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Numerical fits to important rates in high temperature astrophysical plasmas

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Summary. We give simple but accurate numerical fits to various two-body rates in relativistic thermal plasmas, in the temperature range $kT_e \sim 50$ keV–1 MeV. The processes we discuss are bremsstrahlung, Coulomb heating and electron–positron pair production and annihilation. In particular we present a fit to the thermal electron–electron bremsstrahlung spectrum which is accurate to better than 5 per cent. We also include the results of some other workers, for completeness. Our results are suitable for semi-analytic or computer modelling of hot plasmas.

1 Introduction

Active galactic nuclei, gamma-ray bursters and certain compact binary sources (e.g. Cygnus X-1) emit radiation at energies greater than 100 keV. Physical modelling of these sources, therefore, requires a detailed knowledge of electron–photon and electron–particle reactions at relativistic electron energies. Although there has recently been considerable progress in understanding astrophysical plasmas at relativistic energies (Svensson 1982a; Lightman & Band 1981; Stepney 1983), further progress requires accurate, simple formulae for the relevant physical processes. Cross-sections for the important processes are usually available; however, they are often expressed as complicated special functions, or multidimensional integrals, or both. We have computed simple fits to several cross-sections for thermal electron distributions, in particular the electron–electron bremsstrahlung cross-section. Our results, together with those of other workers, which we have included for completeness, can be used to model all the important processes in a thermal plasma analytically or numerically, with the exceptions of particle–particle pair production and electron–positron bremsstrahlung.

2 Bremsstrahlung

2.1 ELECTRON–PROTON BREMSSTRAHLUNG

The cross-section in the Born approximation is given by Heitler (1954). Assuming that T_p is small enough to neglect the protons' motion, the spectral emissivity is then

$$\frac{dE_{ep}}{dV\,dt\,d\omega} = N_p c \int_{1+\omega}^{\infty} \omega \frac{d\sigma}{d\omega} \beta N_e(\gamma)\, d\gamma \qquad (1)$$

where $\omega = h\nu/m_e c^2$ is the dimensionless photon energy, $\theta = kT/m_e c^2$ is the dimensionless temperature, $N_e(\gamma) = N_e \gamma^2 \beta \exp(-\gamma/\theta)/\theta K_2(1/\theta)$ is the Maxwellian electron energy distribution and K_2 is a modified Bessel function. N_e and N_p are the electron and proton number densities. The integral (1) can be evaluated with an error of less than 2 per cent for $\theta > 0.1$ using four-point Gauss–Laguerre quadrature. Górecki & Kluźniak (1981) give a fit to the spectrum which, due to its complexity, has no advantage over direct quadrature.

The total emission can be found by integrating equation (1) over the photon energy. Svensson (1982a) gives the following fit:

$$\frac{dE_{ep}}{dV\,dt} = N_e N_p \sigma_T c \alpha_f m_e c^2 F(\theta) \tag{2}$$

$$= \begin{cases} 4\left(\dfrac{2\theta}{\pi^3}\right)^{1/2} [1 + 1.781\,\theta^{1.34}]; & \theta < 1 \quad (3a) \\ \dfrac{9\theta}{2\pi} [\ln(2\theta \exp(-\gamma_E) + 0.42) + 1.5]; & 1 < \theta \quad (3b) \end{cases}$$

α_f is the fine structure constant, σ_T is the Thomson cross-section and γ_E is Euler's constant ~ 0.5772.

2.2 ELECTRON–ELECTRON BREMSSTRAHLUNG

The cross-section for this process is much more complicated than that for electron–proton bremsstrahlung; however, Haug (1975a) has obtained an expression which can be integrated numerically (see Appendix 1 for details). The photon spectrum is given by:

$$\frac{dN_{ee}}{dV\,dt\,d\omega} = N_e^2 \sigma_T c \alpha_f \exp(-x) G(x, \theta)/x \tag{4}$$

where $x = \omega/\theta$. We have fit $G(x, \theta)$ at 13 temperatures between 50 keV and 1 MeV with an error of less than 2 per cent for $0.05 \leqslant x \leqslant 10.0$. The fit is:

$$G(x, \theta) = (A + Bx)\ln(1/x) + C + Dx; \qquad 0.05 \leqslant x \leqslant 1.1 \tag{5a}$$

$$= \alpha x^2 + \beta x + \gamma + \delta/x; \qquad 1.0 \leqslant x \leqslant 10.0 \tag{5b}$$

where the coefficients α, β, γ, δ and A, B, C, D are given in Table 1. The coefficients can be interpolated to give spectra with errors of less than 5 per cent. The total emission can be obtained either by numerical quadrature of equation (3.20) in Haug (1975b), or by integration of our fit to equation (4) to give:

$$\frac{dE_{ee}}{dV\,dt} = N_e^2 \sigma_T c \alpha_f m_e c^2 \theta^2 (0.797A + 0.164B + 0.632C + 0.264D + 1.839\alpha + 0.736\beta + 0.368\gamma + 0.219\delta). \tag{6}$$

The error in equation (6) is also less than 5 per cent.

For temperatures above 1 MeV the ultra-relativistic approximation (Alexanian 1968) gives spectra with errors of less than 5 per cent. Below 50 keV the non-relativistic formula (Haug 1975b) is accurate to better than 5 per cent for $\omega/\theta \lesssim 1$.

Gould (1980, 1981) has calculated the first-order corrections to the non-relativistic electron–proton bremsstrahlung spectrum due to relativistic effects, electron–electron bremsstrahlung and corrections to the Born approximation. The resulting spectral emissivity is accurate to 1 per cent for $T \sim 10^8$ K ($\theta \sim 0.02$).

Table 1. Coefficients for the fit to the electron–electron bremsstrahlung spectrum $x=\omega/\theta$. The low energy portion is fitted at $x=0.05$, 0.15, 0.5063, 1.139. The high energy portion is fitted at $x=1.139$, 2.563, 5.767, 10.0. In the given ranges the fits (to the computed values) are better than 1 per cent.

Low energy: $0.05<x<1.1$

kT/keV	A	B	C	D
50	1.584	0.578	4.565	2.091
75	1.357	0.437	3.842	1.855
100	1.197	0.291	3.506	1.672
150	1.023	0.204	3.036	1.593
200	0.883	0.0835	2.831	1.487
300	0.700	−0.0494	2.545	1.364
400	0.572	−0.139	2.352	1.254
500	0.484	−0.181	2.175	1.179
600	0.417	−0.209	2.028	1.108
700	0.361	−0.240	1.914	1.030
800	0.322	−0.244	1.795	0.982
900	0.286	−0.257	1.705	0.923
1000	0.259	−0.258	1.617	0.879

High energy: $1.0<x<10.0$

kT/keV	α	β	γ	δ
50	0.0387	0.523	5.319	0.782
75	0.0633	0.540	4.412	0.689
100	0.0862	0.569	3.897	0.633
150	0.128	0.596	3.383	0.523
200	0.159	0.658	2.974	0.532
300	0.208	0.633	2.738	0.326
400	0.234	0.643	2.424	0.302
500	0.245	0.695	2.025	0.394
600	0.248	0.729	1.716	0.453
700	0.247	0.756	1.457	0.500
800	0.243	0.763	1.271	0.515
900	0.239	0.755	1.140	0.508
1000	0.235	0.735	1.060	0.478

3 Electron–positron pair production

3.1 PHOTON–PHOTON

The cross-section for this process can be found in Jauch & Rohrlich (1980). Unfortunately, for a general photon distribution the pair production rate is a complicated six-dimensional integral over both photon momenta. In the case of isotopic photon distributions considerable simplification is possible analytically and Weaver (1976) has evaluated this rate. For cylindrically symmetric photon distributions (e.g. radiative transfer in a slab) we have the reaction rate (per unit volume per unit time):

$$R_{\gamma\gamma}=4\pi c\int n_1(\omega_1,\mu_1)n_2(\omega_2,\mu_2)\,\omega_1\omega_2\,d\omega_1\,d\omega_2\,d\mu_1\,d\mu_2$$

$$\times\int_{\omega_-}^{\omega_+}\frac{\omega^3\sigma(\omega)\,d\omega}{[(\omega_+^2-\omega^2)(\omega^2-\omega_-^2)]^{1/2}} \tag{7}$$

where $\omega_\pm = \omega_1\omega_2[1-\cos(\theta_1 \pm \theta_2)]/2$ and $\mu = \cos\theta$. The ω integral can be evaluated semi-analytically with the aid of a 2-point Gauss–Legendre quadrature to better than 0.5 per cent (see Appendix 2).

3.2 PHOTON–PROTON

The threshold photon energy in the proton's rest frame is $\gamma = 2$. The cross-section is given by (Jost, Luttinger & Slotnick 1950):

$$\sigma(\omega) = \frac{3\alpha_f}{4\pi}\sigma_T \left\{\eta^2\left[2\int_1^{1/\eta} F_1(x)\frac{dx}{x} - F_1(1/\eta)\right] + [-(109+64\eta^2)E(\epsilon) + (42 + 125\eta^2 + 6\eta^4)F(\epsilon)]/27\right\} \tag{8}$$

where

$$\eta = 2/\omega, \quad \epsilon = (1-\eta^2)^{1/2},$$

$$E(\epsilon) = \int_0^{\pi/2}(1-\epsilon^2\sin^2\phi)^{1/2}\,d\phi,$$

$$F(\epsilon) = \int_0^{\pi/2}(1-\epsilon^2\sin^2\phi)^{-1/2}\,d\phi,$$

$$F_1(x) = \int_1^x F[(1-1/\xi^2)^{1/2}]\,d\xi/\xi.$$

We note that the cross-section in Jauch & Rohrlich appears to be incorrect. We have obtained the following fits:

$$\frac{\sigma(\omega)}{\sigma_T} \simeq \alpha_f\frac{\epsilon^6}{32}(1+0.875\,\epsilon^2+0.755\,\epsilon^4+0.661\,\epsilon^6+0.589\,\epsilon^8) \quad \text{for } 2 \leqslant \omega \leqslant 2.4, \tag{9a}$$

$$\simeq (7.6260 - 8.0218\,\omega + 2.5250\,\omega^2 - 0.2047\,\omega^3)\times 10^{-4} \quad \text{for } 2.4 \leqslant \omega \leqslant 4.0, \tag{9b}$$

$$\simeq \frac{7\alpha_f}{6\pi}(\ln 2\omega - 109/42) + (473.65 + 241.26\ln 2\omega - 81.151\ln^2 2\omega + 5.3814\ln^3 2\omega)\times 10^{-5}/\omega \quad \text{for } 4.0 \leqslant \omega. \tag{9c}$$

The error in these fits is less than 0.1 per cent.

3.3 PHOTON–ELECTRON

This cross-section has been calculated by Haug (1975a, 1981) who has also provided the following simple fits:

$$\frac{8\pi}{3\alpha_f}\frac{\sigma(\omega)}{\sigma_T} \simeq [5.6 + 20.4(\omega-4) - 10.9(\omega-4)^2 - 3.6(\omega-4)^3 + 7.4(\omega-4)^4]\times 10^{-3}(\omega-4)^2 \quad 4 \leqslant \omega \leqslant 4.6$$

$$\simeq 0.582814 - 0.29842\,\omega + 0.04354\,\omega^2 - 0.0012977\,\omega^3 \quad 4.6 \leqslant \omega \leqslant 6.0,$$

$$\simeq \frac{3.1247 - 1.3397\,\omega + 0.14612\,\omega^2}{1 + 0.4648\,\omega + 0.016683\,\omega^2} \quad 6 \leqslant \omega \leqslant 18,$$

$$\simeq (84\ln 2\omega - 218)/27 + (-1.333\ln^3 2\omega + 3.863\ln^2 2\omega - 11\ln 2\omega + 27.9)/\omega \quad 14 \leqslant \omega. \tag{10}$$

These fits can be used to find the thermally averaged cross-section with an error of less than 1 per cent with four-point Gauss–Laguerre quadrature.

4 Electron–positron annihilation

Svensson (1982b) has given the spectral emissivity from annihilating Maxwellian electrons and positrons in the form of a single integral over the pair production cross-section:

$$\frac{dN}{dV\,dt\,d\omega}=N_+N_-\sigma_T c\,\frac{2\exp(-\omega/\theta)}{\theta K_2^2(1/\theta)}\,I(\omega\theta) \tag{11}$$

where

$$I(\omega\theta)=\int_1^\infty \frac{3}{8s^2}\left[(2s^2+2s-1)\cosh^{-1}\sqrt{s}-\sqrt{s}\sqrt{s-1}\,(s+1)\right]\exp(-s/\omega\theta)\,ds. \tag{12}$$

He has fitted the integral $I(\zeta)$ by

$$I(\zeta)=\begin{cases}\dfrac{3\sqrt{\pi}}{16}\,\zeta^{3/2}\exp(-1/\zeta)\,C_1(\zeta); & \zeta\leqslant 4 \quad (13a)\\[2ex] \dfrac{3\zeta}{8}\,(\ln 4\zeta-1-\gamma_E)\,C_2(\zeta); & 4\leqslant\zeta \quad (13b)\end{cases}$$

where the polynomials C_1, C_2 are given by his equations (17) and (18). The maximum error is 0.3 per cent. Zdziarski (1980) has also fitted a function to the spectrum (11), which is accurate to 25 per cent for $0.02<\theta<8$.

5 Coulomb heating

Stepney (1983) has derived a general expression for the rate of transfer of energy between populations with Maxwellian distributions in terms of an integral over the scattering cross-section. In the case of hot protons heating cooler electrons the Rutherford cross-section is the relevant one:

$$d\sigma(\gamma,\alpha)=\frac{3}{32\pi\gamma^2\beta^4}\,\frac{d\Omega_{\text{rest}}}{\sin^4\alpha_{\text{rest}}}=\frac{3}{32\pi\gamma^2\beta^4}\left(1+\frac{2\gamma m_e}{m_p}\right)\frac{d\Omega}{\sin^4\alpha} \tag{14}$$

where α is the scattering half angle in the centre of momentum frame and β is the relative velocity.

This gives a heating rate of

$$\frac{dE_e}{dt}=-\frac{3m_e}{2m_p}N_eN_p\sigma_T c\times\frac{(kT_e-kT_p)}{K_2(1/\theta_e)K_2(1/\theta_p)}\ln\Lambda \times\left[\frac{2(\theta_e+\theta_p)^2+1}{\theta_e+\theta_p}K_1\left(\frac{\theta_e+\theta_p}{\theta_e\theta_p}\right)+2K_0\left(\frac{\theta_e+\theta_p}{\theta_e\theta_p}\right)\right] \tag{15}$$

where $\theta_e=kT_e/m_ec^2$, $\theta_p=kT_p/m_pc^2$ and $\ln\Lambda\sim 20$ is essentially a Coulomb logarithm.

6 Discussion

The lack of and poor quality of observations of astrophysical sources above a few hundred keV makes detailed modelling impractical. The thermal assumption minimizes the number of free parameters and so leads to the simplest models.

All the thermally averaged cross-sections relevant at electron temperatures ~100 keV–1 MeV have been considered here, with the exception of particle–particle pair production and electron–positron bremsstrahlung.

Budnev *et al.* (1975) give the cross-section for pair production in particle–particle collisions for electrons with energies $\gtrsim 50$ MeV. As far as the authors are aware the cross-sections nearer threshold are unknown. For plasmas with temperatures less than a few MeV photon–photon pair production will dominate due to the much lower threshold energy for this process, even when the plasma is very optically thin to Thomson scattering. These rates will therefore only be important in the limit of zero optical depth.

The lack of the electron–positron bremsstrahlung spectrum is more serious. In the ultra-relativistic limit it is simply twice the electron–electron spectrum. In the non-relativistic case it is more closely related to the electron–proton spectrum, since both systems radiate via dipole emission. In the case of hard photons (those with energies $\gtrsim 0.5$ MeV) one of the electrons must be relativistic, and so Svensson (1982a) argues that the high energy tail of the spectrum will be the same as that of the electron–proton spectrum. The authors know of no results for electron temperatures $\theta_e \sim 1$.

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Appendix 1: electron–electron bremsstrahlung

The notation used in this appendix is identical to that used by Haug (1975a, b) if no other definition is given. The spectral emissivity is given by equation (2.4) in Haug (1975b):

$$P_{ee}(k,\tau)=\frac{N_e^2 c}{[2\tau K_2(1/\tau)]^2}\int d\epsilon_1 \exp(-\epsilon_1/\tau)\int d\epsilon_2 \exp(-\epsilon_2/\tau)\int d\mu(\mu^2-1)^{1/2}\frac{d\sigma}{dk} \tag{A1}$$

(note that here $\tau = kT/m_e c^2$, $k = h\nu/m_e c^2$ and $P_{ee} = dN/dV\,dt\,d\omega$).

From Haug (1975a) we find:

$$\frac{d\sigma}{dk}=\frac{3\alpha_f}{8\pi^2}\sigma_T\int d\Omega_k\,\frac{k}{\omega\rho}\left[\frac{\rho^2-4}{\omega^2-4}\right]^{1/2}\frac{1}{\pi}\int A\,d\Omega_p. \tag{A2}$$

Choosing coordinates so that the z-axis is parallel to $\mathbf{P}_1 + \mathbf{P}_2$ (the sum of the initial electron three-momenta) and with

$$s = \frac{\epsilon_1 + \epsilon_2 - (k+2)}{\tau}; \qquad t = \epsilon_1 - \epsilon_2 \tag{A3}$$

we find:

$$P_{\mathrm{ee}}(k,\tau) = \frac{3\alpha_{\mathrm{f}}}{16\pi} N_{\mathrm{e}}^2 \sigma_{\mathrm{T}} c \frac{k \exp(-k/\tau)}{\tau[2\exp(1/\tau)K_2(1/\tau)]^2} \int_0^\infty ds \exp(-s) \int_{-\mathrm{T}}^{\mathrm{T}} dt \int_{\mu_1}^{\mu_2} d\mu$$
$$\times \int_0^{\theta_{\max}} \frac{\sin\theta\, d\theta}{[2(\mu+1-x)]^{1/2}} \int_0^\pi d\phi\, \Sigma \tag{A4}$$

where

$$\Sigma = \frac{(\rho^2-4)^{1/2}}{\pi} \int A\, d\Omega_{\mathrm{p}}. \tag{A5}$$

(see equation A1 in Haug 1975a).

The boundary values are found from the condition $\rho^2 \geqslant 4$ which gives:

$$T = \epsilon - 2 \quad \text{if} \quad s_1 \leqslant s$$
$$T = \left[\frac{\mu_{\mathrm{k}}-1}{\mu_{\mathrm{k}}+1}\{\epsilon^2 - 2(\mu_{\mathrm{k}}+1)\}\right]^{1/2} \quad \text{if} \quad s \leqslant s_1 \tag{A6}$$

$$\mu_1 = \mu_{\min}; \quad \mu_2 = \mu_{\max} \quad \text{if} \quad k \leqslant F(\mu_{\min}, \epsilon)$$
$$\mu_1 = \mu_{\mathrm{k}}; \quad \mu_2 = \mu_{\max} \quad \text{if} \quad F(\mu_{\min},\epsilon) < k \leqslant F(\mu_{\max},\epsilon)$$
$$\mu_1 = \mu_{\mathrm{k}}; \quad \mu_2 = 1 + k\,[(\epsilon-k) + \{(\epsilon-k)^2-4\}^{1/2}] \quad \text{if} \quad F(\mu_{\max},\epsilon) < k \tag{A7}$$

$$\cos\theta_{\max} = \mathrm{Max}\left[\frac{\epsilon - (\mu-1)/k}{\{\epsilon^2 - 2(\mu+1)\}^{1/2}}, -1\right], \tag{A8}$$

where

$$\epsilon \equiv \epsilon_1 + \epsilon_2; \quad \mu_{\mathrm{k}} \equiv 1 + k\,[(\epsilon-k) - \{(\epsilon-k)^2-4\}^{1/2}];$$
$$s_1 = \frac{k}{\tau}\left\{\frac{k - [k^2+4k]^{1/2}+3}{2[k^2+4k]^{1/2}+3}\right\}; \mu_{\min} = \epsilon_1\epsilon_2 - P_1P_2; \quad \mu_{\max} = \epsilon_1\epsilon_2 + P_1P_2 \tag{A9}$$

and

$$F(\mu,\epsilon) = \frac{\mu-1}{\epsilon - [\epsilon^2 - 2(\mu+1)]^{1/2}}.$$

In order to evaluate (A4) efficiently, Gaussian quadrature should be used. This is impossible as the integral stands, however, due to the sharp peaks in the integrand for x_1 or x_2 small (this reflects the beaming of radiation in the direction of the electrons' motion). This problem can be overcome by making the following transformations:

$$\sinh\eta = \epsilon_1 Q \tan\left(\frac{\theta-\xi_1}{2}\right) \quad \text{and} \quad \sinh\xi = \sinh\Lambda \tan\phi/2 \tag{A10}$$

to give

$$P_{\mathrm{ee}}(k,\tau)=\frac{3}{2\pi^2}\alpha_{\mathrm{f}}N_{\mathrm{e}}^2\sigma_{\mathrm{T}}c\frac{\exp(-k/\tau)}{k\tau[2\exp(1/\tau)K_2(1/\tau)]^2}\int_0^\infty ds\exp(-s)\int_{-\mathrm{T}}^{\mathrm{T}}dt\,Q^{1/2}\int_{\mu_1}^{\mu_2}d\mu$$

$$\times\int_{\Lambda_1}^{\Lambda_2}d\eta\frac{[1+(\sinh^2\eta)/\epsilon_1^2Q^2]^{1/2}\sin\theta}{\cosh^2\eta[2(\mu+1-\epsilon)]^{1/2}(C+D)^{1/2}}\int_0^\Lambda d\xi\frac{[1+\sinh^2\xi/\sinh^2\Lambda]}{\cosh^3\xi}x_1^2\Sigma \tag{A11}$$

where

$$Q=1+(1-1/\epsilon_1^2)^{1/2};\qquad \sinh\Lambda_1=\epsilon_1Q\tan(\xi_1/2); \tag{A12}$$

$$\sinh\Lambda_2=\epsilon_1Q\tan\left(\frac{\theta_{\max}-\xi_1}{2}\right);\qquad \sinh\Lambda=\left[\frac{1-(1-1/\epsilon_1^2)^{1/2}\cos(\theta+\xi_1)}{1-(1-1/\epsilon_1^2)^{1/2}\cos(\theta-\xi_1)}\right]^{1/2};$$

$$C+D=1-(1-1/\epsilon_1^2)^{1/2}\cos(\theta+\xi_1)$$

and

$$\tan\xi_1=\{2(\mu-1)[\epsilon^2-2(\mu+1)]-2(\mu+1)t^2\}^{1/2}/\{\epsilon^2-2(\mu+1)+\epsilon t\}.$$

All five integrals in equation (A11) can be simply evaluated using Gaussian quadrature. Great care must be taken in evaluating Σ, however, due to rounding error problems (even with 14 significant figures). In particular the terms

$$\frac{L_2}{W_2}\frac{\rho^2-2}{8xx_1}(w^2+\rho^2-4)^2\quad\text{and}\quad\frac{2\rho L_3}{w(w^2-4)^{1/2}x_1}\frac{w^2+\rho^2-4}{4xx_2}(w^2-2)^2 \tag{A13}$$

very nearly cancel and can be very large. This cancellation should be removed analytically.

Appendix 2: photon–photon rates for non-isotropic distribution functions

For a general photon distribution, $dN(\mathbf{k})=N_\gamma n(\mathbf{k})\omega^2d\omega d\mu d\phi$, the reaction rate is given by

$$R_{\gamma\gamma}=\frac{N_\gamma^2c}{2}\int\sigma(\omega)(1-\mu)n_1(\mathbf{k})n_2(\mathbf{k}_2)\omega_1^2d\omega_1d\mu_1d\phi_1\omega_2^2d\omega_2d\mu_2d\phi_2 \tag{B1}$$

where $\sigma(\omega)$ is the cross-section in terms of ω, the centre of momentum energy, and $\mu=\cos\theta$ is the cosine of the angle between the photon directions. Changing variables from ϕ_2 to $\phi=\phi_2-\phi_1$ we have (by geometry)

$$\cos\phi=\frac{\mu-\cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2};\qquad d\phi=\frac{-d\mu}{\sin\theta_1\sin\theta_2\sin\theta}. \tag{B2}$$

The limits of the μ integration are given by $|\cos\phi|\leqslant1$, hence

$$\cos(\theta_1+\theta_2)\leqslant\mu\leqslant\cos(\theta_1-\theta_2). \tag{B3}$$

If the distribution functions are independent of polar angles ϕ_1 and ϕ_2 then the ϕ_1 integration is trivial, and

$$R_{\gamma\gamma} = N_\gamma^2 c\pi \iint_{\mu=\cos(\theta_1+\theta_2)}^{\mu=\cos(\theta_1-\theta_2)} \sigma(\omega)(1-\mu) n_1(\omega_1,\mu_1) n_2(\omega_2,\mu_2) \times \omega_1^2 d\omega_1 d\mu_1 \omega_2^2 d\omega_2 d\mu_2 \frac{d\mu}{\sin\theta_1 \sin\theta_2 \sin\phi}. \quad \text{(B4)}$$

Now change variables from μ to ω, using $2\omega^2 = \omega_1\omega_2(1-\mu)$. The integration limits become $\omega_\pm^2 = \omega_1\omega_2[1-\cos(\theta_1 \pm \theta_2)]/2$. Then $\sin\theta_1 \sin\theta_2 \sin\phi = 2[(\omega_+^2-\omega^2)(\omega^2-\omega_-^2)]^{1/2}/\omega_1\omega_2$ and so

$$R_{\gamma\gamma} = N_\gamma^2 c4\pi \int n_1(\omega_1,\mu_1) n_2(\omega_2,\mu_2)\omega_1\omega_2 d\omega_1 d\omega_2 d\mu_1 d\mu_2 \times \int_{\omega_-}^{\omega_+} \frac{\omega^3 \sigma(\omega) d\omega}{[(\omega_+^2-\omega^2)(\omega^2-\omega_-^2)]^{1/2}}. \quad \text{(B5)}$$

The major contribution to the ω integral comes from the limits, which behave like $1/|\omega-\omega_\pm|^{1/2}$. This contribution can be subtracted off analytically, and the remaining small portion can then be evaluated numerically.