Counting preimages of homogeneous configurations in 1-dimensional cellular automata

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A cellular automaton (CA) is in a *homogeneous configuration* if every cell has the same state. The *preimages* of a configuration s are those configurations which evolve to s within a single time step.

We present two methods of finding the total number of preimages for a given homogeneous configuration. The first is more intuitive, and gives a clear picture of how the number of preimages varies with the number of cells on which the CA operates, but it is only workable for elementary CAs (1-dimensional binary state CAs with neighbourhood size 3). The second method, based on de Bruijn matrices, is more abstract, but more readily extends to general 1-dimensional CAs.

Key words: Homogeneity, preimages, de Bruijn graphs, de Bruijn matrices, spectral graph theory

1 INTRODUCTION

A *cellular automaton (CA)* is a dynamical system, discrete and uniform in space and time. A CA is composed of a number of sites (*cells*), each of which is assigned one of a finite set of *states*. An assignment of states to cells is a *configuration* of the CA, and is analogous to what is usually called the "state" of a dynamical system. The dynamics of the CA is defined by the *local update rule*, a function mapping the "current" state of a cell and its neighbours to the

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"next" state of that cell. The local update rule extends to the *global map*, a function mapping the "current" configuration to the "next" configuration, in the natural way.

In general, the global map is not bijective, so strictly speaking it has no inverse. However, it is often useful to find the *preimages* of a given configuration; that is, those configurations which the global map sends to the given configuration. Intuitively, if the configuration at time t is known, then the preimages of that configuration are the possible configurations at time $t-1$. Wuensche and Lesser [13] give an efficient algorithm for finding preimages.

One way of visualising the dynamics of a CA is by drawing a *transition graph* (or *transition diagram*). This is a directed graph whose vertices represent the configurations of the CA, and whose edges represent the transitions between configurations according to the global map. The in-degree of a vertex in the transition graph is precisely the number of preimages for the corresponding configuration, and so numbers of preimages are a measure of the "branchiness" of the graph.

In [7], we study symmetries (formally, *automorphisms* or self-isomorphisms) of transition graphs. By studying how the number of symmetries varies with the number of cells, we obtain a partial classification of the elementary CAs (1-dimensional, binary state CAs with neighbourhood size 3).

In [8], we specialise our results to *linear CAs*: CAs whose local rules (and thus global maps) are linear functions. We derive an expression for the number of automorphisms of the transition graph for a linear CA, and one of the terms in this expression is the number of preimages for the homogeneous configuration of zeroes.

In this paper, we discuss how to find the numbers of preimages of homogeneous configurations. Even though linear CAs motivate us to carry out this work, the results are equally applicable to nonlinear CAs. After introducing the necessary definitions in Section 2, we describe two different approaches to counting preimages.

The first approach (Section 3) works by considering the possible lengths of sequences of consecutive cells in the same state. When the number of preimages is considered as a function of the number of cells, we see three distinct types of behaviour: constancy, periodicity, and exponential growth. The string length approach gives an intuitive understanding of when and why these behaviours arise.

The second approach (Section 4) applies results from spectral graph theory to the *de Bruijn graphs* of the CA. Specifically, we consider the eigenvalues of the adjacency matrices of the de Bruijn graphs (these adjacency matrices are known as *de Bruijn matrices*). While this method is not quite as intuitive as considering string lengths, it is far more useful for producing numerical results. In Section 4.3, we show that this approach can also be applied to configurations which are heterogeneous but periodic.

2 DEFINITIONS AND NOTATION

Consider a 1-dimensional CA with state set Σ, neighbourhood radius r and local rule $f : \Sigma^{2r+1} \to \Sigma$, operating on a lattice of N cells. In this paper we consider lattices with periodic boundary condition, where the leftmost cell is considered to be adjacent to the rightmost, so that the cells are indexed by the elements of $\mathbb{Z}_N$ (the integers modulo N). This is not the only choice of boundary condition, but the alternatives (treating the boundary cells differently to the rest, or allowing the lattice to be unbounded so that the effective number of cells is infinite) present obvious practical problems.

A configuration of this CA is a mapping from $\mathbb{Z}_N$ to Σ. We will write such a configuration as a string over Σ of length N, and write Σ^N for the set of all configurations. A homogeneous configuration in which each cell has state q is denoted q^N.

The local rule f extends to the global map $F : \Sigma^N \to \Sigma^N$. The *preimages* of a configuration u are those configurations v such that $F(v) = u$. The *successor* of u is the configuration $F(u)$. Note that CAs are deterministic (every configuration has precisely one successor), but not necessarily surjective or injective (a configuration may have zero, one, or many preimages). If a configuration has at least one preimage, it is said to be *reachable*; an unreachable configuration (one with no preimages) is often called a *Garden of Eden* configuration.

A *cyclic shift* is a transformation which shifts the entire configuration by a certain number of cells, respecting the periodic boundary condition of the lattice. For example, applying a cyclic shift to the right by one cell to the configuration

$$x_0 x_1 \ldots x_{N-2} x_{N-1} \tag{1}$$

yields the configuration

$$x_{N-1} x_0 \ldots x_{N-3} x_{N-2} \, . \tag{2}$$

Cyclic shifts are "symmetries" of any CA, in the sense that shifting a configuration cyclically and then applying the CA's global map to the result is equivalent to applying the global map first and shifting the result.

An *elementary CA (ECA)* has neighbourhood radius $r = 1$ and state set $\Sigma = \mathbb{Z}_2 = \{0, 1\}$. There are $2^{2^3} = 256$ possible local rules for an ECA. Following Wolfram [10], we assign each of these rules a number between 0 and 255 inclusive, by considering the string

$$f(1,1,1)f(1,1,0)\dots f(0,0,0) \tag{3}$$

as an 8-bit binary number and converting to decimal notation.

Consider two ECA rules to be "equivalent" if one can be obtained from the other by exchanging states 0 and 1, or by reflecting (reversing) the neighbourhood, or by performing one of these transformations after the other. Then the space of local rules is partitioned into 88 equivalence classes [12, 13]. From each of these classes we choose the rule whose number is smallest, thus obtaining the 88 *essentially different* ECA rules.

ECAs have the advantage that they can be studied exhaustively: compare the total number of ECA rules (256) to the number of $r = 2$, $\Sigma = \mathbb{Z}_2$ rules ($2^{2^5} \approx 4.3 \times 10^9$) or $r = 1$, $\Sigma = \mathbb{Z}_3$ rules ($3^{3^3} \approx 7.6 \times 10^{12}$). Applying the types of equivalence classifications described above does not significantly change the relative magnitudes of these numbers. ECAs also exhibit a wide range of dynamical behaviours: in particular, at least one (rule 110) is Turing complete [1].

3 STRING LENGTHS

In this section, we determine numbers of preimages by considering the possible lengths of sequences of consecutive cells in the same state.

3.1 For ECAs

Every heterogeneous configuration of an ECA is a cyclic shift of a configuration of the form

$$0^{l_{0,1}} 1^{l_{1,1}} \dots 0^{l_{0,k}} 1^{l_{1,k}} \ , \tag{4}$$

for some positive integers $k, l_{0,1}, \dots, l_{1,k}$. In other words, every heterogeneous configuration can be written, modulo cyclic shift, as strings of zeroes alternated with strings of ones. Since the state set has only two elements, heterogeneity is sufficient to ensure that the configuration contains at least one string of each state.

Consider the homogeneous configuration q^N. Let x be a state, and let $\overline{x} = 1 - x$ be the "other state", i.e. the member of the state set $\{0, 1\}$ which is not x. Assume that there exist preimages of q^N which contain the state

$\overline{x}$, and consider the permitted lengths $l_{x,i}$ of strings composed of x in such preimages. The assumption that the preimage contains $\overline{x}$ guarantees that these strings do not cover the entire lattice, so each string has a beginning and an end. By considering the action of the CA rule, we can write down necessary conditions for various string lengths in terms of the entries of f's rule table:

a. $l_{x,i} = 1$ can occur only if $f(\overline{x}, x, \overline{x}) = q$;

b. $l_{x,i} = 2$ can occur only if $f(\overline{x}, x, x) = f(x, x, \overline{x}) = q$;

c. $l_{x,i} \geq 3$ can occur only if $f(\overline{x}, x, x) = f(x, x, x) = f(x, x, \overline{x}) = q$.

Note that condition c implies condition b; that is, if $l_{x,i} \geq 3$ is permitted then $l_{x,i} = 2$ is also permitted. Also note that these conditions are not sufficient: for example, if condition a holds but $f(\overline{x}, \overline{x}, x) \neq q$ and $f(x, \overline{x}, x) \neq q$, then $l_{x,i} = 1$ cannot occur.

We have the following three cases:

1. If none of these conditions are met then our initial assumption, that there are preimages containing both states x and $\overline{x}$, is contradicted. Thus q^N has no heterogeneous preimages.

2. Suppose that, for each choice of x, there is precisely one possibility for $l_{x,i}$; say $l_{0,i} = l_0$ and $l_{1,i} = l_1$. This means that, for each x, either condition a or condition b (but not both, and not condition c) is met. Thus $l_0, l_1 \in \{1, 2\}$. Then a heterogeneous preimage of q^N, modulo cyclic shift, has the form

$$0^{l_0} 1^{l_1} \ldots 0^{l_0} 1^{l_1} \,. \tag{5}$$

   This defines a valid configuration if and only if N is divisible by $l_0 + l_1$. This configuration is unique up to cyclic shifts, and there are precisely $l_0 + l_1$ distinct configurations which are cyclic shifts of this configuration. Thus the number of heterogeneous preimages of q^N in this case is

$$\begin{cases} l_0 + l_1 & \text{if } l_0 + l_1 | N \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

   Note that, since $l_0, l_1 \in \{1, 2\}$, we have $l_0 + l_1 \in \{2, 3, 4\}$.

3. Suppose that there is more than one possibility for $l_{x,i}$, for either value of x (or indeed for both). Each time a string of x appears, there are several choices for its length. As N increases, so does the number of

strings of x and thus the number of choices. A “combinatorial explosion” takes place as N grows larger, and so the number of preimages grows exponentially with respect to N.

These three cases account for the heterogeneous preimages. In addition, there may be zero, one or two homogeneous preimages: specifically, x^N is a preimage of q^N if and only if $f(x, x, x) = q$. The number of homogeneous preimages is thus independent of N.

Note that if we have case 2 with $l_x = 2$ (so that condition b holds for this choice of x), then x^N cannot be a preimage, since $f(x, x, x) = q$ would imply condition c. Thus, in case 2 with $l_x = 2$ for one of the choices of x, there is at most one homogeneous preimage; if $l_x = 2$ for both choices of x, then there are no homogeneous preimages.

In summary, when the number of preimages of the homogeneous configuration q^N is considered as a function of N, there are three possible classes of behaviour:

1. The number of preimages is constant. Indeed, the preimages are themselves homogeneous, and their number can be determined by considering $f(0, 0, 0)$ and $f(1, 1, 1)$ as described above.

2. The number of preimages is periodic, with period 2, 3 or 4.

3. The number of preimages grows exponentially with respect to N.

Furthermore, these three possibilities can easily be distinguished by examining the rule table. The results of this for the 88 essentially different ECAs are presented in Table 1 (Appendix A).

Example 3.1. Period 4 behaviour is relatively rare among the ECAs. Indeed, we can show that only ECA rule 90 exhibits period 4 behaviour for preimages of the configuration 1^N.

For period 4 behaviour, the only permitted string length for both states must be 2. For strings of 0s of length 2 to be permitted, we must have

$$f(1, 0, 0) = f(0, 0, 1) = 1\,. \tag{7}$$

For strings of lengths other than 2 to be forbidden, we must have

$$f(1, 0, 1) = f(0, 0, 0) = 0\,. \tag{8}$$

A similar argument for strings of 1s shows that

$$f(1, 1, 0) = f(0, 1, 1) = 1 \tag{9}$$

and

$$f(0,1,0) = f(1,1,1) = 0\,. \tag{10}$$

But these conditions completely determine the ECA's rule table, and we find that the rule thus defined is rule 90.

By the same method, we can show that the only ECA exhibiting period 4 behaviour for preimages of 0^N is rule 165. This is the rule obtained from rule 90 by exchanging states 0 and 1. By the convention that each equivalence class is represented by its member with the smallest rule number, rule 165 is not among the 88 essentially different ECA rules. $\diamondsuit$

3.2 Beyond ECAs

The situation becomes considerably more complicated when we enlarge the CA's radius. For example, consider the binary state, radius 3 CA whose local rule f is defined by

$$f(0,1,0,1,0,1,0) = 0 \tag{11}$$

$$f(1,0,1,0,1,0,1) = 0 \tag{12}$$

$$f(0,0,0,1,1,1,0) = 0 \tag{13}$$

$$f(1,0,0,0,1,1,1) = 0 \tag{14}$$

$$\vdots$$

$$f(0,0,1,1,1,0,0) = 0 \tag{15}$$

$$f(x_{-3},\ldots,x_3) = 1 \quad \text{otherwise.} \tag{16}$$

Here the preimages of configuration 0^N are those configurations consisting of repetition of the string 01, *or* repetition of the string 000111. In terms of string lengths, there are two distinct cases: either both string lengths are 1, or both string lengths are 3. Thus the possible lengths of strings of 0s depend on the possible lengths of strings of 1s, and vice versa. This kind of interdependence never occurs among the ECAs.

The situation also becomes more complicated if we enlarge the CA's state set. Central to the results for ECAs is the fact that every heterogeneous configuration of a binary state CA can be written, modulo cyclic shift, as strings of 0s alternated with strings of 1s, as in Equation 4. A string of 0s must always be followed by a string of 1s, and vice versa. Furthermore, a heterogeneous configuration must contain, but not consist entirely of, a string of 0s. However, in a ternary state CA, a string of 0s may be followed by a string of 1s or a string of 2s, and indeed there is no guarantee that a heterogeneous configuration must contain a string of 0s at all.

4 DE BRUIJN MATRICES

The approach to counting preimages described in the previous section gives us an intuition for why the number of preimages is sometimes periodic with respect to N: if the possible string lengths are suitably constrained, then they can form a complete configuration only when N is a multiple of the appropriate value. However, the string length approach is somewhat laborious for producing numerical results, and becomes extremely complicated when applied to CAs beyond the ECAs.

McIntosh [6] describes a method of counting preimages for general CA configurations, by means of graphs (or more specifically, adjacency matrices of graphs) first introduced by de Bruijn [3]; see also [9]. Jeras and Dobnikar [5] describe the method more fully. What follows (up to, but not including, Theorem 4.2) is a summary of this.

The *de Bruijn diagram* for state x has vertex set Σ^{2r} (so the vertices are the strings of length $2r$ over Σ), and an edge from vertex $a_1 \dots a_{2r}$ to $b_1 \dots b_{2r}$ if and only if

$$a_2 \dots a_{2r} = b_1 \dots b_{2r-1} \tag{17}$$

and

$$f(a_1, a_2, \dots, a_{2r}, b_{2r}) = x \, . \tag{18}$$

In other words, there is an edge between two vertices if and only if their corresponding strings overlap to form a string of length $2r + 1$, and the local rule maps that string to state x. A *de Bruijn matrix* is the adjacency matrix for a de Bruijn diagram, with the vertices ordered lexicographically.

Example 4.1. The de Bruijn diagrams for an ECA have vertex set $\{00, 01, 10, 11\}$. By Equation 17, the edge set is a subset of

$$\begin{array}{rr} \{00 \to 00, & 00 \to 01 \\ 01 \to 10, & 01 \to 11 \\ 10 \to 00, & 10 \to 01 \\ 11 \to 10, & 11 \to 11\} \end{array} \tag{19}$$

or more succinctly,

$$\{ab \to bc \ : \ a, b, c \in \mathbb{Z}_2\} \ . \tag{20}$$

By Equation 18, the edge set of the de Bruijn diagram for state x is

$$\{ab \to bc \ : \ a, b, c \in \mathbb{Z}_2, f(a, b, c) = x\} \ . \tag{21}$$

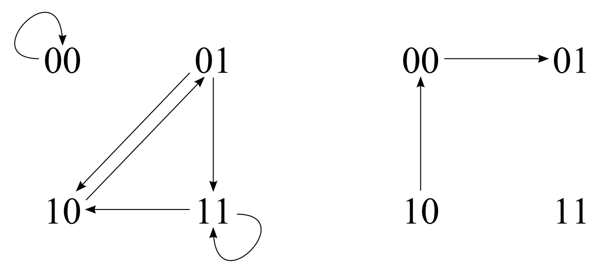

FIGURE 1
De Bruijn diagrams for ECA rule 18, for states 0 (left) and 1 (right).

ECA rule 18 is defined by the following table:

$$\begin{array}{c|cccccccc} abc & 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \hline f(a,b,c) & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \tag{22}$$

The de Bruijn diagrams for rule 18 are shown in Figure 1. The corresponding de Bruijn matrices are the adjacency matrices of these graphs, namely

$$D_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

$\Diamond$

De Bruijn matrices are defined in terms of the local rule, but they can be extended to describe entire configurations. In particular, the number of preimages of a given configuration is the *trace* (sum of diagonal entries) of the matrix obtained by multiplying together the de Bruijn matrices corresponding to the cell states. In other words, the configuration $x_0 \ldots x_{N-1}$ has

$$\mathrm{Tr}\left(D_{x_0} \ldots D_{x_{N-1}}\right) \tag{24}$$

preimages. A proof of this result appears in [5], but we can prove it easily for the special case of a homogeneous configuration.

The following is a well-known property of adjacency matrices: if A is the adjacency matrix of a graph, then the entry in row u, column v of the matrix A^n is the number of distinct paths of length n from vertex u to vertex v.

Let $v_1, \dots, v_N$ be a cycle in the de Bruijn diagram for state q. By definition of the de Bruijn diagram, we have

$$v_1 = a_1 a_2 \dots a_{2r} \tag{25}$$

$$v_2 = a_2 a_3 \dots a_{2r+1} \tag{26}$$

$$\vdots$$

$$v_N = a_N a_1 \dots a_{2r-1} \tag{27}$$

and

$$f(a_{i-2r}, \dots, a_{i+2r}) = q \tag{28}$$

for all i, where the arithmetic in the subscripts takes place modulo N. Thus the configuration $a_1 \dots a_N$ is mapped to the homogeneous configuration q^N by the ECA's global map. In other words, there is a one-one correspondence between cycles of length N in the de Bruijn diagram for state q, and preimages of the homogeneous configuration q^N.

Let D_q be the de Bruijn matrix for state q. The number of cycles of length N which begin and end at vertex u is given by the entry in row u, column u of D_q^N; the total number of cycles is the sum of these diagonal entries, i.e. $\operatorname{Tr} D_q^N$.

Results from spectral graph theory [2] can now be applied to the de Bruijn graphs:

Theorem 4.2. *Let q be a CA state, and let D_q be the corresponding de Bruijn matrix. Suppose that the eigenvalues of D_q are $\lambda_1, \dots, \lambda_k$. Then the homogeneous configuration q^N has exactly*

$$\lambda_1^N + \dots + \lambda_k^N \tag{29}$$

preimages.

The proof of this result uses the following lemmas:

Lemma 4.3. *Suppose that matrices A and B share the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, with corresponding eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_n$ respectively. Then AB has eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, with corresponding eigenvalues $a_1 b_1, \dots, a_n b_n$.*

Proof. Follows immediately from the definition of eigenvalues and eigenvectors. □

Lemma 4.4. *Let A be a square matrix with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and corresponding eigenvalues $a_1, \dots, a_n$. Then, for any positive integer k, the matrix A^k has eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ with corresponding eigenvalues $a_1^k, \dots, a_n^k$.*

Proof. By induction on k, using Lemma 4.3. $\square$

Proof of Theorem 4.2. We have already established that q^N has $\mathrm{Tr}\, D_q^N$ preimages.

It is a well-known result that the trace of a matrix is the sum of its eigenvalues [4, Chapter IV, Section 5]. Thus the number of preimages of q^N is the sum of the eigenvalues of the matrix D_q^N. By Lemma 4.4, the eigenvalues of D_q^N are the Nth powers of the eigenvalues of D_q, namely $\lambda_1^N, \dots, \lambda_k^N$. Therefore q^N has

$$\lambda_1^N + \cdots + \lambda_k^N \tag{30}$$

preimages. $\square$

This result means that, once the eigenvalues of D_q are known, finding the number of preimages of q^N for any value of N is as simple as raising the eigenvalues to the Nth power and summing the results.

4.1 For ECAs

The de Bruijn matrices for an ECA with local rule f are

$$D_1 = \begin{pmatrix} f(0,0,0) & f(0,0,1) & 0 & 0 \\ 0 & 0 & f(0,1,0) & f(0,1,1) \\ f(1,0,0) & f(1,0,1) & 0 & 0 \\ 0 & 0 & f(1,1,0) & f(1,1,1) \end{pmatrix} \tag{31}$$

and

$$D_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} - D_1\,. \tag{32}$$

We can apply Theorem 4.2 to find numbers of preimages for homogeneous configurations of ECAs.

Example 4.5. In Example 4.1, we gave the de Bruijn matrices D_0 and D_1 for ECA rule 18. The eigenvalues of D_1 are all zero, so 1^N has no preimages.

Now consider the homogeneous configuration 0^N. The eigenvalues of D_0 are

$$0, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\,, \tag{33}$$

and so there are

$$1 + \frac{1}{2^N}\left(\left(1+\sqrt{5}\right)^N + \left(1-\sqrt{5}\right)^N\right) \tag{34}$$

preimages.

These results are consistent with those given in Table 1: the number of preimages of 1^N is indeed constant (in fact it is zero); for large N, Equation 34 is dominated by

$$\left(\frac{1+\sqrt{5}}{2}\right)^N \approx 1.618^N \tag{35}$$

and so the number of preimages of 0^N grows exponentially with N. $\Diamond$

Recall that the eigenvalues of a matrix A are the solutions for λ in the equation

$$\det(A - \lambda I) = 0\,, \tag{36}$$

where I is the identity matrix. The expression $\det(A - \lambda I)$ is a polynomial in λ, called the *characteristic polynomial*. If the entries of A are real (as the entries of de Bruijn matrices always are), then the characteristic polynomial has real coefficients, and thus its roots (the eigenvalues of A) must be real or occur in complex conjugate pairs.

If two matrices have the same characteristic polynomial, they have the same eigenvalues. Among the de Bruijn matrices for all 88 essentially different ECAs, there are 23 distinct characteristic polynomials. These are enumerated in Table 2 (Appendix A), their roots are given in Table 3, and the correspondence between characteristic polynomials and ECA rules is given in Table 4. From these tables and Theorem 4.2 we can determine the number of preimages as a function of N.

Figure 2 plots the number of preimages of 1^N against N, for ECA rule 94 (corresponding to characteristic polynomial c_{10}). The overall trend is exponential, but some fluctuation is also apparent. What is the source of this fluctuation?

As N grows large, the expression

$$\lambda_1^N + \cdots + \lambda_k^N \tag{37}$$

is dominated by the terms corresponding to those λ_i whose modulus is maximal. More explicitly, let

$$|\lambda|_{\max} = \max\{|\lambda_i| \,:\, i = 1, \ldots, k\} \tag{38}$$

$$\Lambda = \{\lambda_i \,:\, i = 1, \ldots, k, |\lambda_i| = |\lambda|_{\max}\} \tag{39}$$

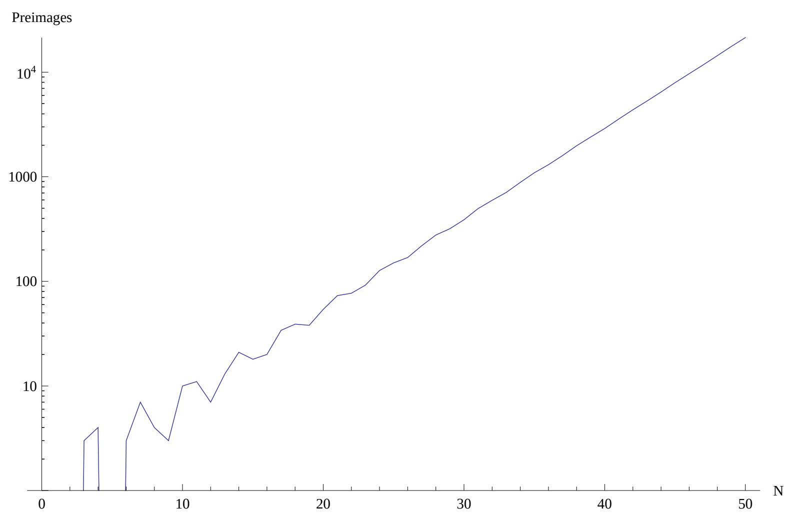

FIGURE 2
Plot of number of preimages in ECA rule 94 for the homogeneous configuration 1^N, against number of cells N. Note that the y-axis scale is logarithmic.

so that $|\lambda|_{\max}$ is the maximal modulus, and Λ is the set of λ_i on which this maximum is attained. Then, for large N,

$$\sum_{i=1}^{k} \lambda_i^N \approx \sum_{\lambda_i \in \Lambda} \lambda_i^N \,. \tag{40}$$

There are several possibilities:

1. If $\Lambda = \{x\}$ for some real $x > 1$, then the sum grows exponentially with N in the manner of x^N.

2. If $\Lambda = \{x\}$ for some real $0 < x < 1$, then the sum decays exponentially with N in the manner of x^N.

3. If $\Lambda = \{1\}$, then the sum approaches 1 as N grows large (the sum over Λ being 1 for all N).

4. If $\Lambda = \{0\}$, then all of the λ_i must be zero and so the sum is zero.

5. If $\Lambda = \{x\}$ for some real $x < 0$, the behaviour is analogous to the first three cases, but the sum oscillates between positive and negative. In other words, the behaviour for $\Lambda = \{x\}$ is the behaviour for $\Lambda = \{-x\}$ multiplied by $(-1)^N$.

6. If $\Lambda = \{z, \bar{z}\}$ for some conjugate pair of complex numbers $z, \bar{z}$, then the sum has the form

   $$2r^N \cos N\theta \,. \tag{41}$$

   This follows immediately from writing $z^N + \bar{z}^N$ in polar form, where r and θ are the modulus and argument of z.

   Ignoring the r^N term (which gives exponential growth or decay depending on whether r is greater or less than 1), this is almost periodic with respect to N; whether it is actually periodic depends on whether $\frac{2\pi}{\theta}$ (the ratio of the oscillation period to the "sampling" period corresponding to the integer values of N) is rational or irrational.

7. If Λ is a union of two or more of these possibilities, then the overall behaviour is the sum of the corresponding individual behaviours.

For our sum $\lambda_1^N + \cdots + \lambda_k^N$, we can only have cases 1 to 4. This is because $\lambda_1^N + \cdots + \lambda_k^N$ is a number of preimages, and so it cannot be negative.

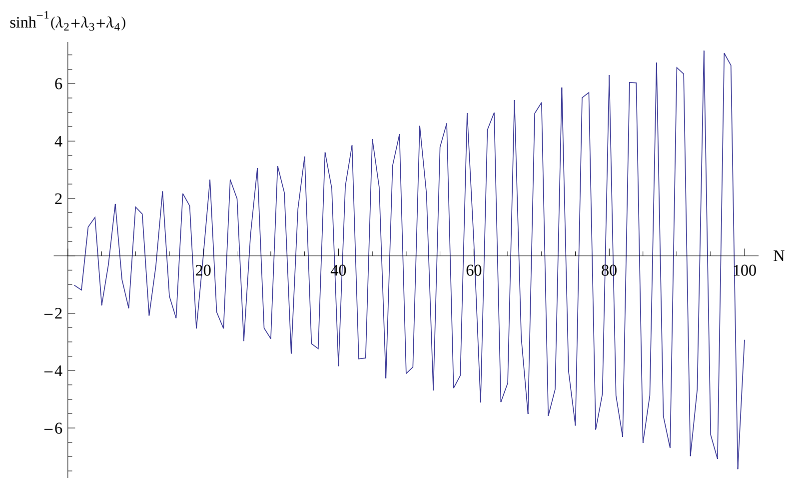

FIGURE 3
As Figure 2, but with the dominant exponential term subtracted. For this plot, we take the inverse hyperbolic sine of the data, to give the y-axis a "signed logarithmic" scale.

In each of the exponential cases (c_{10} to c_{23} inclusive) enumerated in Table 3 (Appendix A), we have case 1. Let λ_1 be the eigenvalue with largest magnitude, and omit it to consider the sum

$$\lambda_2^N + \cdots + \lambda_k^N \,. \tag{42}$$

An example of this is plotted in Figure 3. It is now apparent that the "fluctuations" noted in Figure 2 are in fact oscillations.

The sum without λ_1 can be negative, and so all of the cases enumerated above are possible. In many of the cases listed in Table 3, we have case 6: a conjugate pair of complex numbers dominate, so the overall behaviour is sinusoidal oscillation.

From Equation 41, the magnitude of the oscillation is $2r^N$, so the oscillation decays as N tends to infinity if and only if $r < 1$. This is the case for all but three of the cases in Table 2, the exceptions being c_{10}, c_{18} and c_{19}. Of these, the latter two have $r = 1$ (so the magnitude of the oscillation is constant with respect to N), so only c_{10} gives rise to oscillations whose magnitude grows with N. Thus, among the 88 essentially different ECAs, such growing oscillations only occur with ECA rules 94 and 122 for homogeneous configuration 1^N (they also occur for those rules, not among the 88 essentially different rules, that are equivalent to rule 94 or rule 122).

We can continue to examine the sums in this way until no more terms remain, thus decomposing the overall behaviour into a sum of exponential growths, decays, and sinusoidal oscillations.

4.2 Beyond ECAs

De Bruijn matrices are equally applicable to more general 1-D CAs. In general, for a k-state CA with neighbourhood radius r, the de Bruijn matrices are square matrices with k^{2r} rows and columns. Furthermore, Theorem 4.2 applies to general 1-D CAs, the matrix in question having k^{2r} eigenvalues. When the number of preimages is written as a sum of Nth powers of eigenvalues, the cases enumerated in the previous section still apply, so the qualitative types of behaviour are the same as for ECAs.

4.3 Preimages of heterogeneous periodic configurations

This method is applicable whenever the matrix product

$$D_{x_0} \dots D_{x_{N-1}} \tag{43}$$

can be expressed as a power of a matrix. For example, consider a spatially periodic configuration of the form

$$a_0 \dots a_{p-1} a_0 \dots a_{p-1} \dots a_0 \dots a_{p-1} \, . \tag{44}$$

The spatial period p must be a factor of the number of cells N, so N/p is an integer (in fact N/p is the number of repetitions of the sequence $a_0 \dots a_{p-1}$). The corresponding product of de Bruijn matrices is

$$\left(D_{a_0} \dots D_{a_{p-1}}\right)^{N/p} \, . \tag{45}$$

By a similar argument to the proof of Theorem 4.2, if the eigenvalues of $D_{a_0} \dots D_{a_{p-1}}$ are $\lambda_1, \dots, \lambda_k$, then the number of preimages is

$$\lambda_1^{N/p} + \dots + \lambda_k^{N/p} \, . \tag{46}$$

As before, the λ_i are either real or occur in complex conjugate pairs, so the types of qualitative behaviour exhibited by this expression as a function of N are the same as those described above for preimages of homogeneous configurations for ECAs.

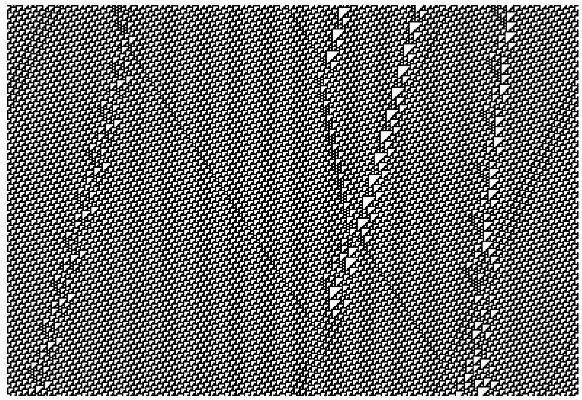

FIGURE 4
A typical space-time diagram for ECA rule 110. Each row of pixels represents a configuration of the CA, with the initial configuration at the top and successive configurations below it. Each column represents a cell. Pixels coloured white and black represent cell states 0 and 1 respectively.

Example 4.6. Figure 4 shows a space-time diagram typical of ECA rule 110. We observe a number of propagating structures ("gliders"), with the rest of the lattice filled by a simple repeating pattern (the "ether"). We may ask whether there is any configuration in which the gliders or other structures present are annihilated, leaving only ether. In other words, does an ether configuration have any preimages which are not themselves ether configurations?

Modulo cyclic shift, an ether configuration is composed of repetitions of the string $e = 00010011011111$. So an ether configuration can only exist when the number of cells N is a multiple of 14. Let

$$D_e = D_0^3 D_1 D_0^2 D_1^2 D_0 D_1^5 \,, \tag{47}$$

where D_0 and D_1 are the de Bruijn matrices for rule 110. Thus an ether configuration has

$$\mathrm{Tr}\, D_e^{N/14} \tag{48}$$

preimages ($N/14$ is the number of repeats of the ether pattern). Direct calculation shows that

$$D_e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 2 \end{pmatrix}, \tag{49}$$

and that the eigenvalues of D_e are $0, 0, 0, 2$. Therefore the number of preimages is $2^{N/14}$.

In the $N = 14$ case, the preimages of e are

$$p = 11110001001101 \tag{50}$$

and

$$q = 11110001110101\,. \tag{51}$$

Configuration p is simply e shifted cyclically by four cells to the right, and is thus also an ether configuration. Configuration q is not an ether configuration.

For larger N, the preimages are all possible sequences composed of repetitions of p and q. For example, for $N = 14 \times 3$, the eight preimages of eee are

$$ppp, ppq, pqp, pqq, qpp, qpq, qqp, qqq\,. \tag{52}$$

Precisely one of the preimages (namely $pp \ldots p$) is an ether configuration, leaving $2^{N/14} - 1$ non-ether preimages.

The de Bruijn matrices corresponding to p and q are

$$D_p = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D_q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{53}$$

These matrices do not have the same eigenvectors, so Lemma 4.3 does not apply. However, direct calculation shows that

$$D_p^2 = 2D_p \tag{54}$$

$$D_q D_p = D_p \tag{55}$$

$$D_p D_q = 2D_q \tag{56}$$

$$D_q^2 = D_q\,, \tag{57}$$

and so any product involving D_p and D_q can be reduced to a scalar multiple of D_p or D_q, whichever appears last in the product. Furthermore, $\operatorname{Tr} D_p = 2$ and $\operatorname{Tr} D_q = 1$, so the trace of a product involving D_p and D_q is 2^c, where c is the number of times D_p occurs. For example, for $N = 14 \times 3$:

1. $\mathrm{Tr}(D_p^3) = 2^3 = 8$, so ppp has eight preimages;
2. $\mathrm{Tr}(D_p^2 D_q) = 2^2 = 4$, so ppq has four preimages;
3. Similarly, pqp and qpp each have four preimages;
4. pqq, qpq and qqp each have two preimages;
5. qqq has one preimage.

The configuration q has one preimage, namely

$$s = 11011111011100\,. \tag{58}$$

Thus the preimages of a configuration composed of p and q are precisely those configurations obtained by replacing each q with s, and each p with the cyclic shift by four cells to the right of either p or q. These configurations are "pre-preimages" of an ether configuration, or configurations from which an ether configuration is reached after two time steps.

Configuration s has five preimages, so this kind of analysis becomes rather more difficult at this point. In terms of transition graphs, we have reached a particularly "branchy" vertex in the tree. Further investigation of these transients is a subject for future work. $\diamondsuit$

5 CONCLUSION

We have given two different methods of counting preimages of homogeneous configurations. The second method is more useful in terms of producing numerical data; the first gives more insight into the causes of the data's qualitative relationship with N, and into what the preimages actually are, but is rather cumbersome as a tool for calculation.

Both methods can explain why the number of preimages sometimes oscillates with respect to N. The first method shows that oscillations occur when the preimages have spatial periodicity, which can only occur when the number of cells is a multiple of the spatial period. The second method's explanation is less illuminating but more general: as a sum of powers of complex numbers occurring in conjugate pairs, the expression for the number of preimages often contains terms of the form $2r^N \cos N\theta$, and $\cos N\theta$ is periodic with respect to N (or "nearly periodic" if θ is not a rational multiple of π).

In studying the subset of initial configurations that eventually lead to a homogeneous configuration, we are effectively identifying the subspace of the

CA's global state space in which the qualitative behaviour is what Wolfram [11] calls *class 1* behaviour. Numbers of preimages for homogeneous configurations do not directly tell us the size of this region (as we do not count those initial configurations that yield homogeneous configurations after more than one time step), but they serve as an indication.

ACKNOWLEDGEMENT

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A TABLES OF RESULTS

TABLE 1: String lengths for ECAs (see Section 3.1)

Rule	$q = 0$							$q = 1$						
	Permitted l_0			Permitted l_1			Behaviour	Permitted l_0			Permitted l_1			Behaviour
	1	2	≥ 3	1	2	≥ 3		1	2	≥ 3	1	2	≥ 3	
0	•	•	•	•	•	•	Exponential							Constant
1	•	•		•	•	•	Exponential							Constant
2	•			•	•	•	Exponential							Constant
3	•			•	•	•	Exponential							Constant
4	•	•	•		•	•	Exponential				•			Constant
5	•	•			•	•	Exponential				•			Constant
6	•				•	•	Exponential				•			Constant
7	•				•	•	Exponential				•			Constant
8	•	•	•	•			Exponential							Constant
9	•	•		•			Exponential							Constant
10	•			•			Period 2							Constant
11	•			•			Period 2							Constant
12	•	•	•				Constant				•			Constant
13	•	•					Constant				•			Constant
14	•						Constant				•			Constant
15	•						Constant				•			Constant

TABLE 1: String lengths for ECAs (continued)

Rule	$q = 0$							$q = 1$						
	Permitted l_0			Permitted l_1			Behaviour	Permitted l_0			Permitted l_1			Behaviour
	1	2	≥ 3	1	2	≥ 3		1	2	≥ 3	1	2	≥ 3	
18	•			•	•	•	Exponential		•					Constant
19	•			•	•	•	Exponential		•	•				Constant
22	•				•	•	Exponential		•		•			Period 3
23	•				•	•	Exponential		•	•	•			Exponential
24	•			•			Period 2							Constant
25	•			•			Period 2							Constant
26	•			•			Period 2		•					Constant
27	•			•			Period 2		•	•				Constant
28	•						Constant				•			Constant
29	•						Constant				•			Constant
30	•						Constant		•		•			Period 3
32		•	•	•	•	•	Exponential	•						Constant
33		•		•	•	•	Exponential	•						Constant
34				•	•	•	Constant	•						Constant
35				•	•	•	Constant	•						Constant
36		•	•		•	•	Exponential	•			•			Period 2
37		•			•	•	Exponential	•			•			Period 2
38					•	•	Constant	•			•			Period 2

TABLE 1: String lengths for ECAs (continued)

Rule	$q=0$ Permitted l_0 1	$q=0$ Permitted l_0 2	$q=0$ Permitted l_0 ≥ 3	$q=0$ Permitted l_1 1	$q=0$ Permitted l_1 2	$q=0$ Permitted l_1 ≥ 3	$q=0$ Behaviour	$q=1$ Permitted l_0 1	$q=1$ Permitted l_0 2	$q=1$ Permitted l_0 ≥ 3	$q=1$ Permitted l_1 1	$q=1$ Permitted l_1 2	$q=1$ Permitted l_1 ≥ 3	$q=1$ Behaviour
40		•	•	•			Exponential	•						Constant
41		•		•			Period 3	•						Constant
42				•			Constant	•						Constant
43				•			Constant	•						Constant
44		•	•				Constant	•			•			Period 2
45		•					Constant	•			•			Period 2
46							Constant	•			•			Period 2
50				•	•	•	Constant	•	•					Constant
51				•	•	•	Constant	•	•	•				Constant
54					•	•	Constant	•	•		•			Exponential
56				•			Constant	•						Constant
57				•			Constant	•						Constant
58				•			Constant	•	•					Constant
60							Constant	•			•			Period 2
62							Constant	•	•		•			Exponential
72	•	•	•	•			Exponential					•		Constant
73	•	•		•			Exponential					•		Constant
74	•			•			Period 2					•		Constant

TABLE 1: String lengths for ECAs (continued)

Rule	$q=0$							$q=1$						
	Permitted l_0			Permitted l_1			Behaviour	Permitted l_0			Permitted l_1			Behaviour
	1	2	≥ 3	1	2	≥ 3		1	2	≥ 3	1	2	≥ 3	
76	•	•	•				Constant				•	•		Constant
77	•	•					Constant				•	•		Constant
78	•						Constant				•	•		Constant
90	•			•			Period 2		•			•		Period 4
94	•						Constant		•		•	•		Exponential
104		•	•	•			Exponential	•				•		Period 3
105		•		•			Period 3	•				•		Period 3
106				•			Constant	•				•		Period 3
108		•	•				Constant	•			•	•		Exponential
110							Constant	•			•	•		Exponential
122				•			Constant	•	•			•		Exponential
126							Constant	•	•		•	•		Exponential
128	•	•	•	•	•		Exponential							Constant
130	•			•	•		Exponential							Constant
132	•	•	•		•		Exponential				•			Constant
134	•				•		Period 3				•			Constant
136	•	•	•	•			Exponential							Constant
138	•			•			Period 2							Constant

TABLE 1: String lengths for ECAs (continued)

Rule	$q=0$ Permitted l_0 1	2	≥ 3	Permitted l_1 1	2	≥ 3	Behaviour	$q=1$ Permitted l_0 1	2	≥ 3	Permitted l_1 1	2	≥ 3	Behaviour
140	•	•	•				Constant				•			Constant
142	•						Constant				•			Constant
146	•			•	•		Exponential		•					Constant
150	•				•		Period 3		•		•			Period 3
152	•			•			Period 2							Constant
154	•			•			Period 2		•					Constant
156	•						Constant				•			Constant
160		•	•	•	•		Exponential	•						Constant
162				•	•		Constant	•						Constant
164		•	•		•		Exponential	•			•			Period 2
168		•	•	•			Exponential	•						Constant
170				•			Constant	•						Constant
172		•	•				Constant	•			•			Period 2
178				•	•		Constant	•	•					Constant
184				•			Constant	•						Constant
200	•	•	•	•			Exponential					•	•	Constant
204	•	•	•				Constant				•	•	•	Constant
232		•	•	•			Exponential	•				•	•	Exponential

TABLE 2: Characteristic polynomials for ECAs. The polynomials are ordered by their roots of largest magnitude (see Table 3), and so that the polynomials are grouped by the classes of behaviour given in Table 3, but other than that the ordering (and thus the numbering) is arbitrary.

c_i	Polynomial
c_1	λ^4
c_2	$\lambda^4 - \lambda^3$
c_3	$\lambda^4 - 2\lambda^3 + \lambda^2$
c_4	$\lambda^4 - \lambda^2$
c_5	$\lambda^4 - \lambda^3 - \lambda^2 + \lambda$
c_6	$\lambda^4 - 2\lambda^3 + 2\lambda - 1$
c_7	$\lambda^4 - \lambda$
c_8	$\lambda^4 - \lambda^3 - \lambda + 1$
c_9	$\lambda^4 - 1$
c_{10}	$\lambda^4 - \lambda - 1$
c_{11}	$\lambda^4 - \lambda^2 - \lambda$
c_{12}	$\lambda^4 - \lambda^3 - \lambda^2 + 1$
c_{13}	$\lambda^4 - \lambda^3 - 1$
c_{14}	$\lambda^4 - \lambda^3 - \lambda$
c_{15}	$\lambda^4 - 2\lambda^3 + \lambda^2 - \lambda + 1$
c_{16}	$\lambda^4 - \lambda^3 - \lambda^2$
c_{17}	$\lambda^4 - 2\lambda^3 + \lambda$
c_{18}	$\lambda^4 - \lambda^3 - \lambda - 1$
c_{19}	$\lambda^4 - 2\lambda^3 + \lambda^2 - 1$
c_{20}	$\lambda^4 - \lambda^2 - 2\lambda - 1$
c_{21}	$\lambda^4 - 2\lambda^3 + \lambda^2 - \lambda$
c_{22}	$\lambda^4 - \lambda^3 - \lambda^2 - \lambda$
c_{23}	$\lambda^4 - 2\lambda^3$

TABLE 3: Roots of characteristic polynomials for ECAs

c_i	Cartesian form	Modulus	Argument$/\pi$	Behaviour
c_1	0	0	0	Constant
	0	0	0	
	0	0	0	
	0	0	0	
c_2	1	1	0	Constant
	0	0	0	
	0	0	0	
	0	0	0	
c_3	1	1	0	Constant
	1	1	0	
	0	0	0	
	0	0	0	
c_4	1	1	0	Period 2
	-1	1	1	
	0	0	0	
	0	0	0	
c_5	1	1	0	Period 2
	1	1	0	
	-1	1	1	
	0	0	0	
c_6	1	1	0	Period 2
	1	1	0	
	1	1	0	
	-1	1	1	
c_7	$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1	$\frac{2}{3}$	Period 3
	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	1	$-\frac{2}{3}$	
	1	1	0	
	0	0	0	
c_8	$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1	$\frac{2}{3}$	Period 3
	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	1	$-\frac{2}{3}$	
	1	1	0	
	1	1	0	

TABLE 3: Roots of characteristic polynomials for ECAs (continued)

c_i	Cartesian form	Modulus	Argument/π	Behaviour
c_9	1	1	0	Period 4
	i	1	$\frac{1}{2}$	
	$-i$	1	$-\frac{1}{2}$	
	-1	1	1	
c_{10}	1.22074	1.22074	0	Exponential
	$-0.24813 + 1.03398i$	1.06334	0.57497	
	$-0.24813 - 1.03398i$	1.06334	-0.57497	
	-0.72449	0.72449	1	
c_{11}	1.32472	1.32472	0	Exponential
	$-0.66236 - 0.56228i$	0.86884	-0.77596	
	$-0.66236 + 0.56228i$	0.86884	0.77596	
	0	0	0	
c_{12}	1.32472	1.32472	0	Exponential
	1	1	0	
	$-0.66236 - 0.56228i$	0.86884	-0.77596	
	$-0.66236 + 0.56228i$	0.86884	0.77596	
c_{13}	1.38028	1.38028	0	Exponential
	$0.21945 + 0.91447i$	0.94044	0.42503	
	$0.21945 - 0.91447i$	0.94044	-0.42503	
	-0.81917	0.81917	1	
c_{14}	1.46557	1.46557	0	Exponential
	$-0.23279 - 0.79255i$	0.82603	-0.59094	
	$-0.23279 + 0.79255i$	0.82603	0.59094	
	0	0	0	
c_{15}	1.46557	1.46557	0	Exponential
	1	1	0	
	$-0.23279 - 0.79255i$	0.82603	-0.59094	
	$-0.23279 + 0.79255i$	0.82603	0.59094	
c_{16}	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	Exponential
	$\frac{1-\sqrt{5}}{2}$	$\frac{\sqrt{5}-1}{2}$	1	
	0	0	0	
	0	0	0	

TABLE 3: Roots of characteristic polynomials for ECAs (continued)

c_i	Cartesian form	Modulus	Argument/π	Behaviour
c_{17}	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	Exponential
	1	1	0	
	$\frac{1-\sqrt{5}}{2}$	$\frac{\sqrt{5}-1}{2}$	1	
	0	0	0	
c_{18}	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	Exponential
	i	1	$\frac{1}{2}$	
	$-i$	1	$-\frac{1}{2}$	
	$\frac{1-\sqrt{5}}{2}$	$\frac{\sqrt{5}-1}{2}$	1	
c_{19}	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	Exponential
	$\frac{1}{2}-\frac{\sqrt{3}}{2}i$	1	$-\frac{1}{3}$	
	$\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1	$\frac{1}{3}$	
	$\frac{1-\sqrt{5}}{2}$	$\frac{\sqrt{5}-1}{2}$	1	
c_{20}	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	Exponential
	$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1	$\frac{2}{3}$	
	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	1	$-\frac{2}{3}$	
	$\frac{1-\sqrt{5}}{2}$	$\frac{\sqrt{5}-1}{2}$	1	
c_{21}	1.75488	1.75488	0	Exponential
	$0.12256-0.74486i$	0.75488	-0.44809	
	$0.12256+0.74486i$	0.75488	0.44809	
	0	0	0	
c_{22}	1.83929	1.83929	0	Exponential
	$-0.41964-0.60629i$	0.73735	-0.69272	
	$-0.41964+0.60629i$	0.73735	0.69272	
	0	0	0	
c_{23}	2	2	0	Exponential
	0	0	0	
	0	0	0	
	0	0	0	

TABLE 4: Characteristic polynomials of de Bruijn matrices for ECAs

Rule	char D_0	char D_1	Rule	char D_0	char D_1
0	c_{23}	c_1	56	c_3	c_1
1	c_{22}	c_2	57	c_2	c_2
2	c_{17}	c_1	58	c_3	c_1
3	c_{16}	c_2	60	c_3	c_4
4	c_{21}	c_1	62	c_3	c_{11}
5	c_{18}	c_2	72	c_{17}	c_1
6	c_{15}	c_1	73	c_{12}	c_2
7	c_{14}	c_2	74	c_6	c_1
8	c_{17}	c_1	76	c_3	c_1
9	c_{12}	c_2	77	c_2	c_2
10	c_6	c_1	78	c_3	c_1
11	c_5	c_2	90	c_6	c_9
12	c_3	c_1	94	c_3	c_{10}
13	c_2	c_2	104	c_{15}	c_7
14	c_3	c_1	105	c_8	c_8
15	c_2	c_2	106	c_3	c_7
18	c_{17}	c_1	108	c_3	c_{11}
19	c_{16}	c_2	110	c_3	c_{11}
22	c_{15}	c_7	122	c_3	c_{10}
23	c_{14}	c_{14}	126	c_3	c_{20}
24	c_6	c_1	128	c_{22}	c_2
25	c_5	c_2	130	c_{12}	c_2
26	c_6	c_1	132	c_{14}	c_2
27	c_5	c_2	134	c_8	c_2
28	c_3	c_1	136	c_{16}	c_2
29	c_2	c_2	138	c_5	c_2
30	c_3	c_7	140	c_2	c_2
32	c_{21}	c_1	142	c_2	c_2
33	c_{14}	c_2	146	c_{12}	c_2
34	c_3	c_1	150	c_8	c_8
35	c_2	c_2	152	c_5	c_2
36	c_{19}	c_4	154	c_5	c_2
37	c_{13}	c_5	156	c_2	c_2
38	c_3	c_4	160	c_{18}	c_2
40	c_{15}	c_1	162	c_2	c_2
41	c_8	c_2	164	c_{13}	c_5

TABLE 4: Characteristic polynomials of de Bruijn matrices for ECAs (continued)

Rule	char D_0	char D_1	Rule	char D_0	char D_1
42	c_3	c_1	168	c_{14}	c_2
43	c_2	c_2	170	c_2	c_2
44	c_3	c_4	172	c_2	c_5
45	c_2	c_5	178	c_2	c_2
46	c_3	c_4	184	c_2	c_2
50	c_3	c_1	200	c_{16}	c_2
51	c_2	c_2	204	c_2	c_2
54	c_3	c_{11}	232	c_{14}	c_{14}

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