CIRCLE MAPPINGS WITH SINGULARITIES

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Abstract. After a short review of the classical theory of homeomorphisms of the circle and smoothness of conjugacy to a rigid rotation for the case of differentiable maps, we describe some results for the case when the homeomorphism is everywhere differentiable with the exception of one or two points. The emphasis is given on the case of two singularities where the results of [3] and [4] are stated.

Introduction

The dynamics of homeomorphisms of the circle has long been studied and the results of Denjoy [5, 6] are now classical in Dynamical Systems. Another classical study is to decide how smooth is a conjugacy between a diffeomorphism of the circle and a rigid irrational rotation of the circle.

For the case of critical points, the study turns to when two given differentiable maps of the circle are smoothly conjugate. This is called universality. In this paper we review some of these results (with emphasis in the case of two singularities). The results of [3] and [4] are the main results of the paper (See Section 4).

1. Homeomorphisms of the Circle

Denote the circle by $S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$, where the extremes 0 and 1 are identified by the covering map $\varphi\colon \mathbb{R} \to S^1$ given by $\varphi(x) = x \pmod 1$. The standard differentiable structure of $\mathbb{R}$ induces (via φ) a standard differentiable structure on S^1. Let $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing homeomorphism of $\mathbb{R}$ satisfying

$$\hat{f}(x+1) = \hat{f}(x) + 1 ,$$

for all $x \in \mathbb{R}$. Then $\hat{f}$ induces an orientation preserving homeomorphism f of S^1 by $f(x) = \hat{f}(x) \pmod 1$. It is a classical result that the following limit exists

$$\rho \;=\; \rho(f) \;=\; \lim_{n\to\infty} \frac{\hat{f}^{\,n}(x_0) - x_0}{n} ,$$

and this limit does not depend on the chosen initial point $x_0 \in \mathbb{R}$. The number $\rho(f) \in [0,1)$ is called the *rotation number* of f.

Denjoy [5] showed that f has a periodic point if and only if the rotation number ρ is rational. Also, when ρ is irrational then f is uniquely ergodic and the unique Borel probability invariant measure μ is non-atomic. The continuous map $h\colon S^1 \to S^1$ given by $h(x) = \mu[0, x)$ is a semi-conjugacy of f to the rigid rotation $R_\rho(x) = x + \rho \pmod 1$, i.e. $h \circ f = R_\rho \circ h$.

One of the classical questions is to know when a given homeomorphism f of the circle is topologically conjugate with the rigid rotation R_ρ. Note that when f has irrational rotation number and there is no wandering intervals for f then h is a homeomorphism. Denjoy [5, 6] was able to show the following result.

Theorem 1. *Suppose f is a C^1 diffeomorphism of the circle such that f has irrational rotation number and the derivative of f and f^{-1} have bounded variation then f has no wandering intervals. In particular, f is topologically conjugate to the rigid rotation R_ρ.*

Denjoy also gave an example where if we discard the condition of bounded variation of the derivative of f then the result fails.

Theorem 2. *There exist C^1 diffeomorphisms f with irrational rotation number such that f has non-empty wandering intervals. In particular, h has no inverse in these cases.*

2. Smooth Conjugacy and Rigidity

Another classical question is whether assuming the homeomorphism f is sufficiently differentiable then the conjugacy h is also differentiable. This study is called rigidity. A number of authors contributed to this question and a few of them will be reviewed in this Section.

Arnol'd [1] showed the following result.

Theorem 3. *There exist analytic diffeomorphisms f with irrational rotation such that f is conjugate to the rigid rotation R_ρ but the conjugacy is not absolute continuous, i.e. the unique invariant Borel probability measure μ of f is not absolutely continuous with respect to Lebesgue measure.*

The example relied on choosing f such that the irrational rotation number ρ is badly approximable by rationals. So this showed that the rigidity properties may be related with diophantine approximation properties of the rotation number.

Yoccoz [23] (improving a result from Herman [12]) showed that

Theorem 4. *A smooth diffeomorphism of the circle f is, for Lebesgue almost all rotation numbers, differentiably conjugate to the rigid rotation R_ρ.*

In fact, a finer construction of the rotation number is described in [23].

Recall some basic results from Number Theory (see [11]). Dirichlet's Theorem tells us that every real number ρ satisfies the inequality

$$\left| \rho - \frac{p}{q} \right| < \frac{1}{q^2} \, ,$$

for infinitely many rationals p/q. The number ρ is said to satisfy a Diophantine condition of order $\beta > 0$ if there are infinitely many rationals p/q such that

$$\left|\rho - \frac{p}{q}\right| < \frac{1}{q^{2+\beta}}\ .$$

Such a number ρ is called *Diophantine*.

Again, sharpening a result of Herman [12], Yoccoz [24] showed the following result. (This is also proved by Stark [20] using renormalisation techniques based on the work of Rand [18, 19].)

Theorem 5. *If f is a C^∞ diffeomorphism of the circle whose rotation number ρ is Diophantine then the diffeomorphism h which conjugates f to the rotation R_ρ is of class C^∞.*

There is also a C^r version of the above Theorem, please check the original papers for further details.

Rand [18, 19] and Stark [20] used renormalisation techniques to study rigidity. I shall introduce the basic concepts of renormalisation, starting from the Gauss transformation and the sequence of best approximations of a number. Rand and Stark used the concept of commuting pairs which is introduced in Section 4.

Let $G\colon [0,1] \to [0,1]$ be the Gauss transformation given by $G(\rho) = 1/\rho - a_0(\rho)$ for $\rho > 0$, and $G(0) = 0$, where $a_0(\rho) = [1/\rho]$ is the greatest integer $\leq 1/\rho$. We also set $a_0(0) = \infty$ and make the conventions that $1/0 = \infty$ and $1/\infty = 0$. Defining $a_n = a_n(\rho) = a_0(G^n(\rho))$ for all $n \geq 0$, we obtain the continued fraction expansion of ρ,

$$\rho = [a_0, a_1, \cdots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots}}}\ .$$

We write $p_0 = p_0(\rho) = 0$, $q_0 = q_0(\rho) = 1$ and for $n \geq 1$ we let

$$p_n/q_n = p_n(\rho)/q_n(\rho) = [a_0, a_1, \cdots, a_{n-1}]$$

denote the truncated expansion of ρ of order n in its irreducible form.

Let $f\colon S^1 \to S^1$ be an orientation preserving homeomorphism of the circle without periodic points and let $\rho = \rho(f) \in [0,1)$ be its irrational rotation number. In what follows we shall omit the dependence on ρ and write simply $a_n(\rho) = a_n$, $p_n(\rho) = p_n$ and $q_n(\rho) = q_n$.

Define $J_n \subseteq S^1$ as the closed interval of endpoints $f^{q_n}(0)$ and $f^{q_{n-1}}(0)$ containing 0 in its interior. We also define $I_n \subseteq J_n$ to be the closed subinterval of endpoints 0 and $f^{q_n}(0)$. Recall, from the basic properties of homeomorphisms of the circle, that $J_{n+1} \subseteq J_n$, and inside J_1 the points $f^{q_n}(0)$ and $f^{q_{n-1}}(0)$ lie on opposite sides of 0 for all $n > 1$. It can be shown that the first return map T of f to J_n satisfies the following: $T = f^{q_{n-1}}$ on I_n and $T = f^{q_n}$ on I_{n-1}. The pair $(f^{q_{n-1}}, f^{q_n})$ (after rescaling to the unit interval) forms a commuting pair as in Section 4.

Renormalisation of f. Let I be the unit interval and let $A\colon J_n \to I$ be the unique affine orientation preserving map carrying J_n onto I. Consider the conjugate map ATA^{-1} defined on I. Through the identification of the points 0 and 1 via the canonical projection $\varphi\colon \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$, this conjugate map becomes a new homeomorphism of the circle $T_\star$ on the circle (the n^{th} *renormalisation* of f). It can be shown that the rotation number of $T_\star$ is $\rho = [a_n{+}1, a_{n+1}, a_{n+2}, \cdots]$ (cf. [16, 8]).

De Faria [8] and de Faria and de Melo [9, 10] study rigidity using renormalisation techniques for the case of critical diffeomorphisms of the circle, i.e. when the homeomorphism is everywhere differentiable but has a jet at 0 of the form $x \mapsto x|x|^\delta + a$, for some $\delta > 0$ and constant a, so the inverse is not differentiable at a. They consider the case when the rotation number $\rho(f)$ is of bounded type, i.e. the numbers $a_n(\rho)$ are a bounded sequence. (See also the work of Yampolsky [22] on this subject.) Conditions are given to guarantee when two such maps are differentiably conjugate (i.e. when universality takes place for such maps), see original papers.

3. One-Break Singularity

Khanin et al [13, 14, 15, 21] consider the case of a homeomorphism of the circle with one-break singularity. This is the case when the map is differentiable everywhere except at one point and at that point the left and right derivatives exist and are different. These papers use renormalisation techniques to obtain rigidity results.

As an example we mention one of their results. Consider the following class of functions f. Let $S^1 = \mathbb{R}/\mathbb{Z}$, $f(y) = y + \tilde{f}(y)$. Suppose that

(1) $\tilde{f}$ is a continuous $\mathbb{Z}$-periodic function such that $0 \le \tilde{f}(0) < 1$.
(2) There exists a unique $x_{cr} \in [0, 1)$, such that $\tilde{f} \in C^{2+\varepsilon}((x_{cr}, x_{cr} + 1))$.
(3) In the point of break x_{cr} there exist one-sided derivatives $f'(x_{cr}-), f'(x_{cr}+) > 0$ and $\sqrt{f'(x_{cr}-)/f'(x_{cr}+)} = c \neq 1$.
(4) $\inf_{y\in(x_{cr},x_{cr}+1)} f'(y) > 0$.

Denote by $\mathcal{F}_c$ the set of functions f satisfying conditions (1)-(4) above.

Theorem 6. *Let $f, g \in \mathcal{F}_c$. Suppose that*

(1) $\rho(f) = \rho(g) = \rho$,
(2) ρ *is a quadratic irrational, i.e.* $\rho = a + \sqrt{b}$, $a, b \in \mathbb{Q}$.

Then there exists $\gamma > 0$ such that f and g are $C^{1+\gamma}$ conjugate, i.e., $f = \Phi \circ g \circ \Phi^{-1}$, where Φ is a $C^{1+\gamma}$ orientation-preserving diffeomorphism.

4. Two-Break Singularities

We start by describing the main result of Coelho et al [3] on piecewise affine homeomorphisms of the circle. Let $a, b \in (0,1)$ and consider the map $f\colon [0,1) \to [0,1)$ given by

$$f(x) = \begin{cases} a + \dfrac{1-a}{b}\, x, & \text{for } 0 \le x < b\ ; \\ \dfrac{a}{1-b}\,(x-b), & \text{for } b \le x < 1\ . \end{cases}$$

The graph of f is shown in Figure 1 (where the origin 0 is placed at $\eta(0)$ and b is at 0).

The map f induces a piecewise affine homeomorphism of the circle which we also denote by f. Since the derivative of f has bounded variation, recall from Denjoy Theorem (Theorem 1) that if the rotation number ρ of f is irrational, then there are no wandering intervals for f, and equivalently, f is conjugate to the rotation by ρ (cf. [5, 6]), where ρ denotes the rotation number of f.

Denote the derivative Df by $\alpha = (1-a)/b$ on $[0,b)$, and by $\beta = a/(1-b)$ on $[b,1)$. In [3] it is proved that

Theorem 7. *Let f have parameters (a,b) and derivatives α, β on $[0,b)$, $[b,1)$ respectively. Then the rotation number ρ of f is given by*

$$\rho(a,b) \;=\; \log\frac{1}{\alpha} \Big/ \log\frac{\beta}{\alpha} \;=\; \log\frac{b}{1-a} \Big/ \log\frac{ab}{(1-b)(1-a)}\ .$$

Further it is also shown that

Theorem 8. *Let f have irrational rotation number. Then the density of the unique invariant probability for f is given by*

$$x \;\mapsto\; \frac{1}{c_{\alpha,\beta}} \cdot \frac{\alpha}{\alpha\, x + \beta\,(1-x)}\ ,$$

where the normalising constant $c_{\alpha,\beta}$ is given by $\dfrac{\alpha}{\beta-\alpha}\,\log\frac{\beta}{\alpha}$.

We should point out that Boshernitzan [2] has also obtained the above result when giving an example of a homeomorphism of the circle with the property that the restriction of the map to the set of rationals $\mathbb{Q} \subseteq [0,1)$ is a bijection onto $\mathbb{Q}$, and the orbit of every rational is dense in the circle. The example is produced by taking rational parameters a and b such that the rotation number ρ of the map is irrational.

The reason for this well behaved family of maps is now understood using the notion of commuting pairs which we will now describe.

A pair (ξ, η) of C^r-real valued functions is called a *commuting pair* if it satisfies the following conditions:

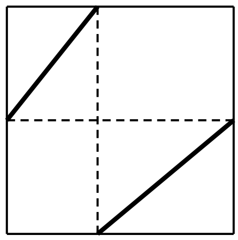
(a) $0 < \xi(0) = \eta(0) + 1 < 1$;
(b) ξ and η are respectively defined on $[\eta(0), 0]$ and $[0, \xi(0)]$;

(c) $\xi' > 0$ on $[\eta(0), 0]$ and $\eta' > 0$ on $[0, \xi(0)]$; and
(d) $(\xi \circ \eta - \eta \circ \xi)^{(i)}(0) = 0$ for $0 \leq i \leq r$.

The condition (d) is the *commuting property* of (ξ, η). We consider the cases $r > 2$, $r = \infty$ or the analytic case $r = \omega$. In the analytic case, condition (d) implies that ξ commutes with η on a neighbourhood of 0, i.e. $\xi \circ \eta = \eta \circ \xi$.

Let S^1 be the interval $[\eta(0), \xi(0)]$ with the extreme points $\eta(0)$ and $\xi(0)$ identified. Given a commuting pair (ξ, η) we define a homeomorphism of the circle $f = f_{\xi,\eta} \colon S^1 \to S^1$ by $f = \xi$ on $x \leq 0$ and $f = \eta$ on $x > 0$. We note that f is C^r except possibly on the points 0 and $f(0)$. The *rotation number* of (ξ, η) is the rotation number of $f_{\xi,\eta}$.

The simplest case of a commuting pair (ξ, η) is the *affine* case, i.e. where ξ and η are affine maps (see Figure 1).



$\eta(0)$ 0 $\xi(0)$

FIGURE 1. An example of an affine commuting pair.

The map $f = f_{\xi,\eta}$ defines a unique C^r-structure $\mathcal{S}_f$ on the circle S^1 and f is C^r with respect to this structure (see proof in [4]). Using this fact together with Theorems 4 and 5, we can prove the following couple of results. (The second result also makes use of Theorem 8.)

Theorem 9. *Suppose (ξ_1, η_1) and (ξ_2, η_2) define the same smooth structure $\mathcal{S}_{f_1} = \mathcal{S}_{f_2}$ and h is a C^0-conjugacy between them. Then for almost every rotation number $\rho(f_1) = \rho(f_2)$, the conjugacy h is smooth with respect to the differentiable structure $\mathcal{S} = \mathcal{S}_{f_1} = \mathcal{S}_{f_2}$.*

Theorem 10. *Let (ξ, η) be a smooth commuting pair. Suppose the rotation number $\rho = \rho(f)$ (for $f = f_{\xi,\eta}$) is Diophantine and suppose there is an affine commuting pair (defining a map R) with the same rotation number ρ and defining the same differentiable structure $\mathcal{S}_f$. Then f and R are smoothly conjugate with respect to $\mathcal{S}_f$. In particular, f has an absolutely continuous invariant measure and the density of the invariant measure is smooth (with the possible exception of at most one point).*

Theorems 9 and 10 will appear in [4]. We should finally mention that Dzhalilov and Liousse [7] considers a class of homeomorphisms of the circle with two-break singularities that have no absolutely continuous invariant measure. This contrasts with the results in [4], but there is no contradiction, the second singularity of the map in [4] is resolved by the first one, i.e. lies in the orbit of the first singularity.

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