A tour of ideas behind restriction and related semigroups

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Outline

- What are restriction semigroups?
- Related semigroups? (some of which came first...)
- Proper restriction semigroups
- Proper covers
- Structure of proper restriction semigroups
Restriction semigroups: what are they?

Restriction semigroups may be obtained as/from:

- Varieties of algebras
- Representation by (partial) mappings
- Generalised Green’s relations
- Inductive categories and inductive constellations
Let $S = (S, \cdot, ^+) \text{ be a semigroup equipped with a unary operation } ^+$ (that is, $S$ is a unary semigroup).

**Definition** $S$ is left restriction if the following identities hold:

\[
x^+ x = x, \quad x^+ y^+ = y^+ x^+, \quad (x^+ y)^+ = x^+ y^+, \quad xy^+ = (xy)^+ x.
\]
Restriction semigroups: varieties

\[ x^+ x = x, \quad x^+ y^+ = y^+ x^+, \quad (x^+ y)^+ = x^+ y^+, \quad xy^+ = (xy)^+ x. \]

Let \( S \) be left restriction and put

\[ E = \{ a^+ : a \in S \}. \]

For any \( a^+ \in E \),

\[ a^+ = (a^+ a)^+ = a^+ a^+ \]

so that we see \( E \) is a semilattice, i.e. commutative semigroup of idempotents.

\( E \) is the distinguished semilattice of \( S \)

Also with \( a^{++} = (a^+)^+ \),

\[ a^+ = a^{++} a^+ = a^+ a^{++} = (a^+ a^+)^+ = a^{++}. \]
Left restriction semigroups form a \textbf{variety} of unary semigroups.

Dually, right restriction semigroups form a variety of unary semigroups, with unary operation denoted by $\ast$, satisfying the left/right duals of the axioms above.

A bi-unary semigroup $S = (S, \cdot, +, \ast)$ is \textbf{restriction} if and only if satisfies the identities for left and right restriction semigroups together with

$$(x^\ast)^+ = x^\ast \text{ and } (x^+)^\ast = x^+. $$
Let $M$ be a monoid and define $a^+ = 1 = a^*$ for all $a \in M$. Then $M = (M, \cdot, ^+, ^*)$ is restriction.

We need to check the identities

\[ x^+ x = x, \quad x^+ y^+ = y^+ x^+, \quad (x^+ y)^+ = x^+ y^+, \quad xy^+ = (xy)^+ x, \]

their duals and

\[ (x^*)^+ = x^* \quad \text{and} \quad (x^+)^* = x^+. \]
Definition A (left) restriction semigroup is **reduced** if $|E| = 1$.

We have seen a monoid is a reduced (left) restriction semigroup (in a different signature).

Conversely, let $S$ be a reduced left restriction semigroup. Then let $E = \{u\}$, so that $u = u^+ = a^+$ for all $a \in S$. Since $a^+ a = a$ for all $a \in S$, we have $ua = a$.

Also,

$$au = au^+ = (au)^+ a = a$$

so that $u$ is an identity for $S$. 

Restriction semigroups: examples
Inverse semigroups
\( R \) and \( L \)

- For any \( a, b \in S \) we have
  
  \[ a R b \iff a S^1 = b S^1 \iff \exists s, t \in S^1 \text{ with } a = bs \text{ and } b = at. \]

- For any \( a, b \in S \) we have
  
  \[ a L b \iff S^1 a = S^1 b \iff \exists s, t \in S^1 \text{ with } a = sb \text{ and } b = ta. \]

- \( R \) (\( L \)) is a left (right) congruence
- \( R \) and \( L \) are the universal relation on any group
- \( R \) and \( L \) are the trivial relation on any semilattice
Definition $S$ is regular if for all $a \in S$ there exists $x \in S$ with $a = axa$.

Notice that if $a = axa$, then $ax, xa \in E(S)$ and

$$ax \mathcal{R} a \mathcal{L} xa.$$ 

Fact $S$ is regular if and only if every $\mathcal{R}$-class (or $\mathcal{L}$-class) contains an idempotent.

Definition $S$ is inverse if $S$ is regular and $E(S)$ is a semilattice.

Fact $S$ is inverse if and only every element has a unique inverse, i.e. for all $a \in S$ there exists a unique $a'$ in $S$ such that

$$a = aa'a \text{ and } a' = a'aa'.$$

Fact $S$ is inverse if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class contains a unique idempotent.
Restriction semigroups: examples
Inverse semigroups

Let $S$ be an inverse semigroup. Recall that $E(S)$ is a semilattice. Put

$$a^+ = aa' \text{ and } a^* = a'a.$$  

Then $S = (S, \cdot, ^+, ^*)$ is restriction with distinguished semilattice $E(S)$. We need to check the identities

$$x^+x = x, \quad x^+y^+ = y^+x^+, \quad (x^+y)^+ = x^+y^+, \quad xy^+ = (xy)^+x,$$

their duals and

$$(x^*)^+ = x^* \text{ and } (x^+)^* = x^+.$$  

Let $a, b \in S$. Then

$$(a^+b)^+ = (aa'b)^+ = aa'b(aa'b)' = aa'bb'aa' = aa'bb' = a^+b^+. $$

Also

$$(ab)^+ a = (ab)(ab)'a = ab(b'a')a = a(bb')(a'a) = a(a'a)(bb') = ab^+. $$
Restriction semigroups: representations

- Every semigroup $S$ embeds in a **full transformation semigroup** $T_X$
- Every group embeds in a **symmetric group** $S_X$
- Every inverse semigroup $S$ embeds (as an inverse semigroup) in the **symmetric inverse semigroup** $I_X$
- $T_X$, $S_X$ and $I_X$ are all subsemigroups of the semigroup $PT_X$ of all partial mappings of $X$.
- Define $^+$ on $PT_X$ by

$$
\alpha^+ = I_{\text{dom } \alpha}.
$$

Then $PT_X$ is left restriction with distinguished semilattice

$$
E = \{I_Y : Y \subseteq X\}.
$$

- $S$ is left restriction if and only if it embeds in some $PT_X$ (Trokhimenko).
Since (left) restriction semigroups form varieties, free algebras exist.

The free (left) restriction semigroup \( \mathcal{FR}(X) \) (\( \mathcal{FLR}(X) \)) on any set \( X \) embeds into the free inverse semigroup \( \mathcal{FI}(X) \) on \( X \) (Fountain, Gomes, G).

The determination of the structure of \( \mathcal{FI}(X) \) by Munn, Schein and Scheiblich is a classical early result of Semigroup Theory.

The structure of \( \mathcal{FLR}(X) \) and \( \mathcal{FR}(X) \) is particularly nice: they are both **proper**.
Let $E \subseteq E(S)$

- The relation $\tilde{\mathcal{R}}_E$ is defined by $a \tilde{\mathcal{R}}_E b$ if and only if
  \[
  ea = a \iff eb = b
  \]
  for all $e \in E$.
- Note if $a \tilde{\mathcal{R}}_E e \in E$, then as $ee = e$ we have $ea = a$.
- For a left restriction semigroup with distinguished semilattice $E$,
  \[
  a \tilde{\mathcal{R}}_E b \text{ if and only if } a^+ = b^+.
  \]

- The relation $\tilde{\mathcal{L}}_E$ is defined dually.
- $\tilde{\mathcal{R}}_E$ and $\tilde{\mathcal{L}}_E$ are equivalence relations.
- If $M$ is a monoid and $E = \{1\}$, then $\tilde{\mathcal{R}}_E$ and $\tilde{\mathcal{L}}_E$ are universal.
- These relations were introduced by El-Qallali in his 1980 thesis (under Fountain) in case $E = E(S)$, later generalised by Lawson.
The relations $\tilde{R}_E$ and $\tilde{L}_E$ - connection to $R$ and $L$

**Fact** For any semigroup $S$ and any $E$

$$R \subseteq \tilde{R}_E.$$

**Proof** Let $aRb$. Then $a = bs$ and $b = at$ for some $s, t \in S^1$.

Hence

$$ea = a \Rightarrow eat = at \Rightarrow eb = b \Rightarrow ebs = bs \Rightarrow ea = a.$$ 

**Fact** If $S$ is regular and $E = E(S)$, then $\tilde{R}_E = R$.

**Proof** If $a\tilde{R}_{E(S)} b$ and $a = axa, b = byb$, then $b = axb$ and $a = bya$. 
Fact A semigroup $S$ is left restriction with distinguished semilattice $E$ iff:

- $E$ is a semilattice;
- every $\tilde{R}_E$-class contains an idempotent of $E$; it is then easy to see that for every $a \in S$ the $\tilde{R}_E$-class of $a$ contains a unique element of $E$, which we call $a^+$;
- the relation $\tilde{R}_E$ is a left congruence and
- the left ample condition (AL) holds:

$$
\text{for all } a \in S \text{ and } e \in E, \ ae = (ae)^+ a.
$$

Similarly for (right) restriction semigroups.
A bit of history

Different schools arrived at (left) restriction semigroups via different directions from 1960s onwards:

- Schweizer, Sklar, Schein, Trokhimenko: **function systems**
  Let $T$ be a subsemigroup of $\mathcal{P}\mathcal{T}_X$ or $\mathcal{B}_X$ (semigroup of binary relations on $X$).
  $T$ may be equipped with additional operations such as $+$, $\cap$, $(f, g) \mapsto f^+g$ etc.
  Can such $T$ be axiomatised by first order formulae? By identities?

- Lawson: **Ehresmann semigroups**
  Lawson found a correspondence between Ehresmann semigroups and certain categories equipped with two orderings. As a special case, restriction semigroups correspond to inductive categories.
A bit more history

- Jackson and Stokes: closure operators
  Introduced ‘twisted $C$-semigroups’, with an axiomatisation equivalent to the one given here.

- Manes, Cockett, Lack: category theory, computer science
  Gave the axioms above. Also interested in restriction categories.

- Fountain: generalisations of regular and inverse semigroups

- Jones: $P$-restriction semigroups obtained from regular $\ast$-semigroups
Summary to date

- We have seen how to define (left) restriction semigroups as
  - varieties
  - by their representations as subalgebras of $\mathcal{PT}_X$
  - by using $\tilde{R}_E$ and $\tilde{L}_E$.

- We have mentioned there is a connection between (left) restriction semigroups and ordered structures

- The approach using $\tilde{R}_E$ and $\tilde{L}_E$ is part of the York approach to studying semigroups via $R^*, L^*, \tilde{R}_E, \tilde{L}_E$ and properties of idempotents.

- We introduced ample, abundant, weakly $U$-abundant semigroups, congruence and ample conditions.
There are three major approaches to structure of inverse semigroups:

- The **Ehresmann-Schein-Nambooripad** association of **inductive groupoids** to inverse semigroups.

- **Munn’s** construction of a **fundamental** inverse semigroup $T_E$ from any semilattice $E$; if $S$ is inverse then $T_{E(S)}$ contains a morphic image $S'$ of $S$ such that $E(S) \cong E(S')$.

- **McAlister’s** approach using **proper covers**: if $S$ is inverse then it has a proper preimage $\hat{S}$ such that $E(\hat{S}) \cong E(S)$ and such that the structure of $\hat{S}$ is known.

We are going to develop the McAlister approach for (left) restriction semigroups.
Proper restriction semigroups: definition

Let $S$ be (left) restriction.

- Recall that $S$ is **reduced** if $|E| = 1$.
- $\sigma_E$ is the least congruence **identifying all the idempotents of** $E$.
- The (left) restriction semigroup $S/\sigma_E$ is reduced.
- A left restriction semigroup $S$ is **proper** if
  \[(a^+ = b^+, a\sigma_E b) \Rightarrow a = b.\]
- A restriction semigroup $S$ is **proper** if
  \[(a^+ = b^+, a\sigma_E b) \Rightarrow a = b \text{ and } (a^* = b^*, a\sigma_E b) \Rightarrow a = b.\]
- Monoids and semilattices are both proper restriction.
- If $S$ is proper left restriction, then $\theta : S \to E \times S/\sigma_E$ given by
  \[s\theta = (s^+, s\sigma_E)\]
  is an injection.
Let $M$ be a monoid and $Y$ a set. Then $M$ acts on the left of $Y$ if there is a map

$$M \times Y \rightarrow Y; \quad (m, y) \mapsto m \cdot y,$$

such that

$$1 \cdot y = y \text{ and } (mn) \cdot y = m \cdot (n \cdot y).$$

Suppose now that $Y$ is a semigroup. Then $M$ acts by morphisms if, in addition,

$$m \cdot (yz) = (m \cdot y)(m \cdot z).$$

In this case, define a product on $Y \times M$ by

$$(y, m)(z, n) = (y(m \cdot z), mn).$$

This product is associative, yielding the semidirect product $Y \rtimes M$. 
Let $M$ be a monoid and let $Y$ be a semilattice.

- $Y * M$ is proper left restriction with $(e, m)^+ = (e, 1)$.
- If $M$ is a group, then $Y * M$ is proper inverse.
- Suppose that $Y$ has a greatest element $1_Y$. We say that $M$ acts **doubly** on $Y$ if $M$ acts by morphisms on the left and right of $Y$ and the compatibility conditions hold, that is

\[(t \cdot e) \circ t = (1_Y \circ t)e \text{ and } t \cdot (e \circ t) = e(t \cdot 1_Y)\]

for all $t \in M$, $e \in Y$.

- In this case

\[Y *_m M = \{(e, t) : e \leq t \cdot 1_Y \} \subseteq Y * M\]

is a proper restriction monoid with identity $(1_Y, 1)$ such that

\[(e, t)^+ = (e, 1) \text{ and } (e, t)^* = (e \circ t, 1).\]
Let $M$ be a monoid acting by morphisms on the right of a semilattice $Y$ such that
\begin{itemize}
  \item $a \circ t = b \circ t \Rightarrow a = b$;
  \item $a \leq b \circ t \Rightarrow a = c \circ t$ for some $c$.
\end{itemize}
Let
$$W = \{(t, a \circ t) : t \in M, a \in Y\} \subseteq M \rtimes Y.$$  
Then $W$ is a proper restriction subsemigroup of the (reverse) semidirect product $M \rtimes Y$ where
$$ (t, a \circ t)^+ = (1, a) \text{ and } (t, a \circ t)^* = (1, a \circ t).$$
(Construction due to Fountain, Gomes and Szendrei).
Proper covers

Let $S$ be (left) restriction.

A **proper cover** of $S$ is a proper (left) restriction semigroup $\hat{S}$ and an onto morphism $\theta : \hat{S} \rightarrow S$ such that $\theta$ separates distinguished idempotents.

**Theorem** Every (left) restriction semigroup has a proper cover (Branco, Fountain, Gomes, G).

**Theorem** Let $S$ be restriction. Then $S$ has a proper cover that is embeddable into a $W$-semigroup (Szendrei).
Let $S$ be a restriction monoid with distinguished semilattice $E$. Define

$$s \cdot e = (se)^+ \text{ and } e \circ s = (es)^*.$$ 

Then these are actions of $S$ on $E$ by morphisms, satisfying the compatibility conditions.

Let $s \in S$ and $e, f \in E$. From the identities $(x^+y)^+ = x^+y^+$ and $xy^+ = (xy)^+x$,

$$s \cdot ef = (sef)^+ = ((se)^+sf)^+ = (se)^+(sf)^+ = (s \cdot e)(s \cdot f).$$

Consequently, $E \star_m S = \{(e, s) : e \leq s^+\} \subseteq E \star S$ is proper restriction.

Define $\theta : E \star_m S \to S$ by $(e, s)\theta = es$. Then $\theta$ is a covering morphism.

Make adaptations for semigroup/one-sided case.
Let $T$ be a monoid acting on the left of a semilattice $\mathcal{X}$ via morphisms. Suppose that $\mathcal{X}$ has subsemilattice $\mathcal{Y}$ with upper bound $\varepsilon$ such that
(a) for all $t \in T$ there exists $e \in \mathcal{Y}$ such that $e \leq t \cdot \varepsilon$
(b) if $e \leq t \cdot \varepsilon$ then for all $f \in \mathcal{Y}$, $e \wedge t \cdot f \in \mathcal{Y}$.

Then $(T, \mathcal{X}, \mathcal{Y})$ is a strong left $M$-triple.

For a strong left $M$-triple $(T, \mathcal{X}, \mathcal{Y})$ we put

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(e, t) \in \mathcal{Y} \times T : e \leq t \cdot \varepsilon\} \subseteq \mathcal{X} \ast T.$$ 

Then $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ is proper left restriction with $(e, s)^+ = (e, 1)$. 
Theorem A left restriction semigroup $S$ is proper if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ (Branco, Gomes, G).

Important point In the above result, we can take

$$T = S/\sigma_E \text{ and } \mathcal{Y} = E.$$
Proper restriction semigroups: structure

- If $S$ is proper restriction, then as $S$ is proper left restriction,
  \[ S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{Y}) \]
  where $T = S/\sigma_E$ and $\mathcal{Y} = E$, and as $S$ is proper right restriction,
  \[ S \cong \mathcal{M}'(\mathcal{Y}, \mathcal{X}', T), \]
  where $\mathcal{M}'(\mathcal{Y}, \mathcal{X}', T)$ is constructed from $T$ acting on the right of a semilattice $\mathcal{X}'$.
- Clearly the left and right actions of $T$ must be connected in some way.
- Let $(T, \mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{X}', T)$ be strong left (right) $\mathcal{M}$-triples such that
  \[ e \leq t \cdot \varepsilon \Rightarrow t \cdot (e \circ t) = e \text{ and } e \leq \varepsilon' \circ t \Rightarrow (t \cdot e) \circ t = e \]
  then
  \[ \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y}) = \mathcal{M}(T, \mathcal{X}, \mathcal{Y}) \cong \mathcal{M}'(\mathcal{Y}, \mathcal{X}', T) \]
  is proper restriction.
Let $S$ be a restriction semigroup.

$S$ satisfies **Condition (EP)** if it satisfies $(EP)^r$ and its dual $(EP)^l$.

$(EP)^r$: for all $s, t, u \in S$, if $s \sigma_E tu$ then there exists $v \in S$ with $t^+s = tv$ and $u \sigma_E v$.

**Theorem** (Cornock, G) Let $S$ be a proper restriction semigroup. Then $S$ is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ if and only if $S$ satisfies $(EP)$.

Free restriction semigroups, and proper inverse semigroups have $(EP)$.

Not all proper restriction semigroups have $(EP)$.
Definition Let $T$ be a monoid, acting partially on the left and right of a semilattice $\mathcal{Y}$, via $\cdot$ and $\circ$ respectively. Suppose that both actions preserve the partial order and the domains of each $t \in T$ are order ideals. Suppose in addition that for $e \in \mathcal{Y}$ and $t \in T$, the following and their duals hold:
(a) if $\exists e \circ t$, then $\exists t \cdot (e \circ t)$ and $t \cdot (e \circ t) = e$;
(b) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $\exists e \circ t$.

Then $(T, \mathcal{Y})$ is a strong M-pair.

We put
$$\mathcal{M}(T, \mathcal{Y}) = \{(e, s) \in \mathcal{Y} \times T : \exists e \circ s\}$$
and define operations by
$$(e, s)(f, t) = (s \cdot (e \circ s \wedge f), st), (e, s)^+ = (e, 1) \text{ and } (e, s)^* = (e \circ s, 1).$$
A structure theorem for proper restriction semigroups: the result

**Theorem** (Cornock, G) If $(T, \mathcal{Y})$ is a strong M-pair, then

$$\mathcal{M}(T, \mathcal{Y}) \cong \mathcal{M}'(\mathcal{Y}, T),$$

where $\mathcal{M}'(\mathcal{Y}, T)$ is constructed dually to $\mathcal{M}(T, \mathcal{Y})$.

**Theorem** (Cornock, G) A semigroup is proper restriction if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{Y})$.

**Corollary** (Petrich and Reilly) A semigroup is proper inverse if and only if it is isomorphic to $\mathcal{M}(G, \mathcal{Y})$ for a group $G$. 


