Beyond the Ehresmann–Schein–Nambooripad Theorem

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Inverse semigroups

An inverse semigroup is a semigroup S in which every element has a unique generalised inverse: for each $s \in S$, there exists a unique $s' \in S$ such that

$$ss's = s \qquad \text{and} \qquad s'ss' = s'.$$

Natural partial order: $s \leq t \Leftrightarrow s = et \Leftrightarrow s = tf$, for idempotents e, f ($e = ss^{-1}$, $f = s^{-1}s$).

Compatible with multiplication and restricts to usual order on idempotents: $e \leq f \Leftrightarrow e = ef$.

Groupoids

Let G be a set, let $\cdot$ be a partial binary operation on G.

If the product $x \cdot y$ is defined, denote this by '$\exists x \cdot y$'.

The identities e of G are those elements which satisfy:

$$\exists e \cdot e = e, \quad [\exists e \cdot x \Rightarrow e \cdot x = x], \quad [\exists x \cdot e \Rightarrow x \cdot e = x].$$

Denote the subset of identities of G by G_o.

Groupoids

Then $(G, \cdot)$ is an groupoid if

(C1) $\exists x \cdot (y \cdot z) \iff \exists (x \cdot y) \cdot z$, in which case $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

(C2) $\exists x \cdot (y \cdot z) \iff \exists x \cdot y$ and $\exists y \cdot z$;

(C3) $\forall x \in G$, $\exists !\, \mathbf{d}(x), \mathbf{r}(x) \in G_o$ such that $\exists \mathbf{d}(x) \cdot x$ and $\exists x \cdot \mathbf{r}(x)$;

(G) $\forall x \in G$, $\exists x^{-1} \in G$ such that

$$\exists x \cdot x^{-1} = \mathbf{d}(x) \quad \text{and} \quad \exists x^{-1} \cdot x = \mathbf{r}(x).$$

$$\exists x \cdot y \Leftrightarrow \mathbf{r}(x) = \mathbf{d}(y).$$

Inductive groupoids

Let $(G, \cdot)$ be a groupoid and suppose that G is partially ordered by $\leq$. Then $(G, \cdot, \leq)$ is an inductive groupoid if

(O1) $a \leq c$, $b \leq d$, $\exists a \cdot b$ and $\exists c \cdot d \Longrightarrow a \cdot b \leq c \cdot d$;

(O2) $a \leq b \Longrightarrow \mathbf{r}(a) \leq \mathbf{r}(b)$ and $\mathbf{d}(a) \leq \mathbf{d}(b)$;

(O3) $f \in G_o$, $a \in G$, $f \leq \mathbf{d}(a) \Longrightarrow \exists!$ restriction $f|a$, such that $f|a \leq a$ and $\mathbf{d}(f|a) = f$;

(O4) $f \in G_o$, $a \in G$, $f \leq \mathbf{r}(a) \Longrightarrow \exists!$ corestriction $a|f$, such that $a|f \leq a$ and $\mathbf{r}(a|f) = f$;

(I) $e, f \in G_o \Longrightarrow e \wedge f$ (w.r.t. $\leq$) exists in G_o.

The 'objects' part of the ESN Theorem

Let S be an inverse semigroup with natural partial order $\leq$. Define the restricted product $\cdot$ in S by

$$a \cdot b = \begin{cases} ab & \text{if } a^{-1}a = bb^{-1}; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $\mathbf{G}(S) = (S, \cdot, \leq)$ is an inductive groupoid with $S_o = E(S)$, $\mathbf{d}(x) = xx^{-1}$ and $\mathbf{r}(x) = x^{-1}x$; $e|a = ea$, $a|e = ae$.

Let G be an inductive groupoid. Define the pseudoproduct in G by

$$a \otimes b = [a|\mathbf{r}(a) \wedge \mathbf{d}(b)] \cdot [\mathbf{r}(a) \wedge \mathbf{d}(b)|b] \,.$$

Then $\mathbf{I}(G) = (G, \otimes)$ is an inverse semigroup.

$\mathbf{G}(\mathbf{I}(G)) = G$ and $\mathbf{I}(\mathbf{G}(S)) = S$.

Morphisms and functors as arrows

Lemma

Let $\varphi : S \to T$ be a morphism of inverse semigroups. Define $\mathbf{G}(\varphi)$ to be the same function on the underlying sets. Then $\mathbf{G}(\varphi) : \mathbf{G}(S) \to \mathbf{G}(T)$ is an inductive functor with respect to the restricted products in $\mathbf{G}(S)$ and $\mathbf{G}(T)$.

Lemma

Let $\psi : G \to H$ be an inductive functor of inductive groupoids. Define $\mathbf{I}(\psi)$ to be the same function on the underlying sets. Then $\mathbf{I}(\psi) : \mathbf{I}(G) \to \mathbf{I}(H)$ is a morphism with respect to the pseudoproducts in $\mathbf{I}(G)$ and $\mathbf{I}(H)$.

A few technical details...

If $\varphi_1 : S \to T_1$ and $\varphi_2 : T_1 \to T_2$ are morphisms, then

- $\mathbf{I}(\mathbf{G}(\varphi_1)) = \varphi_1$;
- $\mathbf{G}(\varphi_1 \varphi_2) = \mathbf{G}(\varphi_1)\mathbf{G}(\varphi_2)$.

Similarly for inductive functors $\psi_1 : G \to H_1$, $\psi_2 : H_1 \to H_2$.

Thus $\mathbf{G}(\cdot)$ and $\mathbf{I}(\cdot)$ are mutually inverse functors.

(Part of)
The Ehresmann–Schein–Nambooripad (ESN) Theorem

Theorem (Various authors: see Lawson 1998)

The category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors.

A little bit of history

Definition (Veblen & Whitehead 1932)

A pseudogroup Γ is a collection of partial homeomorphisms between open subsets of a topological space such that Γ is closed under composition and inverses, where we compose $\alpha, \beta \in \Gamma$ only if im $\alpha =$ dom β.

A little bit of history

What is the abstract structure corresponding to a pseudogroup?

Attempts to 'complete' the operation in a pseudogroup:

- Schouten and Haantjes (1937): compose α, β whenever $\operatorname{im}\alpha \subseteq \operatorname{dom}\beta$.
- Gołab (1939): compose α, β whenever im $\alpha \cap$ dom $\beta \neq \emptyset$.

But 'empty transformation' still missing.

A little bit of history

Final piece of puzzle provided by Viktor Vladimirovich Wagner in 1952: composition of functions is a special case of composition of binary relations — empty transformation now appears naturally.

Studied symmetric inverse semigroup $\mathcal{I}_X$ with α, β composed on

$$\operatorname{dom} \alpha\beta = (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\alpha^{-1}.$$

Led to abstract notion of inverse semigroup (generalised group).

Introduced independently by Gordon Preston in 1954 through study of partial one-one mappings of a set.

A little bit of history

Meanwhile, Charles Ehresmann (from 1957) retained Veblen and Whitehead's partial composition.

Studied local structures — structures defined on toplogical spaces using pseudogroups, by analogy with the use of groups in geometry.

Identified (inductive) groupoid structure.

Connection between inverse semigroups and inductive groupoids made by Boris Schein (1965,1979).

Nambooripad (1979) obtained similar results relating regular semigroups and ordered groupoids.

Nambooripad and Veeramony (1983): an ESN-type theorem for 'premorphisms'.

$\vee$- and $\wedge$-premorphisms

McAlister and Reilly (1977) introduced the following functions between inverse semigroup S and T:

A $\vee$-premorphism is a function $\theta : S \to T$ such that

($\vee$1) $(st)\theta \leq (s\theta)(t\theta)$.

A $\wedge$-premorphism is a function $\theta : S \to T$ such that

($\wedge$1) $(s\theta)(t\theta) \leq (st)\theta$;

($\wedge$2) $(s\theta)^{-1} = s^{-1}\theta$.

$\vee$-premorphisms and inverses

Need not demand explicitly that $\vee$-premorphisms respect inverses:

Lemma (McAlister 1980)

$\vee$-premorphisms respect inverses and the natural partial order.

However, $\wedge$-premorphisms do not automatically preserve inverses or ordering.

Composition of $\vee$-premorphisms

Lemma

The composition of two $\vee$-premorphisms is a $\vee$-premorphism, hence inverse semigroups and $\vee$-premorphisms form a category.

$\vee$-premorphisms and ordered functors

Lemma

Let $\theta : S \to T$ be a $\vee$-premorphism between inverse semigroups. Define $\mathbf{G}(\theta) : \mathbf{G}(S) \to \mathbf{G}(T)$ to be the same function on the underlying sets. Then $\mathbf{G}(\theta)$ is an ordered functor with respect to the restricted products in $\mathbf{G}(S)$ and $\mathbf{G}(T)$.

Lemma

Let $\phi : G \to H$ be an ordered functor between inductive groupoids. Define $\mathbf{I}(\phi) : \mathbf{I}(G) \to \mathbf{I}(H)$ to be the same function on the underlying sets. Then $\mathbf{I}(\phi)$ is a $\vee$-premorphism with respect to the pseudoproducts in $\mathbf{I}(G)$ and $\mathbf{I}(H)$.

The Ehresmann–Schein–Nambooripad (ESN) Theorem

Theorem (Various authors: see Lawson 1998)

The category of inverse semigroups and $\vee$-premorphisms is isomorphic to the category of inductive groupoids and ordered functors; the category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors.

Seek such a theorem for $\wedge$-premorphisms.

Composition of $\wedge$-premorphisms

The composition of two $\wedge$-premorphisms is not necessarily a $\wedge$-premorphism. Call a $\wedge$-premorphism ordered if it is order-preserving. Then:

Lemma

The composition of two ordered $\wedge$-premorphisms is an ordered $\wedge$-premorphism, hence inverse semigroups and ordered $\wedge$-premorphisms form a category.

What function is to correspond to a $\wedge$-premorphism?

Try a 'Gilbert premorphism' (Gilbert 2005):

A function $\psi : G \to H$ between inductive groupoids is a Gilbert premorphism if

(1) $\exists g \cdot h$ in $G \Longrightarrow (g\psi) \otimes (h\psi) \leq (g \cdot h)\psi$;

(2) $(g\psi)^{-1} = g^{-1}\psi$;

(3) ψ is order-preserving.

The composition of two Gilbert premorphisms is a Gilbert premorphism — so inductive groupoids and Gilbert premorphisms form a category.

Correspondence of arrows I

Let $\theta : S \to T$ be an ordered $\wedge$-premorphism of inverse semigroups. Define $\Theta := \mathbf{G}(\theta) : \mathbf{G}(S) \to \mathbf{G}(T)$ to be the same function on the underlying sets.

Conditions (2) and (3) for Gilbert premorphisms are immediate.

Condition (1) is also very easy. Suppose that $\exists g \cdot h$ in $\mathbf{G}(S)$. Then

$$
\begin{aligned}
&(g\theta)(h\theta) \leq (gh)\theta \\
\Longrightarrow &(g\theta) \otimes (h\theta) \leq (g \otimes h)\theta \\
\Longrightarrow &(g\Theta) \otimes (h\Theta) \leq (g \otimes h)\Theta \\
\Longrightarrow &(g\Theta) \otimes (h\Theta) \leq (g \cdot h)\Theta,
\end{aligned}
$$

whence Θ is a Gilbert premorphism.

Correspondence of arrows II

Let $\psi : G \to H$ be a Gilbert premorphism of inductive groupoids. Define $\Psi := \mathbf{I}(\psi) : \mathbf{I}(G) \to \mathbf{I}(H)$ to be the same function on the underlying sets.

Condition ($\wedge$2) (inverses) and order-preservation are immediate.

However, ($\wedge$1) causes problems. All would be OK if ψ satisfied:

(4a) $s\psi|\mathbf{r}(s\psi) \wedge f\psi \leq (s|\mathbf{r}(s) \wedge f)\psi$;

(4b) $f\psi \wedge \mathbf{d}(t\psi)|t\psi \leq (f \wedge \mathbf{d}(t)|t)\psi$,

for $f \in G_o$.

Try a different function...

Let $\psi : G \to H$ be a function between inductive groupoids. We will call ψ an inductive prefunctor if

(1) $\exists g \cdot h$ in $G \Longrightarrow (g\psi) \otimes (h\psi) \leq (g \cdot h)\psi$;

(2) $(g\psi)^{-1} = g^{-1}\psi$;

(3) ψ is order-preserving;

(4a) $s\psi|\mathbf{r}(s\psi) \wedge f\psi \leq (s|\mathbf{r}(s) \wedge f)\psi$;

(4b) $f\psi \wedge \mathbf{d}(t\psi)|t\psi \leq (f \wedge \mathbf{d}(t)|t)\psi$,

for $f \in G_o$.

The composition of two inductive prefunctors is an inductive prefunctor.

Correspondence of arrows again

Lemma

Let $\theta : S \to T$ be an ordered $\wedge$-premorphism between inverse semigroups. Define $\mathbf{G}(\theta) : \mathbf{G}(S) \to \mathbf{G}(T)$ to be the same function on the underlying sets. Then $\mathbf{G}(\theta)$ is an inductive prefunctor with respect to the restricted products in $\mathbf{G}(S)$ and $\mathbf{G}(T)$.

Lemma

Let $\psi : G \to H$ be an inductive prefunctor between inductive groupoids. Define $\mathbf{I}(\psi) : \mathbf{I}(G) \to \mathbf{I}(H)$ to be the same function on the underlying sets. Then $\mathbf{I}(\psi)$ is an ordered $\wedge$-premorphism with respect to the pseudoproducts in $\mathbf{I}(G)$ and $\mathbf{I}(H)$.

Some remaining details

We can clear up a few remaining technicalities:

If $\theta : S \to T$ is an ordered $\wedge$-premorphism and $\psi : G \to H$ is an inductive prefunctor, then

$$\mathbf{I}(\mathbf{G}(\theta)) = \theta \quad \text{and} \quad \mathbf{G}(\mathbf{I}(\psi)) = \psi.$$

If $\theta' : T \to T'$ is another ordered $\wedge$-premorphism and $\psi' : H \to H'$ is another inductive prefunctor, then

$$\mathbf{G}(\theta\theta') = \mathbf{G}(\theta)\mathbf{G}(\theta') \quad \text{and} \quad \mathbf{I}(\psi\psi') = \mathbf{I}(\psi)\mathbf{I}(\psi').$$

The desired result

Theorem

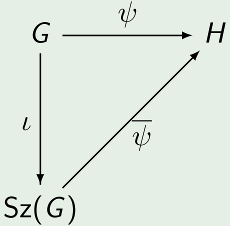
The category of inverse semigroups and ordered $\wedge$-premorphisms is isomorphic to the category of inductive groupoids and inductive prefunctors.

The Szendrei expansion (Gilbert 2005)

Let G be an inductive groupoid. Can define the Szendrei expansion $\mathrm{Sz}(G)$ of G — a new inductive groupoid, built from G. There is an injection $\iota : G \to \mathrm{Sz}(G)$.

Theorem 1

Let $\psi : G \to H$ be a Gilbert premorphism of inductive groupoids. Then there exists a unique inductive functor $\overline{\psi} : \mathrm{Sz}(G) \to H$ such that $\psi = \iota\overline{\psi}$. Conversely, if $\overline{\psi} : \mathrm{Sz}(G) \to H$ is an inductive functor, then $\psi := \iota\overline{\psi}$ is a Gilbert premorphism.



The case of inductive prefunctors

Theorem 2

Let $\psi : G \to H$ be an inductive prefunctor of inductive groupoids. Then there exists a unique inductive functor $\overline{\psi} : \mathsf{Sz}(G) \to H$ such that $\psi = \iota\overline{\psi}$. Conversely, if $\overline{\psi} : \mathsf{Sz}(G) \to H$ is an inductive functor, then $\psi := \iota\overline{\psi}$ is an inductive prefunctor.

Gilbert premorphisms and inductive prefunctors

We know that every inductive prefunctor is a Gilbert premorphism.

Let $\psi : G \to H$ be a Gilbert premorphism.

$\implies \psi = \iota\overline{\psi}$, for some inductive functor $\overline{\psi}$ (Theorem 1).

$\implies \psi$ is an inductive prefunctor (Theorem 2).

Thus,

Lemma

Inductive prefunctors are precisely Gilbert premorphisms.

The desired result, revisited

Theorem

The category of inverse semigroups and ordered $\wedge$-premorphisms is isomorphic to the category of inductive groupoids and Gilbert premorphisms.

The End