A NOTE ON LIVŠIC'S PERIODIC POINT THEOREM

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For Robert Ellis

Abstract. We consider two examples of diffeomorphisms of the connected sum of two anchor rings. The first is a hyperbolic diffeomorphism which induces an automorphism on the first cohomology group (of the attractor) having *1* as an eigenvalue. The second fails to be hyperbolic on two transverse intersecting discs and is non-uniformly hyperbolic on the rest of the attractor. The Livšic periodic point theorem is not true for the latter example.

Introduction

Let M be a compact manifold (possibly with boundary) and let $S\colon M \to S(M) \subseteq \mathrm{int}(M)$ be a diffeomorphism. A closed S-invariant subset $\Omega \subseteq M$ is called a *hyperbolic attractor* if Ω is an attractor for S endowed with a hyperbolic structure. By an *attractor* we mean a maximal and isolated attractor, i.e. there exists an open neighbourhood $\mathcal{U}$ of Ω such that $S(\mathcal{U}) \subseteq \mathcal{U}$; $\cap_{n\geq 0} S^n\mathcal{U} = \Omega$; and the non-wandering set of S restricted to $\mathcal{U}$ is Ω. By a *uniform hyperbolic structure* on Ω we mean there exists a continuous splitting of the tangent bundle restricted to Ω as a direct sum of two subbundles $T_\Omega M = E^s \oplus E^u$, and there are constants $0 < \rho < 1$ and $C > 0$ such that

(a) $DS_x(E^s_x) = E^s_{Sx}$, $DS_x(E^u_x) = E^u_{Sx}$; and

(b) $||DS^n_x(v)|| \leq C\,\rho^n\,||v||$, $||DS^{-n}_x(w)|| \leq C\,\rho^n\,||w||$,

for all $n \geq 0$, $v \in E^s_x$, $w \in E^u_x$, and all $x \in \Omega$. We say the hyperbolic structure on Ω is *non-uniform* if we allow $\rho = \rho_x$ to be dependent on $x \in \Omega$. When M itself has a (uniform) hyperbolic structure then we say that S is an *Anosov* diffeomorphism. We will also use, loosely, the term *pseudo-hyperbolic* to mean an attractor Ω for which a (non-uniform) hyperbolic structure exists on all but a 'small' set of exceptional points and on these exceptional points the Jacobian of S is an isometry.

Bowen ([B2], p.21) mentions a problem first raised by Hirsch [H]: Is there an Anosov map which induces an automorphism on the first homology group having *1* as an eigenvalue? When M is a nilmanifold the answer is negative since Manning [M] shows that eigenvalues of modulus 1 cannot occur. Bowen ([B2], p.27) also remarks that this problem is equivalent to the existence of a continuous function $f\colon M \to \mathbb{R}$ such that $f^n(x) = f(x) + \cdots + f(S^{n-1}x) \in \mathbb{Z}$ whenever $S^n x = x$, but there is no constant $N \in \mathbb{Z}$ such that $f^n(x) = nN$ whenever $S^n x = x$.

In fact this is also the case for a connected hyperbolic attractor as we will show. (Here as elsewhere in this paper, in place of the first homology group of the manifold, we consider the first Čech cohomology group $H^1(\Omega, \mathbb{Z})$ interpreted as the Bruschlinsky group of continuous maps to the circle modulo null homotopic maps.) In connection with this problem we give an example of a hyperbolic diffeomorphism of the connected sum of two anchor rings which induces an automorphism on $H^1(\Omega, \mathbb{Z})$ having *1* as an eigenvalue.

The proof of the equivalence of the two conditions mentioned above depends on the celebrated

Livšic's periodic point theorem. *Let Ω be a hyperbolic attractor for S, and let $f\colon \Omega \to \mathbb{R}$ ($\Omega \to K$, the circle) be Hölder continuous. Then the following are equivalent:*

(i) There exists a continuous map $h\colon \Omega \to \mathbb{R}$ ($\Omega \to K$) such that $f = h \circ S - h$ ($f = h \circ S / h$).
(ii) $f^n(x) = f(x) + \cdots + f(S^{n-1}x) = 0$ ($f^n(x) = f(x) \cdots f(S^{n-1}x) = 1$) for all $x \in \Omega$ such that $S^n x = x$.

The example above does provide us with a real function which satisfies the Bowen condition and this will be made explicit.

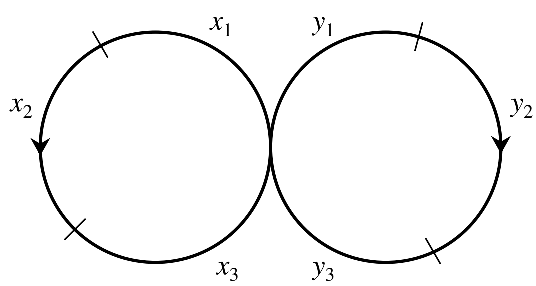
Our second example is also of a diffeomorphism T of the connected sum of two anchor rings but having a pseudo-hyperbolic attractor Ω for which the additive version of Livšic's theorem fails.

Both examples are modelled on maps of the 1-dimensional branched (figure eight) manifold – the wedge of two circles. Both maps expand the circles and wrap each circle around itself twice and once around the other circle, leaving the branch point fixed. The difference between these maps lies in the order of the 'wrappings'. The diffeomorphisms we shall present have for their attractors the 1-dimensional complexes which (up to topological conjugacy) are the inverse limits of the figure eight under these one dimensional maps.

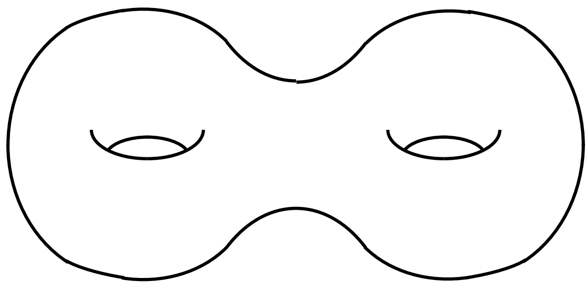
Our presentation of the diffeomorphisms will be geometric and largely descriptive with details of the analysis only indicated. Following these descriptions we establish (rigourously) the claims made above.

§1. The examples

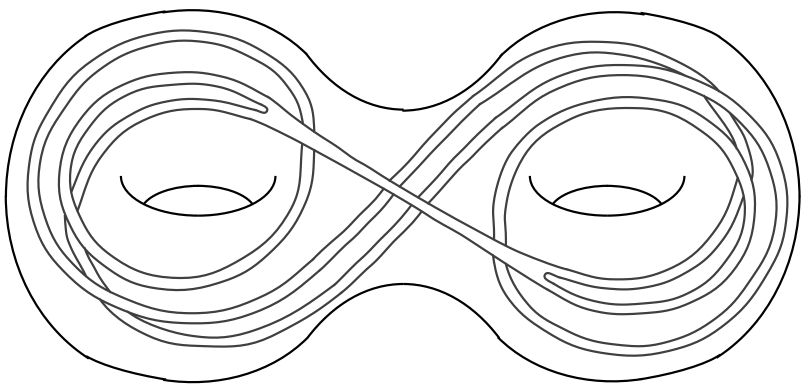
Consider the (oriented) 1-dimensional branched manifold Λ as illustrated in:



and the map σ which expands the first circle (by a factor of 3) and wraps each of x_1, x_2 successively around x ($x = x_1x_2x_3$) and x_3 around y ($y = y_1y_2y_3$) preserving orientation. In short $\sigma\colon x \mapsto x^2y$. Similarly σ maps y accordingly to $\sigma\colon y \mapsto xy^2$. We can mimic this behaviour by constructing a diffeomorphism S of the solid of 'genus' 2:

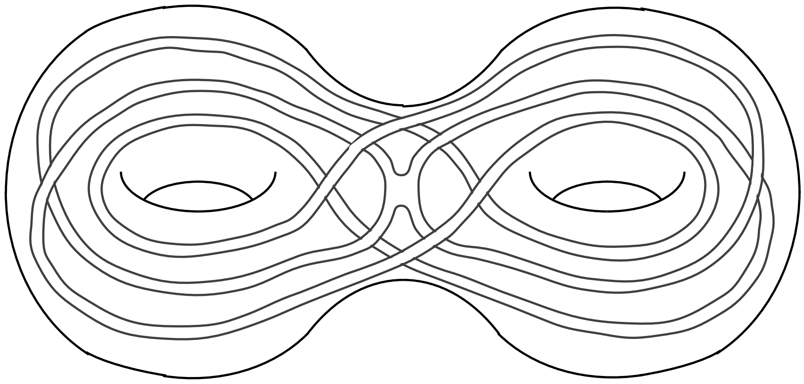


obtained by thickening the one dimensional branched manifold. The diffeomorphism S expands (like σ) in one direction and contracts the transverse discs. Mimicking σ we have the following picture of the manifold and its image under S:



Well known arguments (cf. [W1],[W2]) will show that the attractor $\Omega = \cap S^n M$ is topologically conjugate to the shift $\hat{\sigma}\colon \hat{\Lambda} \to \hat{\Lambda}$ on the inverse limit solenoid defined by $\sigma\colon \Lambda \to \Lambda$ which is locally the direct product of a Cantor set and an arc. The branch point presents no problem here since the image under σ of a (branched) neighbourhood is an arc and S is hyperbolic (cf. [W2]). Clearly there is a projection (contracting transverse discs) which maps M onto $\hat{\Lambda}$ which in turn projects to Λ. Hence any Hölder continuous function defined on Λ can be lifted to $\hat{\Lambda}$ or M.

The second example T is again a diffeomorphism of M onto its image and is also modelled on a map $\tau\colon \Lambda \to \Lambda$. This time τ is symbolically represented by $x \mapsto xyx$, $y \mapsto yxy$. The map τ has a convenient representation as $\tau x = 3x \pmod 1$ defined on the unit interval with 0, 1/2 and 1 identified. One may picture T as follows:



The idea here is again to expand in one direction and contract along the transverse discs. However, one must take care defining T on the 'bridge' joining the 2 anchor rings. The transverse discs bend toward perpendicular discs. Although the manifold expands (in one direction) the expansion is not uniform and in fact the coefficient of expansion decreases to 1 as one approaches the exceptional discs above. One can show that the non-uniformity of expansion is essential for such an example since the branch in a neighbourhood of the fixed point cannot be abolished. In other words, the inverse limit $\hat{\Lambda}$ contains a branched submanifold. Again a Hölder continuous function defined on Λ can be lifted to such a function on $\hat{\Lambda}$ and on M.

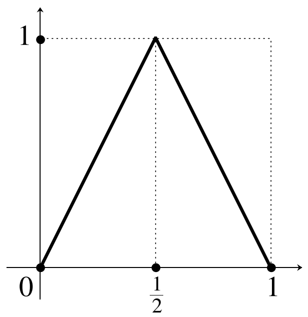
§2. The Bruschlinsky group $H^1(\Omega, \mathbb{Z})$

Since Ω is homeomorphic to $\hat{\Lambda}$ (in each case) we have $H^1(\Omega, \mathbb{Z})$ is the direct limit of

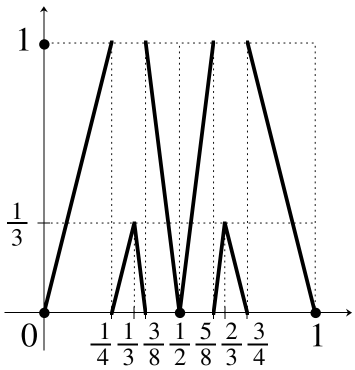
$$H^1(\Lambda, \mathbb{Z}) \longrightarrow H^1(\Lambda, \mathbb{Z}) \longrightarrow \cdots$$

with respect to the endomorphism induced by the appropriate map of the wedge of 2 circles. However, $H^1(\Lambda, \mathbb{Z})$ is $\mathbb{Z}^2$ generated by (maps of) the two circles. Moreover in each case it is easy to see that the induced endomorphism is given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

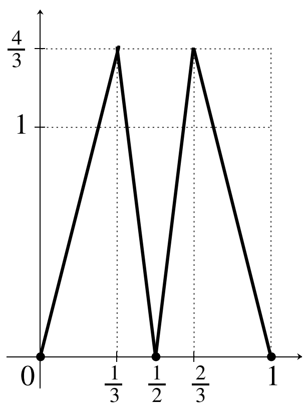
So in each case there is a cohomology class fixed by A, namely $(1, -1)$. A circle valued function representing $(1, -1)$ is the tent function F:



Here we have marked the identification points $\{0, 1/2, 1\}$ for the wedge domain and the identification points $\{0, 1\}$ for the circle image. For our hyperbolic example, $F \circ S/F$ has the symmetric graph



and therefore $F(Sx)/F(x) = e^{2\pi i r(x)}$ where r is the continuous real valued function with graph

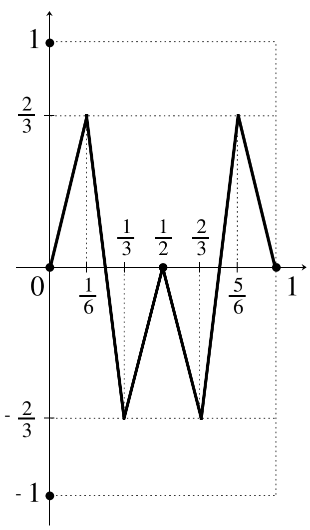


in conformity with the fact that A has a non-trivial fixed vector. Notice that we cannot write $r(x) = h(Sx) - h(x) + c$ (a.e.) for some constant c, where h is integrable with respect to Lebesgue measure (which is preserved by S). This is because integrating this equation we would have c as the integral of r, which is not an integer. Hence such an equation would lead to

$$\frac{F\circ S}{F} = \frac{e^{2\pi i h\circ S}}{e^{2\pi i h}} \cdot e^{2\pi i c}$$

and this would violate the fact that S is mixing. Alternatively, one would conclude that F is an inessential function, which is not the case. Moreover we see that $r^n(x) \in \mathbb{Z}$ when $S^n x = x$ and yet there is no integer N for which $r^n(x) = nN$ when $S^n x = x$, for otherwise we would have $r^1(0) = r(0) = 0$ so that $N = 0$ and $r^1(1/4) = r(1/4) = 0$ which is not the case.

If we now pass to our pseudo-hyperbolic example T we notice that the function F can be written $F(x) = e^{2\pi i u(x)}$, where u is a real valued function defined everywhere except at $\{0, 1/2, 1\}$. We can also write $F(Tx)/F(x) = e^{2\pi i s(x)}$, where s is a continuous real valued function on Ω with graph



Let N be the set $\{x \in \Omega\colon\, T^n x = 0, 1/2, 1,$ for some $n\}$ then we have $s(x) = u(Tx) - u(x)$ for all $x \in \Omega \setminus N$. Notice also that $s^n(x) = s(x) + \cdots + s(T^{n-1}x) = 0$ whenever $T^n x = x$ since for such points $x \notin N$ and $s(x) = u(Tx) - u(x)$, unless $x = 0, 1/2, 1$ where s vanishes.

We show next that Livšic's (additive) periodic point theorem does not hold for T. If Livšic's theorem were true in this context we would conclude that $s = h\circ T - h$ for some continuous function h (defined everywhere on Ω). This implies that

$$\frac{F(Tx)}{F(x)} = \frac{e^{2\pi i h(Tx)}}{e^{2\pi i h(x)}}$$

and therefore $F(x)\, e^{-2\pi i h(x)}$ would be constant, contradicting the fact that F is *not* null-homotopic.

Now for completeness we include the proof of Bowen's observation. In what follows let Ω be a connected hyperbolic attractor for S a diffeomorphism onto its image. Let $f\colon\Omega \to \mathbb{R}$ be a Hölder continuous function and let A be the endomorphism of $H^1(\Omega, \mathbb{Z})$ induced by S.

Theorem (Bowen). *The following are equivalent:*

(i) $\det(A - I) \neq 0$ *;*

(ii) $f^n(x) \in \mathbb{Z}$ *whenever* $S^n x = x$ *implies that* $f^n(x) = nN$ *for some* $N \in \mathbb{Z}$ *whenever* $S^n x = x$ *.*

Proof. Assume (i) and let $f^n(x) \in \mathbb{Z}$ whenever $S^n x = x$, then

$$e^{2\pi i f(x)} \cdots e^{2\pi i f(S^{n-1}x)} \;=\; 1$$

whenever $S^n x = x$. By Livšic's theorem, we have $e^{2\pi i f(x)} = h(Sx)/h(x)$ for some continuous map $h\colon\Omega \to K$. Therefore $A[h] = [h]$ and by assumption $[h]$ is trivial i.e. $h(x) = e^{2\pi i g(x)}$ with g continuous. Thus

$$e^{2\pi i f(x)} \;=\; e^{2\pi i (g(Sx) - g(x))} ,$$

and by connectedness

$$f(x) \;=\; g(Sx) - g(x) + N$$

for some constant $N \in \mathbb{Z}$. Hence $f^n(x) = nN$ whenever $S^n x = x$.

Now assume *not* (i), then there exists a non-trivial cohomology class $[h]$ such that $A[h] = [h]$, i.e. $h(Sx)/h(x) = e^{2\pi i g(x)}$ for some continuous function g. Therefore $e^{2\pi i g^n(x)} = 1$ whenever $S^n x = x$, i.e. $g^n(x) \in \mathbb{Z}$ whenever $S^n x = x$. In order to complete the proof we need only show that nevertheless it is *not* true that $g^n(x) = nN$ for some $N \in \mathbb{Z}$. If on the contrary $g^n(x) = nN$ whenever $S^n x = x$ then $(g - N)^n(x) = 0$ whenever $S^n x = x$ and so by Livšic's theorem $g - N = k\circ S - k$ for some Hölder continuous k. Thus $h\circ S/h = e^{2\pi i k\circ S}/e^{2\pi i k}$ which implies $h\, e^{-2\pi i k}$ is a constant. Therefore $[h]$ is trivial – a contradiction. □

Remark 1. Although Livšic's theorem was needed to prove the Bowen's result it is still true that the Bowen's statements hold for T, even though Livšic's theorem fails in this case. One can see this by lifting the function f to the hyperbolic system consisting of $x \mapsto 3x \pmod 1$ on the circle (where Livšic's is now true). The lift $\tilde{f}$ now satisfies (ii) and it follows from this that f satisfies (ii).

Remark 2. The diffeomorphism T above also serve as an example where Livšic's regularization theorem does not hold. This follows from the fact that the Hölder continuous function $s(x)$ satisfies the equation

$$s(x) \;=\; u(Tx) - u(x)$$

everywhere, except for a countable number of points, and the integrable function u cannot be replaced by a continuous function satisfying this property.

Remark 3. Examples such as S and T can also be realized as hyperbolic (and respectively, pseudo-hyperbolic) diffeomorphisms of the 3-sphere onto itself. This is because the 3-sphere can be viewed as the union of two solids of genus 2 with disjoint interiors. By a standard argument one can extend S say, to the complementary solid $\widetilde{M}$ by making $S\,|_{\widetilde{M}} \simeq S^{-1}|_{S(M)}$ (cf. [G]). The same procedure applies to T producing a pseudo-hyperbolic diffeomorphism of the 3-sphere.

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