Monoids, S-acts and coherency

York February/March 2017

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Semigroups and monoids
What are they?

A **semigroup** S is a non-empty set together with an associative binary operation.

If the binary operation (looks like) $+$ we write '$a + b$', so associativity says

$$(a + b) + c = a + (b + c)$$

A semigroup

The first arithmetic most humans meet involves the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ with operation $+$

Semigroups and monoids
What are they?

For a general semigroup S we write the binary operation as juxtaposition 'ab' so associativity says

$$a(bc) = (ab)c \text{ for all } a, b, c \in S.$$

If $\exists\, 1 \in S$ with $1a = a = a1$ for all $a \in S$, then S is a **monoid**.

A monoid

$\mathbb{N}$ with operation $\times$

- Groups
- Multiplicative semigroups of rings, e.g. $M_n(D)$ where D is a division ring.
- Let X be a set. Then

$$\begin{aligned}\mathcal{T}_X : &= \{\alpha \mid \alpha : X \to X\}\\ \mathcal{S}_X : &= \{\alpha \mid \alpha : X \to X, \alpha \text{ bijective}\}\\ \mathcal{PT}_X : &= \{\alpha \mid \alpha : Y \to Z \text{ where } Y, X \subseteq X\}\\ \mathcal{I}_X : &= \{\alpha \in \mathcal{PT}_X \mid \alpha \text{ is one-one}\}\end{aligned}$$

are monoids under $\circ$, the **full transformation monoid**, the **symmetric group** the **partial transformation monoid** and the **symmetric inverse monoid** on X.
If $X = \underline{n} = \{1, 2, \ldots, n\}$ then we write $\mathcal{T}_n$ for $\mathcal{T}_{\underline{n}}$, etc.

Semigroup Theory
What kind of questions can we ask?

(Algebraic) semigroup theory is a rich and vibrant subject:

- Structure theory for semigroups
- Combinatorial and geometric questions
- Free algebras, varieties and lattices
- S-acts over a monoid S
- The Rhodes/Steinberg school of finite semigroup theory
- Special classes of semigroups e.g. inverse
- Connections with categories
- Semigroup algebras

Throughout many of these, it is the behaviour of **idempotents** *that is significant.*

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Semigroup Theory
Applications/connections with other areas of mathematics

- Automata, languages and theoretical computer science
- Finite group theory
- C^*-algebras and mathematical physics.
- Semigroup algebras, analysis and combinatorics
- Representation theory
- Model theory
- Tropical algebra

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S-acts
Representation of monoid S by mappings of sets

Throughout, S is a **monoid**.

A **(right)** S-**act** is a set A together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

For any $s \in S$, we have a unary operation on A given by $a \mapsto as$.

An S-act A is just a morphism from S to $\mathcal{T}_A$.

S-acts
Standard definitions/Elementary observations

- Of course, S is an S-act via its own operation
- An S-**morphism** from A to B is a map $\alpha : A \to B$ with $(as)\alpha = (a\alpha)s$ for all $a \in A, s \in S$.
- S-acts and S-morphisms form a **category** - products are products, coproducts are disjoint unions
- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.
- If $a \in A$ where A is an S-act, then
$$aS = \{as : s \in S\}$$
- An S-act A is **generated** by $X \subseteq A$ if
$$A = \bigcup_{x \in X} xS.$$
- An S-act A is **finitely generated** if there exists $a_1, \ldots, a_n \in A$ with
$$A = \{a_i s : s \in S\}.$$

Free S-acts

Let X be a set. By general nonsense, the free S-act $\mathcal{F}_S(X)$ on X exists.

Construction of $\mathcal{F}_S(X)$. Let

$$F_S(X) = X \times S$$

and define

$$(x, s)t = (x, st).$$

Then it is easy to check that $F_S(X)$ is an S-act. With $x \mapsto (x, 1)$, we have $F_S(X)$ is free on X.

Free S-acts

We abbreviate (x, s) by xs and identify x with $x1$.

So,

$$F_S(X) = \bigcup_{x \in X} xS.$$

Thus $F_S(X)$ is generated by X and is a **disjoint union of copies of S.**

S is the free S-act on one generator.

An S-act A
Standard definitions/Elementary observations
Congruences -

- A **congruence** ρ on A is an equivalence relation such that
$$a \,\rho\, b \Rightarrow as \,\rho\, bs$$
for all $a, b \in A$ and $s \in S$.
- If ρ is a congruence on A,
$$A/\rho = \{[a] : a \in A\}$$
is an S-act under $[a]s = [as]$.
- ρ is **finitely generated** if ρ is the smallest congruence containing a finite set $H \subseteq A \times A$.
- A is **finitely presented** if
$$A \cong F_S(n)/\rho$$
for some finitely generated free S-act $F_S(n)$ and finitely generated congruence ρ.

The model theory of S-acts
First order languages and L_S

A (first order) language L has alphabet:
variables, connectives (e.g. $\neg, \vee, \wedge, \rightarrow$ etc.), quantifiers $(\forall, \exists)$, $=$, brackets, commas
and some/all of symbols for constants, functions and relations.

There are rules for forming well formed formulae (wff); a **sentence** is a wff with no free variables (i.e. all variables are governed by quantifiers).

The language L_S

has:

no constant or relational symbols (other than $=$)

for each $s \in S$, a unary function symbol ρ_s.

A point of convenience Let us agree to abbreviate $x\rho_s$ in wffs of L_S by xs.

L_S-structures and S-acts

Examples

$\neg(xs = xt)$ is a wff but not a sentence
$(\forall x)\big(\neg(xs = xt)\big)$ is a sentence, $(\exists \vee xsx$ is not a wff.

An L-**structure** is a set D equipped with enough distinguished elements (constants), functions and relations to 'interpret' the abstract symbols of L.

An L_S-structure is simply a set with a unary operation for each $s \in S$.

Clearly an S-act A is an L_S-structure where we interpret ρ_s by the map $x \mapsto xs$.

Model theory = algebra + logic

A **theory** is a set of sentences in a first order language.

Model theory provides a range of techniques to study algebraic and relational structures etc. via properties of their associated **languages** and **theories**.

Model theory of R**-modules** is a well developed subject area (**e.g. Eklof and Sabbagh, Bouscaren, Prest**)

Model theory of S**-acts** - much less is known - authors include **Ivanov, Mustafin, Stepanova** .

Stability is an area within model theory, introduced by **Morley 62**. Much of the development of the subject is due to **Shelah**; the definitive reference is **Shelah 90** though (quote from Wiki) *it is notoriously hard even for experts to read*.

Model theory lite: axiomatisability

Axiomatisability

A class $\mathcal{A}$ of L_S-structures is **axiomatisable** if there is a theory Σ such that for any L_S-structure A, we have $A \in \mathcal{A}$ if and only if every sentence of Σ is true in A, i.e. A is a **model** of Σ.

This is saying is that $\mathcal{A}$ can be captured exactly within the language L_S.

Let Σ_S be the theory

$$\Sigma_S = \big\{(\forall x)((xs)t = x(st)) : s, t \in S\big\} \cup \{(\forall x)(x\,1 = x)\}.$$

Then Σ_S axiomatises the class of S-acts (within all L_S-structures).

Many of the 'natural' classes of S-acts - such as free, projective etc. - are axiomatisable if and only if S **satisfies a finitary condition**.

Finitary condition

A condition satisfied by a finite monoid, e.g., every element has an idempotent power

Finitary conditions were introduced by **Noether** and **Artin** in the early 20th Century to study rings; they changed the course of algebra entirely.

Coherency

This is a finitary condition of importance to us today

Definition

S is (right) coherent if every finitely generated S-subact of every finitely presented S-act is finitely presented.

Coherency is a very weak finitary condition.

Algebraically and existentially closed S-acts

Let A be an S-act. An **equation** over A has the form

$$xs = xt,\ xs = yt \text{ or } xs = a$$

where x, y are variables, $s, t \in S$ and $a \in A$.

An **inequation** is of the form $xs \neq xt$, etc.

Consistency

A set of equations and inequations is **consistent** if it has a solution in some S-act $B \supseteq A$.

Algebraically/existentially closed

A is **algebraically closed** or **absolutely pure** if every finite consistent set of equations over A has a solution in A.
A is **existentially closed** if every finite consistent set of equations and inequations over A has a solution in A.

Let $\mathcal{E}$ denote the class of existentially closed S-acts.

When is $\mathcal{E}$ axiomatisable?

Let T, T^* be theories in a first order language L. Then T^* is a **model companion** of T if every model of T embeds into a model of T^* and vice versa, and embeddings between models of T^* are elementary embeddings.

Wheeler 76

Σ_S has a model companion Σ_S^* precisely when $\mathcal{E}$ (the class of existentially closed S-acts) is axiomatisable and in this case, Σ_S^* axiomatises $\mathcal{E}$.

So, the question of when does Σ_S^* exist? is our question, when is $\mathcal{E}$ axiomatisable?

When is $\mathcal{E}$ axiomatisable?

Let A be an S-act and let $z \in A$. We define

$$\mathbf{r}(z) = \{(u, v) \in S \times S : zu = zv\}.$$

Notice that $\mathbf{r}(z)$ is a right congruence on S.

Theorem: Wheeler 76, G 87, 92, Ivanov 92

The f.a.e. for S:

1. Σ_S^* exists;
2. S is (right) coherent;
3. every finitely generated S-subact of every S/ρ, where ρ is finitely generated, is finitely presented;
4. for every finitely generated right congruence ρ on S and every $a, b \in S$ we have $\mathbf{r}([a])$ is finitely generated, and $[a]S \cap [b]S$ is finitely generated.

Which monoids are right coherent?
20th century

S is **weakly right noetherian** if every right ideal is finitely generated.

S is **right noetherian** if every right congruence is finitely generated.

Theorem **Normak 77** If S is right noetherian, it is right coherent.

Example **Fountain 92** There exists a weakly right noetherian S which is not right coherent.

Old(ish) results: **G**

The following monoids are right coherent:

1. the free commutative monoid on X;
2. Clifford monoids;
3. regular monoids for which every right ideal is principal.

Which monoids are right coherent?
21st century

We have seen free commutative monoids are coherent - it is known that free rings are coherent (**K.G. Choo, K.Y. Lam and E. Luft, 72**). I could *not* show free monoids X^* are coherent.
Notice

$$X^* = \{x_1x_2\ldots x_n : n \geq 0, x_i \in X\}$$

with

$$(x_1x_2\ldots x_n)(y_1y_2\ldots y_m) = x_1x_2\ldots x_ny_1x_2\ldots y_m.$$

Theorem: G, Hartmann, Ruškuc (2015)

Any free monoid X^* is coherent

Which monoids are right coherent?
21st century
For the semigroupers here

We can also show, together with others including Yang that the following monoids are right coherent:

1. regular monoids for which certain 'annihilator right ideals' are finitely generated, e.g. $(\mathbb{Z} \times \mathbb{Z})^1$ with 'bicyclic' multiplication;
2. combinatorial Brandt semigroups with 1 adjoined;
3. primitive inverse semigroups, with 1 adjoined.

BUT free inverse monoids are NOT right coherent.

On the other hand,

Theorem: G, Hartmann 2016

Free left ample monoids are right coherent.

Algebraic closure and injectivity
Strongly related

An S-act T is **injective** if for any S-acts A, B and S-morphisms

$$\phi : A \to B, \psi : A \to T$$

with ϕ one-one, there exists an S-morphism $\theta : B \to T$ such that

$$\phi\theta = \psi.$$

Theorem: G 19■ ■

An S-act T is injective if and only if every consistent system of equations over T has a solution in T.

Restrictions on the A, B give restricted notions of injectivity and these are related to solutions of special consistent systems of equations.

Algebraic closure and injectivity
Strongly related

Recall and S-act C is **absolutely pure/algebraically closed** if every finite consistent system of equations over C, has a solution in C.

Proposition: G

An S-act C is algebraically closed if and only if for any S-acts A, B and S-morphisms

$$\phi : A \to B, \psi : A \to C$$

with ϕ one-one, B finitely presented and A finitely generated, there exists an S-morphism $\theta : B \to C$ such that

$$\phi\theta = \psi.$$

Absolute purity vs almost purity

Algebraically closed a.k.a. absolutely pure

A is **algebraically closed** or **absolutely pure** if every finite consistent set of equations over A has a solution in A.

1-algebraically closed a.k.a. almost pure

A is **1-algebraically closed** or **almost pure** if every finite consistent set of equations over A **in one variable** has a solution in A.

The Question:

Definition A monoid S is **completely right pure** if all S-acts are absolutely pure.

Theorem: G

Suppose that all S-acts are almost pure. Then S is completely right pure.

Question

Does there exist a monoid S and an S-act A such that A is almost pure but not absolutely pure?????

Completely (right) injective/pure monoids

Definition A monoid S is **completely right injective** if all S-acts are injective.

Theorem: (Fountain, 1974)

A monoid S is completely right injective if and only if S has a left zero and S satisfies
(*) for any right ideal I of S and right congruence ρ on S, there is an $s \in I$ such that for all $u, v \in S, w \in I$, $sw \, \rho \, w$ and if $u \, \rho \, v$ then $su \, \rho \, sv$.

Theorem (Gould, 1991)

A monoid S is completely right pure if and only if S has a local zeros and S satisfies
(**) for any finitely generated right ideal I of S and finitely generated right congruence ρ on S, there is an $s \in I$ such that for all $u, v \in S, w \in I$, $sw \, \rho \, w$ and if $u \, \rho \, v$ then $su \, \rho \, sv$.

Absolute purity vs almost purity

Theorem: G, Yang Dandan, Salma Shaheen (2016)

Let S be a finite monoid and let A be an almost pure S-act. Then A is absolutely pure

Theorem: G, Yang Dandan(2016)

Let S be a right coherent monoid and let A be an almost pure S-act. Then A is absolutely pure

Questions

1. There is a way of writing down a complicated condition on chains of finitely presented acts that are equivalent to S having the property that every 1-algebraically closed S-act is algebraically closed. Can we use this to show there exist an almost pure S-act that is not absolutely pure?????
2. Connections of right coherency to products of (weakly, strongly) flat *left* S-acts? These hold in the ring case but are only partially known for monoids.
3. Other finitary conditions arise in stability theory of S-acts, including that of being **ranked** on the lattice of right congruences of S. A number of open questions remain concerning their correlation and their simplification.

- For an arbitrary S, does being ranked imply being weakly right noetherian?
- Can we describe ranked groups?

Thank you very much for your time.

Any questions?