c/A) = \{\phi(x) \in L(A) : \mathbf{M} \models \phi(c)\}$$

is a *(complete 1-)type over A*. Clearly $\mathrm{tp}(c/A)$ is a complete set $p(x)$ of sentences of $L(A,x)$ such that a model exists for $p(x) \cup \mathrm{Th}(\mathbf{M},a)_{a\in A}$. Conversely, if $p(x)$ is a set of formulae satisfying these conditions, then properties of $\mathbf{M}$ (concerned with saturation) give that $p(x) = \mathrm{tp}(b/A)$ for some $b \in \mathbf{M}$. The *Stone space* $S(A)$ of A is the collection of all types over A; $S(A)$ is equipped with a natural topology, which comes into play in the definition of Morley rank (see Section 5).

For a cardinal κ, T is *κ-stable* if for every subset A of a model of T with $|A| \leq \kappa$ we have $|S(A)| \leq \kappa$. If T is κ-stable for some infinite κ, then T is *stable* and T is *superstable* if T is κ-stable for all $\kappa \geq 2^{|T|}$. If T is not stable, then it is said to be *unstable*. Morley argued that a theory T in a countable language is ω-stable if and only if T is κ-stable for every infinite κ [22].

From now on we shall concentrate on the theory T_S for a fixed right coherent monoid S. The purpose of this section is to give a straightforward characterisation of types over S-acts. We do not present the proofs, as they involve quite standard concepts. Some of these ideas appear implicitly in [17]; explicit proofs may be found in the unpublished notes [10] of the authors. Ivanov [17] shows that T_S is a stable theory, and also characterises those monoids S such that T_S is superstable. By making the characterisation of types explicit, we have both an alternative approach to these results of Ivanov, and a solid tool with which to characterise ranks of types, needed for our later discussions.

If A is an S-act, then an *A-triple* is a triple (I,ρ,f) such that $(I,\rho) \in \mathcal{C}$ and $f : I \to A$ is an S-morphism with $\mathrm{Ker} f = \rho \cap (I \times I)$. We denote the set of all A-triples by $\mathcal{T}(A)$.

Let $\mathcal{T} = (I,\rho,f)$ be an A-triple and let $\Sigma_{\mathcal{T}}$ be the union of the following sets of formulae of $L_S(A)$:

$$\{xs = a : a = f(s), s \in I\}, \{xs \neq a : s \notin I, a \in A\},$$

$$\{xs = xt : (s,t) \in \rho\}, \{xs \neq xt : (s,t) \notin \rho\}.$$

An easy argument using quantifier elimination and the fact that the class of S-acts has the strong amalgamation property yields the following.

Lemma 3.1. [10] *Let A be an S-act and let $\mathcal{T}$ be an A-triple. Then there is an embedding of A into an existentially closed S-act E, and an element $c \in E$ such that $tp(c/A) = p_{\mathcal{T}}$ is the unique type over A containing $\Sigma_{\mathcal{T}}$.*

Conversely, given $p \in S(A)$ we obtain an A-triple $\mathcal{T}_p$.

Lemma 3.2. [10] *Let p be a type over an S-act A. Let*

$$I_p = \{s \in S : xs = a \in p \textit{ for some } a \in A\},$$

$$\rho_p = \{(s,t) \in S \times S : xs = xt \in p\},$$

and

$$f_p : I_p \to A \text{ be defined by } f_p(s) = a \text{ where } xs = a \in p.$$

Then $\mathcal{T}_p = (I_p, \rho_p, f_p)$ *is an* A*-triple.*

The next result is crucial. Essentially, it allows us to translate arguments involving types, and ranks thereof, into arguments internal to our monoid S.

Proposition 3.3. [10] *The maps*

$$p \mapsto \mathcal{T}_p, \;\; \mathcal{T} \mapsto p_{\mathcal{T}}$$

are mutually inverse bijections between $S(A)$ *and* $\mathcal{T}(A)$*.*

The corollary below is an immediate consequence of the proposition.

Corollary 3.4. [10] *(1) Let* A *be an* S*-act and let* $p, q \in S(A)$*. Then* $p = q$ *if and only if* $I_p = I_q, \rho_p = \rho_q$ *and* $f_p = f_q$*.*

(2) There is a bijection between the set of right congruences on S *and* $S(\emptyset)$*.*

(3) For any congruence pair (I, ρ) *on* S *there is an* S*-act* A *and a type* p *over* A *with* $I_p = I$ *and* $\rho_p = \rho$*.*

(4) Let p *be a type over an* S*-subact* A *of* B*. Then there is a type* q *over* B *such that* $I_p = I_q, \rho_p = \rho_q$ *and* $f_q = jf_p$ *where* $j : A \to B$ *is the inclusion map.*

Let A be an S-act and I be a right ideal of S. The number of S-morphisms from I to A is at most $|A|^{|S|}$, the number of right ideals of S is at most $2^{|S|}$ and the number of right congruences on S is at most $2^{|S|^2}$. Hence the number of A-triples is at most $2^{|S|}2^{|S|^2}|A|^{|S|}$. Thus, if we take κ =max$\{\aleph_0, 2^{|S|}\}$ and $|A| \leq \kappa$, then $|\mathcal{T}(A)| \leq \kappa$ and, in view of Proposition 3.3, $|S(A)| \leq \kappa$.

Now consider an arbitrary subset B of the S-act $\mathbf{M}$. It is easy to see that $|S(B)| = |S(A)|$, where A is the S-subact of $\mathbf{M}$ generated by B (indeed, the Stone spaces are homeomorphic, see [1, 19]). We can therefore deduce that the theory T_S is stable.

We can do better than this when every right ideal of S is finitely generated, that is, when S is *weakly right noetherian*. Then, for any right ideal I, the number of S-morphisms from I to A is at most max$\{\aleph_0, |A|\}$ so that there are no more than $2^{|S|}$max$\{\aleph_0, |A|\}$ A-triples. Hence for any infinite cardinal κ with $2^{|S|} \leq \kappa$ we have that if $|A| \leq \kappa$, then $|S(A)| \leq \kappa$. Now $|T_S|$ =max$\{\aleph_0, |S|\}$ so that T_S is superstable [17].

If we assume that S has at most max$\{\aleph_0, |S|\}$ right congruences in addition to being weakly right noetherian, then we see that the number of A-triples is at most max$\{\aleph_0, |S|\}^2$max$\{\aleph_0, |A|\}$. Thus for any infinite cardinal κ with $|S| \leq \kappa$ we have that if $|A| \leq \kappa$, then $|S(A)| \leq \kappa$. Hence, for a countable S which is weakly right noetherian and has only countably many right congruences we have that T_S is ω-stable. In particular, T_S is ω-stable for any finite monoid S.

A monoid S is *right noetherian* if every right congruence on S is finitely generated; since every right ideal of S is determined by a right congruence, it follows that such a monoid is weakly right noetherian. Moreover, every right noetherian monoid is right coherent [12]. Thus if S is a countable, right noetherian monoid, then T_S is ω-stable.

If S is countably infinite and T_S is ω-stable, then $|S(\emptyset)| \leq \aleph_0$ so that by Corollary 3.4, S has only countably many right congruences.

The following result summarises the above discussion; (1) and (2) are also consequences of results in [17].

Proposition 3.5. [17, 10] *Let S be a right coherent monoid. Then*

(1) the theory T_S is stable;

(2) if S is weakly right noetherian, then T_S is superstable;

(3) if S is weakly right noetherian and has at most max$\{\aleph_0, |S|\}$ right congruences, then T_S is κ-stable for all κ with max$\{\aleph_0, |S|\} \leq \kappa$;

(4) if S is countable, then if S is weakly right noetherian and has at most $\aleph_0$ right congruences, T_S is ω-stable;

(5) if S is finite, then T_S is ω-stable;

(6) if S is countable and right noetherian, then T_S is ω-stable;

(7) if S is countable and T_S is ω-stable, then S has at most $\aleph_0$ right congruences.

The converses of (2) and (4) of the above proposition will be obtained in Section 4.

By an *extension* of a type p in $S(A)$ we mean a type q in $S(B)$ where A is an S-subact of B and $p \subseteq q$. The proof of the following result follows easily from Lemma 3.1.

Proposition 3.6. [10] *Let A be an S-subact of B, $p \in S(A)$ and $q \in S(B)$. Then q is an extension of p if and only if*

(i) $I_p \subseteq I_q$, (ii) $f_q|I_p = f_p$, (iii) $f_q^{-1}(A) = I_p$ and (iv) $\rho_p = \rho_q$.

A consequence of Proposition 3.6 is that if p and q are as in Corollary 3.4 (4), then q is an extension of p.

For the final result of this section we again make use of the fact that the class of S-acts has the strong amalgamation property.

Proposition 3.7. [10] *Let A be an S-act and $p \in S(A)$. Let J be a ρ_p-saturated right ideal containing I_p. Then there is an S-act B containing A and an extension q of p in $S(B)$ such that $I_q = J$. Moreover, B can be chosen to be existentially closed.*

4. U-rank and superstability of T_S

Rank notions are an important tool in determining stability properties of theories. In this section we relate the U-rank of a type p, introduced by Lascar in [20], to what we call the ρ_p-rank of the right ideal I_p. It is then straightforward to prove the converse of (2) of Proposition 3.5. As in Section 3 we omit most arguments, which may be found in detail in [10].

First we recall the *foundation* rank on a set $\mathcal{S}$ partially ordered by $\leq$. We define subclasses $\mathcal{S}_\alpha$ of $\mathcal{S}$ for each ordinal α by transfinite induction:

(I) $\mathcal{S}_0 = \mathcal{S}$;

(II) $\mathcal{S}_\alpha = \bigcap\{\mathcal{S}_\beta : \beta < \alpha\}$, if α is a limit ordinal;

(III) $x \in \mathcal{S}_{\alpha+1}$ if and only if $x < y$ for some $y \in \mathcal{S}_\alpha$.

We thus obtain a nested sequence of subclasses of $\mathcal{S}$ indexed by the ordinals. The *foundation rank* of $x \in \mathcal{S}$, denoted by $\mathrm{R}(x)$, can now be defined as follows:

If $x \in \mathcal{S}_\alpha$ for all ordinals α, then we write $\mathrm{R}(x) = \infty$. Otherwise, $\mathrm{R}(x) = \alpha$ where α is the (unique) ordinal such that $x \in \mathcal{S}_\alpha \setminus \mathcal{S}_{\alpha+1}$; in this case we say that *x has R-rank.*

The convention that $\alpha < \infty$ for all ordinals α simplifies the statements of the following standard proposition (see for example [27], p. 35).

Proposition 4.1. *(i) For any $x \in \mathcal{S}$ and any ordinal α*

$$R(x) \geq \alpha \text{ if and only if } x \in \mathcal{S}_\alpha.$$

(ii) Let $x, y \in \mathcal{S}$ where $x < y$. If $R(y)$ is an ordinal then $R(x) > R(y)$. Moreover, if $R(x)$ is an ordinal then so is $R(y)$.

(iii) For any $x \in \mathcal{S}$, $R(x)$ is an ordinal if and only if there are no infinite chains of the form

$$x = x_0 < x_1 < \dots .$$

For the first application of foundation rank, consider a right congruence ρ on S and put

$$\mathcal{S} = \mathcal{S}_\rho = \{J : (J, \rho) \in \mathcal{C}\}.$$

The relation $\leq$ is taken as the usual inclusion order of right ideals. If $J \in \mathcal{S}_\rho$ then $\mathrm{R}(J)$ is said to be the *ρ-rank* of J and is written as ρ-$\mathrm{R}(J)$.

Corollary 4.2. *Let $(I, \rho) \in \mathcal{C}$. Then ρ-$R(I)$ is an ordinal if and only if S has the ascending chain condition on ρ-saturated right ideals containing I.*

Our second application of foundation rank is to obtain the *U-rank* $\mathrm{U}(p)$ of a type $p \in S(A)$, where $A \subseteq \mathbf{M} \models T$ and T is a complete, stable theory in a first order language L. First we review some definitions associated with types of T; for more details the reader can consult one of the standard texts.

If $p \in S(A)$, where $A \subseteq \mathbf{M}$, then the *class* of p, written $\mathrm{cl}(p)$, is the set

$$\mathrm{cl}(p) = \{\phi(x, y_1, \dots, y_n) \in L : \text{ for some } a_1, \dots, a_n \in A, \phi(x, a_1, \dots, a_n) \in p\}$$

and C_p is the set

$$C_p = \{\mathrm{cl}(q) : p \subseteq q, q \in S(M), A \subseteq M \models T\}.$$

It is a fact that C_p has a unique minimum element (under inclusion) denoted by $\beta(p)$. Clearly, if $p \in S(M)$ where $M \models T$, then $\mathrm{cl}(p) = \beta(p)$. For $A \subseteq B$ and an extension $q \in S(B)$ of p, it is obvious that $\beta(p) \subseteq \beta(q)$. Then q is a *non-forking extension* of p if $\beta(p) = \beta(q)$; otherwise, q is a *forking extension* of p.

Put

$$\mathcal{S} = \{\beta(p) : p \in S(A) \text{ for some } A \subseteq \mathbf{M}\}.$$

Clearly $\mathcal{S}$ is partially ordered by set inclusion. The *U-rank* of $p \in S(A)$, denoted $\mathrm{U}(p)$, is the foundation rank of $\beta(p)$. If $\mathrm{U}(p)$ is an ordinal, then we say that p *has U-rank*. Clearly, in our discussion of U-rank, we can assume that all types are over L-substructures of models of T.

Our objective in this section is to characterise those monoids S for which T_S is superstable or ω-stable. In other words, our goal is to prove the converses of (2) and (4) of Proposition 3.5. In fact, the converse of (4) follows easily from that of (2) so our effort is directed towards showing that if T_S is superstable, then S is weakly right noetherian. To do this, we use the characterisation of superstable theories in terms of U-rank of types. Then, by associating the U-rank of a type $p \in S(A)$ with ρ_p-$\mathrm{R}(I_p)$, we are able to achieve our goal.

Theorem 4.3. [20] *Let T be a complete, stable theory in a first order language. Then T is superstable if and only if all types have U-rank.*

Turning our attention to the theory T_S, we have the following characterisation of forking. Recall from Proposition 3.6 that if q is an extension of a type p, then $I_p \subseteq I_q$ and $\rho_p = \rho_q$.

Lemma 4.4. [10] *Let $A \subseteq B$ be S-acts, and let $q \in S(B)$ be an extension of $p \in S(A)$. Then q is a forking extension of p (equivalently, $U(p) > U(q)$) if and only if $I_p \subset I_q$.*

From Proposition 3.7 we know that if for an S-act A we have $p \in S(A)$ and $I_p \subset J$ for some ρ_p-saturated right ideal J, then there is an S-act $B \supseteq A$ and $q \in S(B)$ with $p \subseteq q$. From Lemma 4.4, $\mathrm{U}(p) > \mathrm{U}(q)$.

We can now associate the U-rank of types over T_S with ranks assigned to members of $\mathcal{C}$.

Proposition 4.5. [10] *For any S-act A and $p \in S(A)$,*

$$UR(p) = \rho_p\text{-}R(I_p).$$

Corollary 4.6. *For any S-act A and $p \in S(A)$, p has U-rank if and only if the set of ρ_p-saturated right ideals containing I_p satisfies the ascending chain condition.*

Part (1) of the following theorem is also a consequence of [17] (Theorem 2.4).

Theorem 4.7. *Let S be a right coherent monoid.*
(1) The theory T_S is superstable if and only if S is weakly right noetherian.
(2) If S is countable, then the theory T_S is ω-stable if and only if S is weakly right noetherian and has only countably many right congruences.

Proof. (1) If S is weakly right noetherian, then T_S is superstable by (2) of Proposition 3.5. Alternatively, this follows from Theorem 4.3 and Corollary 4.6.

Conversely, if T_S is superstable, then applying Corollary 4.6 to the type in $S(\emptyset)$ corresponding to the identity congruence gives that S is weakly right noetherian.

(2) Suppose that S is countable. If T_S is ω-stable, then it is superstable by [22] and so by (1), S is weakly right noetherian. Also we must have $|S(\emptyset)| \leq \aleph_0$ and hence by Corollary 3.4, the number of right congruences on S is countable. The converse is (4) of Proposition 3.5. $\square$

This theorem allows us to give examples of monoids to illustrate the various possibilities. Thus if $S = \{1\} \cup I$ where 1 acts as an identity and I is an infinite set with $ab = a$ for all $a, b \in I$, then I is a right ideal of S which is not finitely generated; moreover, it is easy to see that S is right coherent. Hence T_S exists, but is not superstable.

On the other hand, T_G is superstable for any group G. But, for example, the group of rationals Q has $2^{|Q|}$ subgroups (and hence $2^{|Q|}$ (right) congruences) so that T_Q is not ω-stable.

Both the infinite cyclic group and the quasi-cyclic group $Z(p^\infty)$ (p a prime number) have $\aleph_0$ subgroups so they provide specific examples of infinite groups G such that T_G is ω-stable.

Of course, for any *finite* monoid S we have that T_S is ω-stable.

5. Total transcendence of T_S

Having considered U-rank of types in the previous section we now turn our attention to another rank, the Morley rank $\mathrm{MR}(p)$, of a type p. This rank is used to define totally transcendental theories; to be precise a complete theory T is *totally transcendental* if and only if for all subsets A of models of T, all types over A have Morley rank.

For a countable theory T, it is a fact that T is totally trancendental if and only if T is ω-stable [22]. There are, however, uncountable theories T which are not totally transcendental but are κ-stable for all κ with $|T| \leq \kappa$.

When T is a theory of modules, if p is a type over a subset of a model of T such that $\mathrm{MR}(p)$ is defined, then $\mathrm{MR}(p) = \mathrm{U}(p)$ [28]. For S-acts, however, the picture is different and rather subtle. In this section we investigate those monoids S for which $\mathrm{MR}(p) = \mathrm{U}(p) < \infty$ for all types p over subsets of models of T_S, introducing a condition (MU). We also refer the reader to [16], where Ivanov presents a condition bearing some resemblance to (MU) that will imply $\mathrm{MR}(p) = \mathrm{U}(p)$. In a subsequent article [13] the second author builds on the techniques developed here to consider the more general question of for which monoids S do we have $\mathrm{U}(p) \leq \mathrm{MR}(p) < \infty$. Our algebraic characterisation of such monoids allows us to give examples of S such that T_S is totally transcendental but is such that $\mathrm{U}(p) < \mathrm{MR}(p)$ for some type p. We remark that for a complete, stable theory T, if $p \in S(A)$ and $q \in S(B)$ with $A \subseteq B$, $p \subseteq q$ and $\mathrm{MR}(p)$ an ordinal, then $\mathrm{U}(p) = \mathrm{U}(q)$ if and only if $\mathrm{MR}(p) = \mathrm{MR}(q)$ [27].

The two conditions on monoids used in the characterisation theorem are the right noetherian property (that is, all right congruences are finitely generated) and the condition (MU) which we now explain. Let S be a monoid and let (I, ρ) be a congruence pair, that is, $(I, \rho) \in \mathcal{C}$. We say that (I, ρ) is *critical* if there is a finite subset K of $(S \times S) \setminus \rho$ such that for all right congruences θ which saturate I, contain ρ and agree with ρ on I (i.e. $\theta \cap (I \times I) = \rho \cap (I \times I)$) we have

$$K \subseteq (S \times S) \setminus \theta \text{ implies } \rho = \theta \text{ or } \theta\text{-R}(I) < \rho\text{-R}(I).$$

We then say that S *satisfies* (MU) if every congruence pair of S is critical.

Note that for any right congruence ρ, the congruence pair (S, ρ) is critical. In the very special case where S is a group, to show that S satisfies (MU) we need only show that $(\emptyset, \rho)$ is critical for every right congruence ρ. In this case, for *any* right congruence θ, we have that $\theta\text{-R}(\emptyset) = 1$. Thus to show that $(\emptyset, \rho)$ is critical, we need to find a finite set $K \subseteq (S \times S) \setminus \rho$ such that if $\rho \subseteq \theta$ and $K \subseteq (S \times S) \setminus \theta$, then $\rho = \theta$.

For any right coherent monoid S, if $(I, \rho) \in \mathcal{C}$ and $\{s\rho : s \notin I\}$ is finite it is then easy to see that the pair (I, ρ) is critical.

Lemma 5.1. *For any right ideal I of a monoid S with S/I finite, every congruence pair (I, ρ) is critical. In particular, every finite monoid satisfies (MU).*

We now consider a useful sufficient condition for a monoid to satisfy (MU).

Proposition 5.2. *Let $C_r(S)$ be the lattice of right congruences of a monoid S. If $C_r(S)$ satisfies the minimal condition and each $\rho \in C_r(S)$ has only a finite number of covers, then S satisfies (MU).*

Proof. Let (I, ρ) be a congruence pair. If $S = I$, then we have already noted that the pair is critical. Otherwise, ρ cannot be universal since I is ρ-saturated and so the set of right congruences strictly containing ρ contains minimal members which are covers of ρ. Let $\rho_1, \dots, \rho_t$ be these covers. For each $i \in \{1, \dots, t\}$ choose a pair (a_i, b_i) in $\rho_i \setminus \rho$. Now put

$$K = \{(a_1, b_1), \dots, (a_t, b_t)\}.$$

Suppose that $\theta \in C_r(S)$ and $\rho \subseteq \theta$. If $\rho \neq \theta$, then it follows from the minimal condition that $\rho_i \subseteq \theta$ for some i. Thus $(a_i, b_i) \in \theta$ and consequently K is not contained in $(S \times S) \setminus \theta$. Hence the pair (I, ρ) is critical and consequently S satisfies (MU). □

For groups the converse of Proposition 5.2 is true as we now demonstrate.

Proposition 5.3. *A group G satisfies (MU) if and only if the lattice $L(G)$ of subgroups of G satisfies the minimal condition and every subgroup has only finitely many covers in $L(G)$.*

Proof. Suppose that G satisfies (MU) and let

$$\rho_1 \supseteq \rho_2 \supseteq \dots$$

be a decreasing sequence of right congruences. Put $\rho = \bigcap\{\rho_i : i \in \omega\}$. By assumption, $(\emptyset, \rho)$ is critical and so there is a finite set K such that $K \subseteq (G \times G) \setminus \rho$ and if $K \subseteq (G \times G) \setminus \rho_m$, then $\rho = \rho_m$. If $(a, b) \in K$, then $(a, b) \notin \rho_t$ for some t and since K is finite, it follows that for some m we do have $K \subseteq (G \times G) \setminus \rho_m$. Hence $\rho_m = \rho_{m+1} = \ldots$ and $C_r(G)$ satisfies the minimal condition.

In view of the minimal condition, every $\rho \in C_r(G)$ except $G \times G$ actually does have covers. If $\{\rho_\lambda : \lambda \in \Lambda\}$ is the set of covers of ρ, then $\rho_\lambda \cap \rho_\mu = \rho$ for each $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$. Hence, if $(a, b) \notin \rho$, then (a, b) can belong to at most one of the covers of ρ. Since $(\emptyset, \rho)$ is critical, there is a finite set K such that $K \subseteq (G \times G) \setminus \rho$ and K is not contained in $(G \times G) \setminus \rho_\lambda$ for any cover ρ_λ of ρ. But any given pair in K is in at most one cover of ρ and so there are only finitely many covers of ρ.

Now use the fact that the lattice of right congruences on a group is isomorphic to the lattice of subgroups. $\square$

We note that the quasi-cyclic group $Z(p^\infty)$ where p is a prime number satisfies the conditions of Proposition 5.3 and thus satisfies (MU). On the other hand the infinite cyclic group does not satisfy the minimal condition for subgroups and hence does not satisfy (MU). It is, in fact, easy to show that the congruence pair $(\emptyset, \iota)$ is not critical in this case.

We have introduced the condition (MU) to help in our discussions of Morley rank. To define the latter we use make use of the natural topology on Stone spaces of types.

Let T be a complete theory and let $A \subseteq \mathbf{M}$. Then $S(A)$ may be made into a topological space by specifying the sets

$$\langle \phi(x) \rangle = \{p \in S(A) : \phi(x) \in p\}$$

as a basis of open sets, where $\phi(x)$ is a formula of $L(A)$. The space $S(A)$ has a basis of clopen sets $\langle \phi(x) \rangle$, and is compact and Hausdorff.

If T is a theory which has elimination of quantifiers (for example, T_S), then a routine argument gives that the sets $\langle \theta(x) \rangle$ where $\theta(x)$ is a conjunction of atomic and negated atomic formulae form a basis for the topology of $S(A)$.

Let T be a complete theory in a first order language L and let A be a subset of a model of T. Subsets $MR^\alpha(A)$ of $S(A)$ are defined by induction on the ordinal α as follows:

(I) $MR^0(A) = S(A)$.

(II) If α is a limit ordinal, then

$$MR^\alpha(A) = \bigcap\{MR^\beta(A) : \beta < \alpha\}.$$

(III) For any α, $MR^{\alpha+1}(A) = MR^{\alpha}(A) \setminus X^{\alpha}$, where

$$X^{\alpha} = \{p \in MR^{\alpha}(A) : \text{ for all } B \supseteq A \text{ and all extensions } q \text{ of } p \text{ on } B,$$

$$q \notin MR^{\alpha}(B) \text{ or } q \text{ is isolated in } MR^{\alpha}(B)\}.$$

We may take B to be an L-substructure of a model of T.

For $p \in S(A)$, the *Morley rank* of p is $\mathrm{MR}(p)$ where, if $p \in MR^{\alpha}(A)$ for all α, then $\mathrm{MR}(p) = \infty$ and otherwise $\mathrm{MR}(p)$ is α where $p \in MR^{\alpha}(A) \setminus MR^{\alpha+1}(A)$. If $\mathrm{MR}(p) < \infty$, then we say that *p has Morley rank*.

It is a standard result that for all types p, $\mathrm{U}(p) \leq \mathrm{MR}(p)$ [27]; we need this in the proof of the main result of this section. We first note that for any type p over an S-act A, $\mathrm{MR}(p) = 0$ if and only if $I_p = S$, that is $\mathrm{U}(p) = 0$. For if $I_p = S$ and $p \subseteq q$ where $q \in S(B)$, then since $1 \in I_q$, $x = b \in q$ for some $b \in B$ and $\{x = b\}$ isolates q in $S(B)$. Thus $p \notin MR^1(A)$ so that $\mathrm{MR}(p) = 0$. The converse is clear.

We can now state the main result of this section; the interested reader may wish to compare our result with that of [17], where some conditions are given which imply that for a relevant type p, $\mathrm{MR}(p) = \mathrm{U}(p)$.

Theorem 5.4. *For every type p over an S-act A, $MR(p) = U(p) < \infty$ if and only if S is right noetherian and satisfies (MU).*

Proof. Suppose first that the condition on ranks of types holds. Let (I, ρ) be a congruence pair. By Corollary 3.4, there is an S-act A and a type p over A with $I_p = I, \rho_p = \rho$. Let the associated A-triple be (I, ρ, f) and let p have Morley rank α. Then there is an open set U in $S(A)$ such that $p \in U$ and $\mathrm{MR}(q) < \alpha$ for all q in $U \setminus \{p\}$. Let $U = \langle \phi(x) \rangle$ where $\phi(x)$ is a conjunction of sets of formulae:-

$$\{xr_i = a_i : i \in \Lambda_1\}, \{xs_j = xt_j : j \in \Lambda_2\},$$

$$\{xu_k \neq xv_k : k \in \Lambda_3\}, \{xw_\ell \neq b_\ell : \ell \in \Lambda_4\}$$

where the index sets $\Lambda_1, ..., \Lambda_4$ are all finite. Since $p \in \langle \phi(x) \rangle$, each r_i is a member of I and each pair (s_j, t_j) is in ρ.

Let θ be any right congruence on S which saturates I, properly contains ρ and agrees with ρ on I. Then (I, θ, f) is an A-triple; let $\overline{p}$ be the associated type over A. Certainly each pair (s_j, t_j) is in θ since $\rho \subseteq \theta$. Thus we see that the sets $\{xr_i = a_i : i \in \Lambda_1\}$ and $\{xs_j = xt_j : j \in \Lambda_2\}$ are contained in $\overline{p}$. If the formula $xw_\ell = b_\ell$ is in $\overline{p}$ for some $\ell \in \Lambda_4$, then $w_\ell \in I$ and $f(w_\ell) = b_\ell$ and consequently, $xw_\ell = b_\ell$ is in p, a contradiction. Thus each inequation $xw_\ell \neq b_\ell$ is in $\overline{p}$ and we see that $\phi(x) \in \overline{p}$ if and only if $xu_k \neq xv_k$ is in $\overline{p}$ for each $k \in \Lambda_3$.

Let $K = \{(u_1, v_1), ..., (u_m, v_m)\}$; since $xu_k \neq xv_k$ is in p we certainly have that $K \subseteq (S \times S) \setminus \rho$. If $K \subseteq (S \times S) \setminus \theta$, then we have $\phi(x) \in \overline{p}$ so that $\overline{p} \in \langle \phi(x) \rangle$ and hence $\mathrm{MR}(\overline{p}) < \mathrm{MR}(p)$. But $\mathrm{U}(\overline{p}) = \mathrm{MR}(\overline{p})$ and $\mathrm{U}(p) = \mathrm{MR}(p)$ so that θ-$\mathrm{R}(I) < \rho$-$\mathrm{R}(I)$. Thus (I, ρ) is critical and hence S satisfies (MU).

To see that S is right noetherian we consider the case $I = \emptyset$. Let σ be the right congruence on S generated by $\{(s_j, t_j) : j \in \Lambda_2\}$. Certainly $\sigma \subseteq \rho$ and if p_1

is the type over $\emptyset$ associated with σ, then clearly $p_1 \in \langle \phi(x) \rangle$. Hence, using our assumption on ranks,

$$\mathrm{MR}(p) = \mathrm{U}(p) = \rho\text{-R}(\emptyset) \leq \sigma\text{-R}(\emptyset) = \mathrm{U}(p_1) = \mathrm{MR}(p_1) \leq \mathrm{MR}(p),$$

that is, $\mathrm{MR}(p_1) = \mathrm{MR}(p)$. By the choice of $\langle \phi(x) \rangle$, we have that $p = p_1$ so that $\rho = \sigma$ and ρ is finitely generated.

Conversely, suppose that S is right noetherian and satisfies (MU). By Theorem 4.7, T_S is certainly superstable so that for every S-act A, every type p in $S(A)$ has U-rank. We show by induction that for every p, $\mathrm{MR}(p) = \mathrm{U}(p)$.

If $\mathrm{U}(p) = 0$, then $I_p = S$ and so, as already noted, $\mathrm{MR}(p) = 0$.

Now let $p \in S(A)$ and $\mathrm{U}(p) = \alpha$ and suppose that for all S-acts B and all types $q \in S(B)$ with $\mathrm{U}(q) < \alpha$ we have $\mathrm{MR}(q) = \mathrm{U}(q)$. Let $I = I_p$, $\rho = \rho_p$. Certainly $\mathrm{U}(p) \leq \mathrm{MR}(p)$ so we have $p \in MR^{\alpha}(A)$ and we wish to show that $p \notin MR^{\alpha+1}(A)$, that is, for every S-act B containing A and every extension q of p over B we want either $q \notin MR^{\alpha}(B)$ or q is isolated in $MR^{\alpha}(B)$.

So let $q \in S(B)$ where B is an extension of A and $q|A = p$. Suppose that $q \in MR^{\alpha}(B)$. We have to find an open set U such that $MR^{\alpha}(B) \cap U = \{q\}$. By Proposition 3.6, we have $I \subseteq I_q$ and $\rho = \rho_q$. Now $\alpha \leq \mathrm{MR}(q)$ and so by the inductive assumption we cannot have $\mathrm{U}(q) < \alpha$. But $\mathrm{U}(q) \leq \mathrm{U}(p) = \alpha$ so that $\mathrm{U}(q) = \alpha$. Now by the definition of U-rank, we must have that q is a non-forking extension of p and so by Lemma 4.4, $I = I_q$.

As S is right noetherian, $I = \bigcup\{w_i S : i \in \Lambda\}$ for some finite set Λ and ρ is generated by a finite subset H of $S \times S$. For each $i \in \Lambda$, let $a_i = f_q(w_i)$. By assumption, the pair (I, ρ) is critical. Let K be the finite subset of $(S \times S) \setminus \rho$ required in the definition of criticality and let $\xi(x)$ be the formula obtained by taking the conjunction of the following sets of formulae:

$$\{xw_i = a_i : i \in \Lambda\}, \{xs = xt : (s,t) \in H\}, \{xu \neq xv : (u,v) \in K\}.$$

Then $q \in \langle \xi(x) \rangle$. Let $r \in \langle \xi(x) \rangle$ and suppose that $\mathrm{MR}(r) \geq \alpha$. Our aim is to show that $r = q$ and this will complete the proof that $\mathrm{U}(p) = \mathrm{MR}(p)$ and hence prove the result by induction.

Note that $I \subseteq I_r$ and $\rho \subseteq \rho_r$ so that I_r is ρ-saturated. If $I \neq I_r$, then, by Proposition 3.7, q has an extension $\overline{q}$ with $I_{\overline{q}} = I_r$ and by Lemma 4.4, $\overline{q}$ is a forking extension of q. Hence, $\mathrm{U}(\overline{q}) < \mathrm{U}(q) = \alpha$. By Proposition 3.6, $\rho_{\overline{q}} = \rho$ and thus

$$\mathrm{U}(r) = \rho_r\text{-R}(I_r) \leq \rho_{\overline{q}}\text{-R}(I_{\overline{q}}) = \mathrm{U}(\overline{q}) < \alpha.$$

The inductive assumption gives $\mathrm{MR}(r) < \alpha$, a contradiction, so that we may suppose that $I = I_r$.

Since f_q and f_r agree on the set of generators $\{w_i : i \in \Lambda\}$ of I, it follows that $f_q = f_r$ and $\rho_r \cap (I \times I) = \ker f_r = \ker f_q = \rho \cap (I \times I)$.

If $\rho \neq \rho_r$, then as $K \subseteq (S \times S) \setminus \rho_r$ we have $\rho_r\text{-R}(I_r) < \rho\text{-R}(I)$ so that $\mathrm{U}(r) < \mathrm{U}(q) = \alpha$ and the inductive assumption gives $\mathrm{MR}(r) < \alpha$, a contradiction. Thus $\rho_r = \rho$ and, as $f_r = f_q$, Corollary 3.4 now gives $r = q$ as desired. $\square$

We have noted already that the infinite cyclic group does not satisfy (MU) although, of course, it is (right) noetherian. On the other hand the group $Z(p^\infty)$ is not (right) noetherian but does satisfy (MU). Thus the two conditions in the theorem are independent. Furthermore, these observations also show that there are monoids S such that T_S is totally transcendental (ω-stable) but such that for some S-act A there is a type p in $S(A)$ with $\mathrm{U}(p) < \mathrm{MR}(p)$.

We can be more precise with our two examples. For any group G and any type p over a G-set A we have $I_p = G$ or $I_p = \emptyset$. In the former case $\mathrm{U}(p) = \mathrm{MR}(p) = 0$ and in the latter case $\mathrm{U}(p) = 1$. It is not difficult to see that if $p \in S(\emptyset)$ (so that necessarily $I_p = \emptyset$), then for any G-set A there is exactly one extension p_A of p in $S(A)$ with $I_{p_A} = \emptyset$. A simple argument using transfinite induction shows that for all ordinals $\alpha \geq 1$, $\mathrm{MR}(p) \geq \alpha$ if and only if $\mathrm{MR}(p_A) \geq \alpha$ for all G-sets A. It follows that $\mathrm{MR}(p) = \alpha$ if and only if $p \in MR^\alpha(\emptyset)$ and p is isolated in $MR^\alpha(\emptyset)$. Moreover, $\mathrm{MR}(p) = \mathrm{MR}(p_A)$ for all G-sets A.

It is now not difficult to show that for the infinite cyclic group G with generator g, if p_n is the type in $S(\emptyset)$ corresponding to the subgroup generated by g^n, then $\mathrm{MR}(p_n) = 1$ for $n \geq 1$ and $\mathrm{MR}(p_0) = 2$. Thus $\mathrm{U}(p_0) < \mathrm{MR}(p_0)$.

Similarly, if $G = Z(p^\infty)$ is regarded as the group of all p^n-th roots of unity for all $n \geq 1$ and if for each n, p_n is the type in $S(\emptyset)$ corresponding to the subgroup generated by a primitive p^n-th root of one, then $\mathrm{MR}(p_n) = 1$. For the type p_∞ in $S(\emptyset)$ corresponding to G itself we find that $\mathrm{MR}(p_\infty) = 2$ so that $\mathrm{U}(p_\infty) < \mathrm{MR}(p_\infty)$.

6. Right noetherian monoids which satisfy (MU)

The main result of the preceding section makes it natural to consider the monoids of the title. As the condition (MU) is rather complicated it is far from clear precisely which monoids satisfy (MU). Of course, any finite monoid is right noetherian and also, by Lemma 5.1, satisfies (MU). One of the main results of this section shows that the converse is true for an extensive class of monoids, namely the weakly periodic monoids. However, not every right noetherian monoid which satisfies (MU) is finite. We will show that an infinite example is the free commutative monoid on one generator.

Our first objective is to show that (right) noetherian groups which satisfy (MU) are finite. To this end we need the lemma below which can be deduced from König's Lemma, but which is very easy to prove directly in much the same way that König's Lemma is proved.

Lemma 6.1. *Let Y be a lattice satisfying the finite chain condition. If every member of Y has only finitely many covers, then Y is finite.*

Proof. Since Y satisfies the descending chain condition, it has a least element x_0. If Y is infinite, then since x_0 has only finitely many covers, x_0 has a cover x_1 such the filter above x_1 is infinite. But x_1 has only finitely many covers, so there must be one of these, say x_2, such that the filter above x_2 is infinite. Continuing in this way we find an infinite chain

$$x_0 < x_1 < x_2 < \dots$$

of elements of Y, contradicting the ascending chain condition. $\square$

Corollary 6.2. *Let G be a right noetherian group which satisfies (MU). Then G is finite.*

Proof. By Proposition 5.3, the lattice $\mathcal{L}(G)$ of subgroups of G satisfies the minimal condition and every subgroup has only finitely many covers in $\mathcal{L}(G)$. Since $\mathcal{L}(G)$ also satisfies the maximal condition, it has the finite chain condition and by Lemma 6.1, $\mathcal{L}(G)$ is finite. As pointed out on pp.170-171 of [2], it follows easily that G is finite. $\square$

The next stage in our argument is to show that any subgroup of a monoid which is right noetherian and satisfies (MU) inherits these properties. To do this we utilise some classical semigroup theory, in particular, basic results about Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$. The relation $\mathcal{L}$ is defined on a monoid S by the rule that for any $a, b \in S$, $a \,\mathcal{L}\, b$ if and only if $Sa = Sb$. The relation $\mathcal{R}$ is defined dually; $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. Note that $\mathcal{L}$ ($\mathcal{R}$) is a right (left) congruence. Details may be found in any of the standard texts. We recommend [15].

Lemma 6.3. *If the monoid S is right noetherian, then so is every subgroup.*

Proof. Let G be a subgroup of S. For any right congruence ρ on G, let $\overline{\rho}$ denote the right congruence *on* S generated by ρ. If $a, b \in S$ and $a \,\overline{\rho}\, b$, then $a = b$ or there exists a sequence

$$a = c_1 t_1, d_1 t_1 = c_2 t_2, \dots, d_\ell t_\ell = b,$$

where $(c_i, d_i) \in \rho, 1 \leq i \leq \ell$. Notice in particular that $a \,\mathcal{L}\, b$. Suppose now that $a, b \in G$. We claim that $\overline{\rho} \cap (G \times G) = \rho$. Let e be the identity of G. Then we certainly have

$$a = c_1(et_1), d_1(et_1) = c_2(et_2), \dots, d_\ell(et_\ell) = b,$$

Taking inverses in G we have

$$et_1 = c_1^{-1} a \in G.$$

This gives that $a \,\rho\, d_1(et_1)$. Now

$$et_2 = c_2^{-1} d_1(et_1) \in G,$$

so that $a \,\rho\, d_2(et_2)$. Continuing in this manner we obtain $a \,\rho\, b$. Thus G is $\overline{\rho}$-saturated and $\overline{\rho} \cap (G \times G) = \rho$ as required. It is now easy to see that if S is right noetherian, so also is G. $\square$

Lemma 6.4. *If the monoid S is right noetherian and satisfies (MU), then so does every maximal subgroup.*

Proof. Let G be a maximal subgroup of S, so that G is a group $\mathcal{H}$-class. We already know from Lemma 6.3 that G is (right) noetherian. Suppose now that S satisfies (MU). To show that G satisfies (MU) we need only prove that the pair $(\emptyset, \rho)$ is critical for any right congruence ρ on G.

Let e be the identity of G, let

$$I = \cup\{SaS : SaS \subset SeS\} \text{ and } J = SeS.$$

Then I and J are ideals of S. From Theorem 1.3 of [14] we know that the principal factor J/I is completely 0-simple or completely simple. Let $\overline{\rho}$ be defined as in Lemma 6.3; since $\rho \subseteq \mathcal{L}$ and $\mathcal{L}$ is a right congruence, we have that $\overline{\rho} \subseteq \mathcal{L}$. Thus any ideal of S is $\overline{\rho}$-saturated. Let ν_I be the Rees congruence associated with I, so that for any $a, b \in S$, $a \,\nu_I\, b$ if and only if $a = b$ or $a, b \in I$. Since I is $\overline{\rho}$-saturated and ν_I-saturated, it is clear that $\tilde{\rho} = \overline{\rho} \cup \nu_I$ is a right congruence saturating I. Moreover, for any $a, b \in S$, if $a \neq b$ and $a \,\tilde{\rho}\, b$, then either $a, b \in I$ or $a, b \in J \setminus I$. In the latter case, we have $a \overline{\rho}\, b$ and so, since J/I is completely (0-)simple, it follows that $a \,\mathcal{H}\, b \,\mathcal{R}\, e$. Consequently, any right ideal containing I is $\tilde{\rho}$-saturated. Thus if θ is any right congruence on G, then $\tilde{\rho}$-R$(I) = \tilde{\theta}$-R(I).

The congruence pair $(I, \tilde{\rho})$ is critical; let $K \subseteq (S \times S) \setminus \tilde{\rho}$ be a finite set of pairs guaranteed by the fact that $(I, \tilde{\rho})$ is critical. We need to pick a set of pairs of elements of G that will enable us to show that $(\emptyset, \rho)$ is critical.

For any pair

$$(a, b) \in K \cap \mathcal{H} \cap (R_e \times R_e)$$

choose and fix $c = c_{(a,b)} \in J \setminus I$ with $ac, bc \in G$. It follows from the fact that J/I is completely (0)-simple that $(ac, bc) \notin \rho$. We now put

$$H = \{(ac, bc) : (a, b) \in K \cap \mathcal{H} \cap (R_e \times R_e)\},$$

so that $H \subseteq (G \times G) \setminus \rho$.

Let θ be a right congruence on G containing ρ and such that $H \subseteq (G \times G) \setminus \theta$. Certainly $\tilde{\rho} \subseteq \tilde{\theta}$, I is $\tilde{\theta}$-saturated and $\tilde{\rho} \cap (I \times I) = \tilde{\theta} \cap (I \times I)$. If $K \nsubseteq (S \times S) \setminus \tilde{\theta}$, then there exists $(a, b) \in K \cap \tilde{\theta}$. But $(a, b) \notin \tilde{\rho}$, so we are forced to deduce that $a, b \in R_e$ and $a \,\mathcal{H}\, b$. Consequently,

$$(ac, bc) \in \tilde{\theta} \cap (G \times G) = \overline{\theta} \cap (G \times G) = \theta.$$

But $(ac, bc) \in H$, a contradiction. Thus $K \subseteq (S \times S) \setminus \tilde{\theta}$. Now by the definition of critical pair, $\tilde{\rho} = \tilde{\theta}$ or $\tilde{\rho}$-R$(I) < \tilde{\theta}$-R(I). But the latter is impossible by previous comments on saturation of right ideals. We conclude that $\tilde{\rho} = \tilde{\theta}$ and consequently, $\rho = \theta$ as required. $\square$

From Lemmas 6.2, 6.4 we deduce the following.

Theorem 6.5. *If S is a right noetherian monoid which satisfies (MU), then all subgroups of S are finite.*

A semigroup S is *weakly periodic* if for every element s of S there is a positive integer $n = n(s)$ such that $I^2 = I$ where $I = S^1 s^n S^1$. If S is a semigroup which satisfies the minimal condition for principal ideals or for principal right (or left) ideals or if S is periodic, then S is weakly periodic. Regular and eventually regular (some power of any element is regular) semigroups are weakly periodic as are semisimple semigroups, that is, semigroups with no null principal factors.

Corollary 6.6. *If S is a weakly periodic right noetherian monoid which satisfies (MU), then S is finite.*

Proof. By Theorem 6.5, all subgroups of S are finite. Hence by Theorem 2.3 of [14], S is finite. □

Corollary 6.7. *Let S be a right noetherian monoid which satisfies (MU). If the relation $\mathcal{R}$ is a congruence on S and there are only finitely many trivial $\mathcal{R}$-classes, then S is finite.*

Proof. We show that S is weakly periodic so that the result follows from Corollary 6.6. Let $a \in S$ and consider the sequence $S \supseteq aS \supseteq a^2S \supseteq \ldots.$ Let $I = \bigcap\{a^iS : i \in \omega\}$, ρ be the Rees right congruence associated with I and ρ_i that associated with a^iS. If $I = \emptyset$, then we take ρ to be ι. The pair (I, ρ) is critical and so there is a finite subset K of $(S \times S) \setminus \rho$ such that for any right congruence θ with $K \subseteq (S \times S) \setminus \theta$ where θ saturates I, agrees with ρ on I and contains ρ, we have either $\rho = \theta$ or θ-$\mathrm{R}(I) < \rho$-$\mathrm{R}(I)$. Since K is finite, $K \subseteq (S \times S) \setminus \rho_n$ for some n. By hypothesis, $a^pS = I$ for some p, or there is an element a^m with $n \leq m$ whose $\mathcal{R}$-class is non-trivial.

In the latter case, suppose that $a^mS \neq I$. Let x, y be distinct elements in the $\mathcal{R}$-class of a^m and let ν be the right congruence generated by the set $\rho \cup \{(x, y)\}$. It is easy to see that if $(u, v) \in \nu$ and $u \neq v$, then $u, v \in a^mS$ and either $u \,\mathcal{R}\, v$ or $u, v \in I$. Thus $\rho \subset \nu \subseteq \rho_m$ and hence $K \subseteq (S \times S) \setminus \nu$. Furthermore, ν saturates I and agrees with ρ on I and consequently, ν-$\mathrm{R}(I) < \rho$-$\mathrm{R}(I)$. But all right ideals which contain I are both ρ-saturated and ν-saturated since as noted above, if $(u, v) \in \nu$ and $u, v \notin I$, then $u \,\mathcal{R}\, v$. Hence ν-$\mathrm{R}(I) = \rho$-$\mathrm{R}(I)$, a contradiction. It follows that if $a \in S$ then the descending chain of principal right ideals $S \supseteq aS \supseteq a^2S \supseteq \ldots$ is finite. Thus $a^qS = I$ for some q so that $a^qS = a^{q+1}S = \ldots.$ Hence $a^q = a^{2q}s$ for some $s \in S$ and so $a^qS = (a^qS)^2$. It follows that $Sa^qS = (Sa^qS)^2$, and S is weakly periodic. □

On a commutative monoid the relations $\mathcal{H}, \mathcal{R}$ and $\mathcal{L}$ coincide and $\mathcal{R}$ is automatically a congruence. The following result is thus an immediate consequence of Corollary 6.7.

Corollary 6.8. *Let S be a noetherian commutative monoid which satisfies (MU). If S has only finitely many trivial $\mathcal{H}$-classes, then S is finite.*

We now give an example of an infinite noetherian commutative monoid which satisfies (MU). Of course, in view of Corollary 6.8, our example must have infinitely many trivial $\mathcal{H}$-classes.

Proposition 6.9. *The additive monoid* $\mathbb{N}$ *of non-negative integers is noetherian and satisfies (MU).*

Proof. It is well known and easy to show directly that $\mathbb{N}$ is noetherian. If I is a non-empty ideal of $\mathbb{N}$, then $\mathbb{N}/I$ is finite so that it follows from Lemma 5.1 that any congruence pair (I, ρ) is critical. It remains to consider pairs $(\emptyset, \rho)$. If $\rho = \iota$, then ι-R$(\emptyset) = \omega$. When $\rho \neq \iota$, let r, m be the smallest integers such that $(r, r+m) \in \rho$ and $m \geq 1$. In fact, from page 137, Exercise 5 of [7] we know that ρ is generated by $(r, r+m)$. It is then easy to see that ρ-R$(\emptyset)$ is finite so that $(\emptyset, \iota)$ is critical by choosing $K = \emptyset$. Further, putting

$$K = \{(s, s+n) : 0 \leq s \leq r, 0 \leq n \leq m\} \setminus \{(r, r+m)\},$$

it is clear that $K \subseteq (S \times S) \setminus \rho$. But if $\rho \subset \theta$, then $K \cap \theta \neq \emptyset$ and consequently the pair $(\emptyset, \rho)$ is critical. Thus $\mathbb{N}$ satisfies (MU). $\square$

In our final result we show that $\mathbb{N}$ is the only infinite commutative cancellative principal ideal monoid which is both noetherian and satisfies (MU).

Proposition 6.10. *Let S be a commutative, cancellative principal ideal monoid. Then S is noetherian and satisfies (MU) if and only if S is a finite group or is isomorphic to* $\mathbb{N}$.

Proof. Suppose that S is noetherian and satisfies (MU). If S is finite, then since it is cancellative, it must be a group.

If S is infinite, then by Corollary 6.8, S must have infinitely many trivial $\mathcal{H}$-classes. Let a be a unit of S so that $a\,\mathcal{H}\,1$. For any element $c \in S$, we have $ac\,\mathcal{H}\,c$ since $\mathcal{H}$ is a congruence on S. If $a \neq 1$ then $ac \neq c$ since S is cancellative and so H_c is non-trivial unless $a = 1$. Thus the group of units of S is trivial. It follows from Theorem 12 of [8] that S is isomorphic to $\mathbb{N}$. $\square$

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