Notions of properness for semigroups

York Semigroup
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A Farewell to CAUL

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Gracinda
Notions of properness for which semigroups? Ehresmann semigroups.

The classical background.

Some candidates for propriety.

Using one candidate: $S$-labelled trees.

The set of idempotents of any semigroup $S$ is denoted by $E(S)$. 

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A **unary** semigroup is a semigroup equipped with a unary operation, normally denoted by

\[ a \mapsto a^+. \]

A **biunary** semigroup is a semigroup equipped with two unary operations, normally denoted by

\[ a \mapsto a^+ \text{ and } a \mapsto a^*. \]

We regard unary [biunary] semigroups as algebras with signature \((2,1)\) \([(2,1,1)]\).

Similarly for unary and biunary **monoids**.
1. Ehresmann semigroups: 
Unary and biunary semigroups

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1. Ehresmann semigroups: Inverse semigroups

A semigroup $S$ is **inverse** if for each $a \in S$ there exists a unique $a' \in S$ such that

\[ a = aa'a \quad \text{and} \quad a' = a'aa'. \]

If $S$ is inverse, then for any $a \in S$ we have $aa', a'a \in E(S)$ and

\[ ef = fe \quad \text{for all} \quad e, f \in E(S). \]

It follows that $E(S)$ is a **semilattice** i.e. a commutative semigroup of idempotents.

A semilattice is **partially ordered** under

\[ e \leq f \quad \text{if and only if} \quad ef = e \]

and $ef$ is the meet of $e$ and $f$. \\n
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1. Ehresmann semigroups:
Inverse semigroups

Clearly, an inverse semigroup is a unary semigroup under

\[ a \mapsto a'. \]

An inverse semigroup is also \textbf{biunary} where

\[ a \mapsto a^+ = aa' \text{ and } a \mapsto a^* = a'a. \]
1. Left Ehresmann semigroups: A variety of unary semigroups

**Definition** A unary semigroup \((S, \cdot, ^+)\) is **left Ehresmann** if it satisfies the identities \(\Sigma_\ell:\)

\[
\begin{align*}
 a^+a^+ &= a^+, & a^+b^+ &= b^+a^+, & (a^+b^+)^+ &= a^+b^+, & a^+a &= a, & (ab^+)^+ &= (ab)^+.
\end{align*}
\]

Let

\[
E = \{a^+ : a \in S\}.
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Then \(E\) is a **semilattice**, the semilattice of **projections** of \(S\).

**Example 1** Inverse semigroups are left Ehresmann under \(a^+ = aa'\).

**Example 2** Any monoid is left Ehresmann with \(a^+ = 1\) for all \(a \in M\). It is a **reduced** left Ehresmann semigroup.
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1. Ehresmann semigroups:
A variety of biunary semigroups

**Definition** A biunary semigroup \((S, \cdot, ^+, ^*)\) is **Ehresmann** if it satisfies the identities \(\Sigma_\ell\), the dual identities \(\Sigma_r\) and

\[(a^*)^+ = a^*, \ (a^+)^* = a^+.

If \(S\) is Ehresmann then

\[E = \{a^* : a \in S\} = \{a^+ : a \in S\}.

**Example 1** Inverse semigroups are Ehresmann under \(a^+ = aa'\) and \(a^* = a'a\).

**Example 2** Any monoid is Ehresmann with \(a^+ = 1 = a^*\) for all \(a \in M\). Such an Ehresmann semigroup is called **reduced**.
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1. Ehresmann semigroups: Observations, examples

- The name ‘Ehresmann’ was coined by Lawson, 1991; he first established the connection between Ehresmann semigroups and the bi-ordered categories of C. Ehresmann.
- Inverse semigroups are Ehresmann and inverse semigroups are important!!
- As Ehresmann semigroups are varieties, they are closed under H,S,P; free algebras exist.
- Any biunary subsemigroup of an inverse semigroup is Ehresmann.
- Type A semigroups (later called ample) are Ehresmann; restriction semigroups are Ehresmann.
- Ehresmann semigroups are the variety generated by the quasi-variety of adequate semigroups.
- Above in one-sided case and monoid case.
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- $\mathcal{PT}_X$ is Ehresmann where $\alpha^+ (\alpha^*)$ is the identity map in the domain (range) of $\alpha$; in fact, $\mathcal{PT}_X$ is left restriction Trokhimenko, 1973.

- $\mathcal{B}_X$ is Ehresmann under

  $$\rho^+ = \{(a, a) : a \in \text{dom } \rho\} \text{ and } \rho^* = \{(a, a) : a \in \text{im } \rho\}.$$  

- Any semidirect product $Y \rtimes M$, where $Y$ is a semilattice and $M$ a monoid is left restriction, hence left Ehresmann.

- Let $Y$ be a semilattice. Then the free idempotent generated semigroup $\text{IG}(Y)$ is adequate, hence Ehresmann. G, Yang, 2013.
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1. Ehresmann semigroups: The bigger picture: Classes of biunary semigroups with semilattices of idempotents

- **ample identities**
- **no ample identities**

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ample: quasi-variety
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```
adequate: quasi-variety
```

```
restriction: variety
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- **inverse**
  - $ab^+ = (ab)^+ a$
  - $b^* a = a(ba)^*$

- **Ehresmann**
  - variety
1. Left Ehresmann semigroups: The bigger picture:
Classes of unary semigroups with semilattices of idempotents

ample identities

no ample identities

left ample: quasi-variety

inverse

\[ ab^+ = (ab)^+a \]

left adequate: quasi-variety

left restriction: variety

left Ehresmann: variety
2. The classical background

Proper inverse semigroups

Let $S$ be an inverse semigroup.

- $\sigma = \langle E(S) \times E(S) \rangle$ is the least group congruence on $S$.
- $S$ is proper if $(a^+ = b^+ \text{ and } a \sigma b)$ implies $a = b$;

  *this definition is left/right dual.*
- Free inverse semigroups are proper.
- If $S$ is proper, $S \to E(S) \times S/\sigma$ given by

  $$s \mapsto (s^+, s\sigma)$$

  is clearly a SET embedding.

The McAlister Theorems, 1974 Let $S$ be an inverse semigroup.

(i) $S$ is proper if and only if $S$ is isomorphic to a $P$-semigroup;

(ii) $S$ has a proper cover. That is, there exists a proper inverse semigroup $\hat{S}$ and an idempotent separating morphism $\hat{S} \to S$. 
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2. The classical background
Proper inverse semigroups and generalisations

- Let $S$ be Ehresmann; put $\sigma = \langle E \times E \rangle$.
- $S/\sigma$ is reduced.
- A restriction semigroup $S$ is proper if the following condition and its dual holds:
  \[ (a^+ = b^+ \text{ and } a \sigma b) \implies a = b. \]
- The free restriction semigroup is proper.
- Results for proper restriction semigroups involving semidirect products, analogous to those in the inverse case hold where \textbf{group} is replaced by \textbf{monoid} Branco, Cornock, El Qallali, Fountain, Gomes, G, Lawson, Szendrei; more recently, Kudryavtseva, Jones.
- The above has analogues in the one-sided case and ample case.
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2. The classical background

What makes such results involving semidirect products work?

Let $M$ be a left Ehresmann monoid.

1. Suppose that $M = \langle X \rangle_{(2,1,0)}$. Put $T = \langle X \rangle_{(2,0)}$ so that $T$ is the monoid generated by $X$.

2. $M = \langle T \cup E \rangle_{(2)}$ so that any $s \in M$ can be written as

$$s = t_0 e_1 t_1 \ldots e_n t_n,$$

for some $t_0, \ldots, t_n \in T$ and $e_1, \ldots, e_n \in E$.

3. If the ample identities hold, e.g. in the inverse case or restriction case, then $s = f t_0 t_1 \ldots t_n$ for some $f \in E$, so that $M = ET$.

4. The above is what is behind results connecting (left) restriction/ample/inverse monoids to semidirect products $Y \rtimes T$ of a semilattice $Y$ and a monoid $T$. 
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3. Some candidates for propriety:
What do we know from former work?

Let $M$ be left Ehresmann and let $T$ be a submonoid. Then $T$ acts on $E$ by order-preserving maps via

$$t \cdot e = (te)^+.$$ 

If $M$ is inverse/left ample/left restriction, then this action is by morphisms of the semilattice $E$. 
3. Some candidates for propriety: What are we looking for?

The old notion of ‘proper’ is no good - it leads inexorably to a semidirect product construction, which is no longer appropriate.

- Want condition \( P \) for left Ehresmann monoids such that:
  1. left Ehresmann monoids satisfying \( P \) have their structure described by monoids acting on semilattices;
  2. if \( M \) is left Ehresmann then there exists a left Ehresmann \( \hat{M} \) satisfying \( P \) and a projection-separating morphism \( \hat{M} \rightarrow M \), i.e. \( \hat{M} \) is a cover of \( M \);
  3. free left Ehresmann monoids satisfy \( P \).
  4. \( P \) plays a role in defining categories and varieties of left Ehresmann monoids.
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  (iii) free left Ehresmann monoids satisfy $P$.
  (iv) $P$ plays a role in defining categories and varieties of left Ehresmann monoids.
3. Some candidates for propriety: Generators and $T$-normal form
Branco, Gomes, G

Let $M$ be a left Ehresmann monoid.

Suppose that $M = \langle E \cup T \rangle_{(2)}$ where $T$ is a submonoid of $M$.

Any $x \in M$ can be written as

$$x = t_0 e_1 t_1 \cdots e_n t_n,$$

where $n \geq 0$, $e_1, \ldots, e_n \in E$, $t_1, \ldots, t_{n-1} \in T \setminus \{1\}$, $t_0, t_n \in T$ and for $1 \leq i \leq n$

$$e_i < (t_i e_{i+1} \cdots t_n)^+.$$

Such an expression is in $T$-normal form and may be effectively calculated.

$M$ has uniqueness of $T$-normal forms if every $x \in M$ has a unique such expression.
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$M$ is said to be **strongly $T$-proper** if for all $u, v \in T$,

$$u \sigma v \Rightarrow u = v;$$

$M$ is said to be **very $T$-proper** if for all $u, v \in T$,

$$u \sigma v \Rightarrow u^+ v = v^+ u;$$

$M$ is said to be **$T$-proper** if for all $u, v \in T, e, f \in E$

$$(ue)^+ = (ve)^+ \text{ and } ue \sigma ve, \text{ then } ue = ve.$$ 

**Note** If $M$ is left restriction, then $M$ is (very) $M$-proper if and only if it is proper.
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Strongly $T$-proper

**Theorem** Branco, Gomes, G & Jones Let $M = \langle T \cup E \rangle_{(2)}$ be left Ehresmann, where $T$ is a $T$ submonoid of $M$. Then $M$ has a strongly $T$-proper cover.
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3. Some candidates for propriety: 
Uniqueness of $T$-normal forms

**Theorem Branco, Gomes, G** Let $T$ be a monoid acting on the left of a semilattice $E$ with identity, via order-preserving maps. Then there is a left Ehresmann monoid $\mathcal{P}_\ell(T, E)$ such that:

- $\mathcal{P}_\ell(T, E) = \langle T \cup E \rangle_2$;
- $\mathcal{P}_\ell(T, E)$ has uniqueness of $T$-normal forms;
- $\mathcal{P}_\ell(T, E)/\sigma \cong T$;
- the free left Ehresmann monoid on $X$ is of the form $\mathcal{P}_\ell(X^*, E)$.

**Theorem Branco, Gomes, G** Let $M = \langle T \cup E \rangle_2$ be left Ehresmann, where $T$ is a submonoid of $M$. Then $\mathcal{P}_\ell(T, E)$ is a cover for $M$. 
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4. **S-labelled trees**: G, Hartmann and Wang

A category of left Ehresmann monoids
Let $X \neq \emptyset$ and let $\mathcal{C}(X)$ be the category such that

(i) **objects** are triples $(M, X, \mu)$ where $M$ is left Ehresmann and $\mu : X \to M$ is a map such that $M = \langle X\mu \rangle_{(2,1,0)}$;

(ii) an **arrow** $\theta : (M, X, \mu) \to (N, X, \tau)$ is a morphism $\theta : M \to N$ such that $\tau = \mu \theta$.

Then $\mathcal{C}(X)$ is the **category of $X$-generated left Ehresmann monoids**.
4. $S$-labelled trees:

A category of left Ehresmann monoids

Let $X \neq \emptyset$, let $S$ be a monoid let $\tau : X \to S$ such that $S = \langle X \tau \rangle$.

Let $C(X, \tau, S)$ be the full subcategory of $C(X)$ such that an object $(M, X, \mu)$ of $C(X)$ lies in the subcategory if there is a morphism $\kappa : M \to S$ such that $\operatorname{Ker} \kappa = \sigma$ and $\mu \kappa = \tau$, and $M$ is strongly $T$-proper, where $T$ is the monoid $\langle X \mu \rangle$:

$$
\begin{array}{ccc}
X & \xrightarrow{\mu} & M \\
\downarrow{\tau} & \downarrow{\kappa} & \downarrow{\kappa} \\
S & & \\
\end{array}
$$
4. S-labelled trees:  
A category of left Ehresmann monoids

Let $F(X)$ be the free left Ehresmann monoid on $X$, with $\iota : X \to F(X)$. $S$ is a monoid, $\tau : X \to S$ such that $S = \langle X\tau \rangle$.

**Theorem** The category $\mathcal{C}(X, \tau, S)$ has initial object

$$(F(X)/\rho, X, \iota\rho^\dagger)$$

where

$$\rho = \langle (u\nu, v\iota) : u, v \in X^*, u\tau = v\tau \rangle.$$

**Theorem** The left Ehresmann monoid $F(X)/\rho$ is isomorphic to $\mathcal{P}_\ell(E, S)$ and hence has uniqueness of $S$-normal forms.
4. S-labelled trees:
   Free left Ehresmann monoid $F(X)$ Kambites 2011

$X$-labelled trees with root ‘start’ vertex and an ‘end’ vertex

Tree $\Gamma$: word $(y(xy)^+z^+)^+(xy)^+xx^+$

$\Gamma\Delta$: glue end of $\Gamma$ to start of $\Delta$

for $+$ take $\otimes$ to $\circ$

Take equivalence classes under $\sim$, where $\Gamma \sim \Delta$ if $\Gamma, \Delta$ have a common retract
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4. *S*-labelled trees: 

G, Hartmann and Wang

Relabel edges by elements of $S$: here $a = x\tau$, $b = y\tau$, $c = z\tau$

Delete vertices of degree 2

![Diagram of S-labelled trees](image-url)
4. *S*-labelled trees

Relabel edges by elements of *S*
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4. S-labelled trees

Foldings:

If \( pq = sk \), for some \( k \), ‘fold’ the branch labelled \( s \) to the path labelled \( pq \):
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$kr = u \ell = vw$; fold the branches labelled $u$ and $v$ to the path labelled $kr$
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**Theorem G, Hartmann, Wang** Let \( \Sigma, \Delta \) be idempotent \( X \)-trees and let \( \Sigma_S, \Delta_S \) be the corresponding \( S \)-trees. Then \( \Sigma_S = \Delta_S \) in \( F(X)/\rho \) if and only if \( \Sigma_S \) folds to \( \Delta_S \) and vice versa.

**Consequently** as \( F(X)/\rho \) has uniqueness of \( S \)-normal forms, and we have an effective procedure to obtain such, the word problem in \( F(X)/\rho \) is solvable (modulo solving systems of equations in \( S \)).
Questions:

1. The word problem in the corresponding very $T$-proper case.
2. Are the subalgebras of $\mathcal{P}_\ell(T, E)$ exactly those satisfying some properness condition?
3. Is there an analogue of the McAlister $P$-theorem?
4. Closure properties of classes of left Ehresmann monoids having covers over given varieties of monoids.
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