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Boundary Properties and Construction Techniques in General Topology

by

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Abstract

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The aim of this thesis is twofold. First, we investigate spaces defined by asserting that their nowhere dense subsets have certain properties. Secondly, we develop some techniques for the construction of topological spaces.

We consider spaces where the nowhere dense sets are asserted to have some property $\mathcal{P}$, calling such spaces boundary-$\mathcal{P}$. We show that if there are no Lusin spaces then every compact boundary-metrizable space is metrizable. Boundary-separability is also studied and we show that if there are no L-spaces then every boundary-separable space is separable.

By adapting the absolute dimension function of Arhangel'skiĭ, we define the new concept of cohesion. We show that every compact cohesive and every Hausdorff, sequential cohesive space is scattered. However, we construct regular, crowded spaces of all finite cohesions though there are no regular spaces of transfinite cohesion. We consider too the preservation of cohesion under various mappings and under the formation of products.

Turning to construction, we consider the class of compact monotonically normal spaces. It is well-known that it contains the class of spaces which are the continuous images of compact ordered spaces but it is still open as to whether they are actually distinct classes. Using Watson's resolutions, we give a method for constructing monotonically normal spaces. Though this also preserves continuous images of arcs, we show that it is because of a powerful result of Cornette rather than any trivial observation.

We also examine more closely monotone normality in images of compact ordered spaces using the Collins-Roscoe structuring mechanism. From this, we extract a strong instance of the mechanism, linear chain (F), which is held by all images of ordered compacta and all proto-metrizable spaces and implies Junnila's concept of utter normality.

Elementary submodels are an important tool in the construction of topological spaces. We develop a general method for applying them in varying circumstances and illustrate it by constructing three examples: Balogh's Q-set space, Rudin's normal but not collectionwise Hausdorff space and Balogh's small Dowker space.

Contents

Chapter 1

Introduction

1.1 An outline of the thesis

This thesis divides into three main sections: the first consisting of Chapters 2 and 3 deals with spaces where the nowhere dense subsets are asserted to have certain properties; the second (Chapter 4 and 5) addresses some topics concerning compact monotonically normal spaces; the third (Chapter 6) develops the use of elementary submodels in topology.

Nowhere dense subsets are fundamental objects in topological spaces and their importance is evident in such powerful results as the Baire Category Theorem and the topological reformulation of Martin's Axiom, see [K_1] p.62. Despite this, there are few instances where classes of spaces have been defined by asserting that the nowhere dense subsets have certain properties.

The most prominent example of a theory which is founded on the properties of nowhere dense sets is dimension theory. Here, the small inductive dimension *ind* and the large inductive dimension *Ind* are defined as follows:

Definition 1.1.1 For a space X, $\mathrm{ind}X = -1 = \mathrm{Ind}X$ if and only if $X = \emptyset$. Then inductively, for $n \in \omega$, $\mathrm{ind}X \leqslant n$ if and only if X has a basis of open sets whose boundaries have small inductive dimension strictly less than n. Similarly, $\mathrm{Ind}X \leqslant n$ if and only if, for every closed subset C of X and open set U containing C, there is an open set V in X for which $C \subseteq V \subseteq U$ and the boundary of V has large inductive dimension strictly less then n.

Thus, these dimension functions can be regarded as asserting that certain closed nowhere dense subsets (the boundaries referred to in the definitions) fit into the given inductive hierarchy. However, even in spaces with well-defined dimension in both senses, there are still many nowhere dense subsets which are entirely unrestricted by this definition. Because of this, the inductive dimensions are rarely considered in terms of nowhere dense sets.

As a strengthening of these definitions, Arhangel'skiĭ [A_2] defined the notion of absolute dimension. This gives the same meaning to zero-dimensional as the small inductive dimension but the higher dimensions are modified so that, for $n \in \omega$, a space has absolute dimension at most n if every nowhere dense subset has absolute dimension strictly less than n. Absolute dimension was defined as a tool by which to study cleavability over the reals but only a few facts about it were used.

Possibly the most well-known example of a space all of whose nowhere dense sets have a given property is a Lusin set. This is an uncountable subset of the real line for which every nowhere dense subset is countable (see [M]). Lusin sets are easily constructed under the Continuum Hypothesis but it is also consistent, for instance under Martin's Axiom and the negation of the Continuum Hypothesis, that there are none. However, they are very useful, occurring in a number of different places in topology. Kunen

[K_2] has also extended the definition to give the more general concept of a Lusin space.

There is a much less well-known body of work which considers spaces all of whose nowhere dense subsets have a given covering property. This work was initiated by Katětov [Ka] who proved that T_1 spaces without isolated points in which every closed nowhere dense subset is compact are themselves compact. Mills and Wattel [MW] and Blair [Bl] have generalised this result to encompass a wide range of different covering properties including countable compactness, the Lindelöf property and pseudo-compactness.

Our aim in Chapter 2 is to study spaces whose closed nowhere dense subsets all have $\mathcal{P}$ for various topological properties $\mathcal{P}$. Such spaces are said to have the new property boundary-$\mathcal{P}$. We consider three main cases of $\mathcal{P}$ being metrizability, separability and scattered-ness, in each case giving conditions which determine when boundary-$\mathcal{P}$ spaces have $\mathcal{P}$. We obtain the rather surprising results that if there are no Lusin spaces then every compact boundary-metrizable space is metrizable and if there are no L-spaces then every boundary-separable space is separable.

In Chapter 3, we define a dimension-like function which we call cohesion. This is based on absolute dimension but, in order to avoid imposing atypical, good behaviour in the lower "dimensions", the base case of the definition is altered: a space has cohesion zero if and only if it is discrete. We examine many properties of cohesion, its effect on scattered, crowded and compact spaces applying some results from the previous chapter. The preservation of cohesion under various topological constructions is also considered.

Monotonically normal spaces have proven themselves to be an important class of spaces ever since their definition in 1973 [HLZ]. They include many of the major types of spaces such as metrizable spaces and ordered spaces. Yet, even with such diversity, they have a great deal of structure. For some of the more elegant results on monotone normality, see [HLZ], [G], [Mi] *inter alia.*

One of the most exciting problems concerning monotonically normal spaces comes from a quite unexpected direction, namely, from attempting to generalise the Hahn-Mazurkiewicz Theorem. The classical Hahn-Mazurkiewicz Theorem states that a Hausdorff space is the continuous image of the unit interval in the real line if and only if it is a non-empty, metrizable, locally connected continuum. As the unit interval is the unique metrizable arc, the natural generalisation would be to say that a space is the continuous image of an arc if and only if it is a locally connected continuum. However, this is not the case as is demonstrated by an example of Mardešić [Ma]. Actually, counter-examples are easily found on noting that continuous images of arcs must be monotonically normal. Thus, any locally connected continuum which is not monotonically normal, for example the product of the closed unit interval with the one point compactification of the long line, is a counter-example. This does not render Mardešić's example obsolete because it still has many other nice properties, such as arc-wise connectivity, which might conceivably have provided alternative characterisations of the continuous images of arcs. But this does provoke the question: is every locally connected, monotonically normal continuum the continuous image of an arc (arctic)? Or more generally, there is the famous question of Nikiel, is every monotonically normal compactum the continuous image of a compact LOTS (CICLOTS)?

Various classes of spaces have been shown to be arctic, see [Wa_1, N_2]. Nikiel, following on from the work of Treybig [Tr] and Ward [Wa_2], has provided a number of characterisations of arctic spaces [N_1]. These have been extremely useful in determining many of the properties of arctic spaces and also those of CICLOTS. For a good, brief summary of the main results in this area, see Section 6 of [MO]. However, the question of whether there is a compact monotonically normal space which is not a CICLOTS is still open.

There are two main difficulties in answering this problem. The first is that it is very difficult to construct monotonically normal compacta from which to obtain possible counter-examples to the problem. The second is that the characterisations of CICLOTS which have been obtained seem to bear little relation to the monotone normality structure of these spaces.

In Chapter 4, we will turn our attention to constructing monotonically normal spaces by applying Watson's recently developed theory of resolutions [W]. We give a new type of resolution which does preserve monotone normality when resolving over a locally connected, monotonically normal continuum.

In Chapter 5, we analyse CICLOTS in a new way using the Collins-Roscoe structuring mechanism. The structuring mechanism has much in common with monotone normality and properties derived from it, such as acyclic monotone normality [MRRC] and Borges normality [St_2]. Also, it plays a key rôle in many aspects of the study of generalised metric spaces. We will give a full discussion of the structuring mechanism and define a new and rather strong instance of it. This is possessed not only by CICLOTS, as required, but also by all proto-metrizable spaces. As well as this, it implies Junnila's newly formulated notion of utter normality.

Another recent development in the construction of topological spaces is the use of elementary submodels. There have been a number of results which have used them in an essential way for simplifying and elucidating otherwise convoluted proofs. However, as yet there is no standard technique for using elementary submodels, indeed it seems as if there are as many techniques as topologists who use them!

In Chapter 6, we have developed a method for utilising elementary submodels in a number of different circumstances. We illustrate the method by applying it in the construction of three important, yet varied, examples. The three examples are a Q-set space [B_1], a "small" normal but not collectionwise Hausdorff space [R_3] and a small Dowker space [B_2]. By way of an introduction to reflection techniques, we also give elementary submodel proofs of some of the basic tools which will be needed in the exposition of the examples. The work of this chapter has been done jointly with Chris Good and Will Pack and I am very grateful to them for allowing me to incorporate this work into my thesis.

1.2 Definitions, notation and elementary results

Any terms and notation not explained in this section may be found in [E] or [KV].
Throughout the thesis, all topological spaces are assumed to be T_1.

Some elementary topology

For a topological space X, τX will denote the topology on X. To avoid confusion, when two distinct spaces have the same underlying set the spaces will be denoted by different symbols. For the remainder of this chapter, X and Y are topological spaces.

For $A \subseteq X$, $\mathrm{int}_X A$ denotes the interior of A in X, $\overline{A}^X$ denotes the closure of A in X and the *boundary* of A in X, $\mathrm{bd}_X A$, is defined by $\mathrm{bd}_X A = \overline{A}^X \setminus \mathrm{int}_X A$. Where no ambiguity can arise, the X will be omitted from this notation. A is said to be *nowhere dense* in X if $\mathrm{int}\overline{A} = \emptyset$.

If a non-empty space X has a basis of clopen sets then X is *zero-dimensional*. More generally, for a topological property $\mathcal{P}$, X has *rim-$\mathcal{P}$* if it has a basis of sets whose boundaries have property $\mathcal{P}$.

For $x \in X$, x is an *isolated point* of X if $\{x\} \in \tau X$ otherwise it is an *accumulation point*. X is *discrete* if every point of X is isolated and X is *crowded* if it has no isolated points. This latter term was introduced by van Douwen [vD_3] and is preferred by the author over the more usual terms "dense-in-itself", which is quite clumsy to use, and "perfect", which has other topological meanings. In fact, we use *perfect* to mean that every closed set is a *G_δ-set*, that is, a countable intersection of open sets.

We make explicit a well-known property of isolated points:

Proposition 1.2.1 *If A is a subset of X and $a \in A$ is an isolated point of A then a is an isolated point of $\overline{A}$.*

Corollary 1.2.2 *If $A \subseteq X$ is crowded then $\overline{A}$ is also crowded.*

A *scattered space* is one in which every subspace has an isolated point of itself. Taking X^d to denote the set of accumulation points of X, define for each ordinal α:

$$X^{(0)} = X$$
$$X^{(\alpha+1)} = (X^{(\alpha)})^d$$
$$X^{(\alpha)} = \bigcap_{\beta\in\alpha} X^{(\beta)} \text{ for } \alpha \text{ a limit ordinal}$$

It is clear that X is scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. In this case, the *scattered length* of X, denoted $\mathrm{sl}(X)$, is the least α for which this holds.

Although nowhere dense sets are natural and familiar topological objects, there are few places in the literature which explicitly state their basic properties. We therefore set out some of the more elementary results which will be useful later on. The easier proofs are left to the reader.

Proposition 1.2.3 *If $A \subseteq Y \subseteq X$ and Y is nowhere dense in X then A and $\overline{A}^X$ are nowhere dense in X.*

Proposition 1.2.4 *If D is a discrete collection of points in a crowded space X then $\overline{D}$ is nowhere dense.*

Proof If, for some $U \in \tau X$, $U \subseteq \overline{D}$ and U is non-empty then there exists $d \in D \cap U$. D is discrete so $\{d\} \in \tau D$ and, by Proposition 1.2.1, $\{d\} \in \tau\overline{D}$. Thus, there exists $V \in \tau X$ such that $\{d\} = V \cap \overline{D}$. As $U \subseteq \overline{D}$, $\{d\} = V \cap U$ which is an open set in X. This contradicts the fact that X is crowded.

$\square$

Proposition 1.2.5 *If $\mathcal{U}$ is a disjoint collection of open subsets of a crowded space X and, for all $U \in \mathcal{U}$, x_U is some point in U then $\{x_U : U \in \mathcal{U}\}$ is nowhere dense.*

Proof $\mathcal{U}$ is a collection of open sets witnessing that $\{x_U : U \in \mathcal{U}\}$ is discrete. The result now follows from Propositions 1.2.4 and 1.2.3.

$\square$

Proposition 1.2.6 *If $\mathcal{U}$ is a maximal disjoint collection of non-empty open sets X then $X \setminus \bigcup\mathcal{U}$ is closed and nowhere dense.*

Proposition 1.2.7 *For a scattered space X with $sl(X) = \alpha$ for some ordinal α, if $\beta < \alpha$ then $X^{(\beta+1)}$ is nowhere dense in $X^{(\beta)}$.*

Proof It suffices to show that X^d is nowhere dense in X. X^d is closed in X as it is the complement of all the isolated points in X. If $U \in \tau X$ is non-empty, by the definition of scattered, there is a point $x \in U$ which is isolated in U. As an open subset of an open subset of X, $\{x\} \in \tau X$ and U contains an isolated point of X. Thus, no non-empty open set in X is a subset of X^d. That is, X^d is nowhere dense in X.

$\square$

Proposition 1.2.8 *If X is scattered and $Y \subseteq X$ is nowhere dense then $Y \subseteq X^d$.*

Proof No nowhere dense subset of X can contain any isolated points of X because isolated points are open in X.

$\square$

Mappings

A continuous mapping $f : X \to Y$ is said to be *irreducible* if f is surjective and for no closed subset A of X, $f|_A$ is surjective; *open* if for all $U \in \tau X$, $f(U) \in \tau Y$; *closed* if for all C closed in X, $f(C)$ is closed in Y; *perfect* if f is closed and $f^{-1}(y)$ is compact for all $y \in Y$.

For $A \subseteq X$, define the *small image* of A under f, denoted $f^*(A)$, to be $\{y \in Y : f^{-1}(y) \subseteq A\}$.

Closed and irreducible maps are not commonly used in topology but they have some useful properties. We reproduce here two results which will be important later. Both results are taken from [P].

Proposition 1.2.9 *For a surjection $f : X \to Y$*

1. *f is irreducible if and only if, for every non-empty open subset of X, $f^*(U)$ is non-empty*
2. *f is closed if and only if, for every open subsets of X, $f^*(U)$ is open*

Proof Note first that, for $U \subseteq X$, $f^*(U) = Y \setminus f(X \setminus U)$. Using this, both statements follow naturally.

f is irreducible if and only if, for every proper closed subset A of X, $f(A) \neq Y$ if and only if, for every non-empty open subset U of X, $f(X \setminus U) \neq Y$ if and only if, for every non-empty open subset U of X, $f^*(U)$ is non-empty.

f is closed if and only if, for every closed subset A of X, $f(A)$ is closed in Y if and only if, for every open subset U of X, $f(X \setminus U)$ is closed in Y if and only if, for every open subset U of X, $f^*(U)$ is open in Y. $\square$

Proposition 1.2.10 *If $f : X \to Y$ is perfect then there exists $A \subseteq X$ which is closed in X such that $f|_A : X \to Y$ is irreducible and perfect.*

Proof Take $\mathcal{U} = \{U \subseteq X : U$ is open in X and $f^*(U) = \emptyset\}$ and order it by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{U}$, take $V = \bigcup \mathcal{C}$. V is necessarily open. We wish to show that $V \in \mathcal{U}$.

If $y \in f^*(V)$ then $f^{-1}(y) \subseteq V$ and $\mathcal{C}$ is an open cover of $f^{-1}(y)$ in X. As f is perfect, $f^{-1}(y)$ is compact. Find a finite subcover for $f^{-1}(y)$ from $\mathcal{C}$, say $\{U_i : i = 1, \ldots, k\}$ for some $k \in \omega$. The U_i are linearly ordered by inclusion as $\mathcal{C}$ is a chain and so there is a largest one, say U_j for some $j \in \{1, \ldots, k\}$. But then $f^{-1}(y) \subseteq U_j$, that is $y \in f^*(U_j)$ which contradicts the fact that $U_j \in \mathcal{U}$.

Hence $V \in \mathcal{U}$ which means that every chain in $\mathcal{U}$ has an upper bound in $\mathcal{U}$ and by Zorn's Lemma, $\mathcal{U}$ has a maximal element, W say.

Take $A = X \setminus W$. A is closed from which it easily follows that $g = f|_A$ is perfect. If $y \notin g(A)$ then $f^{-1}(y) \cap A = \emptyset$ and $f^{-1}(y) \subseteq W$ which contradicts W being in $\mathcal{U}$. Thus g is surjective.
Suppose C is a proper closed subset of A. Since $X \setminus A \subseteq X \setminus C$, $X \setminus C$ is an open set strictly containing W so $X \setminus C \notin \mathcal{U}$. Hence $f^*(X \setminus C) \neq \emptyset$ and there exists $y \in Y$ such that $f^{-1}(y) \subseteq X \setminus C$. Hence $y \notin g(C)$ and $g|_C$ is not surjective. Therefore g is irreducible. $\square$

Set-theoretic notation

We shall always work in ZFC, that is, the Zermelo-Fraenkel axioms with the Axiom of Choice, unless explicitly stated otherwise. The standard, that is ZFC, set-theoretic universe is denoted by V. CH is the Continuum Hypothesis, MA is Martin's Axiom.

As usual, $\mathbb{R}$ denotes the real line, $\mathbb{Q}$ the rationals and I the closed unit interval in $\mathbb{R}$.

Cardinals are identified with initial ordinals, ω denoting the first infinite cardinal and the set of natural numbers, ω_1 is the first uncountable ordinal, $\mathfrak{c}$ the cardinality of the continuum.

For $f, g \in \omega^\omega$, $f < g$ means that $f(m) < g(m)$ for all $m \in \omega$, $f <_n g$ means that $f(m) < g(m)$ for all $m \in \omega \setminus n$ and $f <^* g$ means $f <_n g$ for some $n \in \omega$. $\mathfrak{b}$ denotes the least cardinality of a subset of ω^ω which is unbounded in $(\omega^\omega, <^*)$. $\mathfrak{b}$ is an uncountable regular cardinal between ω_1 and $\mathfrak{c}$ and, regardless of the value of $\mathfrak{c}$, these are the only restrictions on the value of $\mathfrak{b}$. For more details on $\mathfrak{b}$ see [vD$_2$].

To avoid confusion with intervals in lines, ordered pairs and n-tuples will be denoted by angle braces, for example, $\langle x, y \rangle$.

For a set A, $|A|$ denotes the cardinality of A, $\mathcal{P}(A)$ the power set of A. For a cardinal κ, $[A]^\kappa$ is the set of subsets of A of size κ and $[A]^{<\kappa}$ is the set of subsets of A of size strictly less than κ.

Cardinal functions

The *weight* of X, denoted $w(X)$, is the least cardinality of a basis for τX. If $w(X) = \omega$ then X is *second countable.* The *(pseudo-)character* of a point x of X, denoted $\chi(x, X)$ ($\psi(x, X)$), is the least cardinality of a local (pseudo-)basis for x in X. The (pseudo-)character of X, denoted $\chi(X)$ ($\psi(X)$), is the supremum of the (pseudo-)characters of all points in X. If $\chi(X) = \omega$ then X is *first countable.* The *density* of X, denoted $d(X)$, is the least cardinality of a dense subset of X. If $d(X) = \omega$ then X is *separable.* The *Lindelöf degree* of X, denoted $L(X)$, is the least upper bound on the minimum size of a subcover of any open cover of X. If $L(X) = \omega$ then X is simply *Lindelöf.* The *cellularity* of X, denoted $c(X)$, is the supremum of the cardinalities of families of disjoint open sets in X. If $c(X) = \omega$ then X satisfies the *countable chain condition* or, more simply, is ccc. The *spread* of X, denoted $s(X)$, is the supremum of the cardinalities of the discrete subsets of X.

If κ is a cardinal function on X then $h\kappa(X) = \sup\{\kappa(Y) : Y \subseteq X\}$ and $hcl\kappa(X) = \sup\{\kappa(Y) : Y$ is a closed subset of $X\}$. For a topological property $\mathcal{P}$, X is hereditarily $\mathcal{P}$ if every subset of X is $\mathcal{P}$.

Compacta and continua

We shall assume that all compact spaces are Hausdorff and we shall use the term compactum interchangeably with compact space. βX denotes the Stone-Čech compactification of X. X is *locally compact* if it has a basis of open sets whose closures are compact.

A *continuum* is a connected compactum. A *locally connected* space is one with a basis of connected open sets. A point x of a connected space X is a *cut-point* if $X \setminus \{x\}$ is not connected. Given two points a, $b \in X$, a cut-point x *separates* a and b if $X \setminus \{x\}$ decomposes into two disjoint open sets, one of which contains a and the other b. A *cyclic element* of a connected space is a subset which is maximal with respect to the property of having no cut-point of itself. A cyclic element is *trivial* if it consists of only one point.

A *dendron* is a locally connected continuum all of whose cyclic elements are trivial. Equivalently, any two points of a dendron are separated by a third.

Cyclic elements are a powerful tool in the study of locally connected continua. They were originally defined by Whyburn [Wh] for metrizable continua but the theory has been more recently developed for use in all continua, see [C], [N$_1$]. They are a crucial concept in Nikiel's characterisation of continuous images of arcs [N$_1$].

Ordered spaces

Suppose $(X, <)$ is a linearly ordered set. For $a \in X$, $(a, \rightarrow)_X = \{x \in X : a < x\}$ and $(\leftarrow, a)_X = \{x \in X : x < a\}$. Other intervals in X are denoted using the usual conventions of round and square brackets. If there is a possibility of confusion as to which ordered set is meant, a subscript will be added as in the above notation.

X is a *linearly ordered topological space*, or LOTS, if $\{(\leftarrow, a) : a \in X\} \cup \{(a, \rightarrow) : a \in X\}$ is a sub-basis for τX. X is a *generalised ordered space*, or GO-space, if it has a basis of sets which are convex with respect to $<$. Alternatively, a GO-space is a subspace of a LOTS. An *arc* is a connected, compact LOTS. A *jump* in X is a pair $\langle x, y \rangle \in X^2$ such that $x < y$ and $(x, y) = \emptyset$. A *jump-point* is one half of a jump.

For two LOTS, $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$, the *lexicographic order* $\preccurlyeq$ on $X \times Y$ is defined by: for $\langle a, b \rangle$, $\langle x, y \rangle \in X \times Y$, $\langle a, b \rangle \preccurlyeq \langle x, y \rangle$ if $a <_X x$ or $a = x$ and $b \leqslant_Y y$.

We collect here a couple of elementary properties of ordered spaces which may be found in [E].

Proposition 1.2.11 *A separable LOTS is metrizable if and only if it has countably many jumps.*

Proposition 1.2.12 *A compact GO-space is a LOTS.*

Proposition 1.2.13 *Every subset of a compact LOTS has an infimum and supremum.*

A *CICLOTS* is the continuous image of a compact LOTS and an *arctic* space is the continuous image of an arc.

Monotone normality and generalised metric spaces

X is *monotonically normal* [HLZ] if there exists an operator $G : X \times \tau X \to \tau X$, such that:

1. If, for $x \in U \in \tau X$ and $y \in V \in \tau X$, $G(x, U) \cap G(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$

Such an operator is called a monotone normality operator. Note that it suffices to define G only on a basis of X.

X is *acyclic monotonically normal* [MRRC] if there exists an operator $H : X \times \tau X \to \tau X$, such that:

2. If $x \in U \in \tau X$ and $U \subseteq V \in \tau X$ then $H(x, U) \subseteq H(x, V)$
3. For all x,$y \in X$, $H(x, X \setminus \{y\}) \cap H(y, X \setminus \{x\}) = \emptyset$
4. For all $n \in \omega$, and all distinct $x_0, x_1, \ldots, x_{n-1} \in X$ with $x_n = x_0$, $\bigcap_{i=0}^{n-1} H(x_i, X \setminus \{x_{i+1}\}) = \emptyset$

Such an H is called an acyclic monotone normality operator. It is easy to see that, given an operator H on X satisfying (2) and (3), there is an operator G on X satisfying (1) and *vice versa*. Thus, all acyclic monotonically normal spaces are also monotonically normal. GO-spaces are acyclic monotonically normal [MR] and, as acyclic monotone normality is preserved by closed maps, so too are CICLOTS.

In the course of our study of monotonically normal compacta, two well-known classes of spaces emerge as natural to consider.

The first class is the non-Archimedean spaces. A space X is *non-Archimedean* if it has a rank-1 base. That is, there is a base, $\mathcal{B}$, for the topology on X such that if B, $B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$ then either $B \subseteq B'$ or $B' \subseteq B$. Nyikos proved the following:

Theorem 1.2.14 **[Ny]** *Every non-Archimedean space is a GO-space.*

In the same article, Nyikos defined the second class of spaces which we consider - the *proto-metrizable spaces*. The most well-known formulation of proto-metrizability is in terms of the *scattering process*: for a class $\mathcal{C}$ of spaces, transfinitely construct spaces, at successor stages, by isolating a subset of points and replacing these points with members of $\mathcal{C}$ and, at limit stages, by taking a subspace of the inverse limit of the construction so far. The class of spaces so defined is denoted $\mathcal{S}(\mathcal{C})$. A space is proto-metrizable if and only if it is in $\mathcal{S}(\text{METRIZABLE})$. Despite this peculiar definition, proto-metrizability has flourished not only because it is a good generalisation of metrizability sharing many of the key properties of metrizable spaces but also because of the wealth of different characterisations of proto-metrizable spaces. We give now a sample of these.

Recall that a *pair-base* $\mathcal{B} = \{B = (B_1, B_2) : B \in \mathcal{B}\}$ for X is a subset of $(\tau X)^2$ such that for all $B \in \mathcal{B}$, $B_1 \subseteq B_2$ and for all $x \in U \in \tau X$, there exists $B \in \mathcal{B}$ for which $x \in B_1 \subseteq B_2 \subseteq U$.

Theorem 1.2.15 *The following are equivalent:*

1. *X is proto-metrizable*
2. **[GZ]** *X has a rank-one pair-base, that is, a pair-base $\mathcal{B}$ such that if B, $B' \in \mathcal{B}$ and $B_1 \cap B'_1 \neq \emptyset$ then either $B_1 \subseteq B'_2$ or $B'_1 \subseteq B_2$*
3. **[F]** *X is the perfect image of a non-Archimedean space*
4. **[GM]** *X has a continuous monotone normality operator, that is, a monotone normality operator H such that if $x \in U \in \tau X$ then there exists $V \in \tau X$ which contains x and, for all $y \in V$, $V \subseteq H(y, U)$*
5. **[GM]** *X is monotonically paracompact, that is, if Σ is the set of open covers of X, then there exists an $m : \Sigma \to \Sigma$ such that, for all $\mathcal{U}$, $\mathcal{V} \in \pm$:*
   (a) *$m(\mathcal{U})$ star-refines $\mathcal{U}$*
   (b) *$m(\mathcal{U})$ refines $m(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$*

Some exotica

We now give some details of some of the more arcane topological objects which we will encounter in the course of this work.

A *Souslin line* is a non-separable, ccc LOTS. Souslin's Hypothesis (SH) is the assertion that there are no Souslin lines. It is well-known that SH is both independent and consistent with ZFC. More specifically, under $\diamondsuit$, there is a Souslin line whilst under MA + $\neg$CH, SH holds.
If there is a Souslin line, there is one which is also compact and connected - simply take the Dedekind completion of the given Souslin line.

A pair $\langle \mathcal{T}, \leqslant \rangle$ is a tree if $\mathcal{T}$ is a set partially ordered by $\leqslant$ in such a way that, for all $t \in \mathcal{T}$, $\{s \in \mathcal{T} : s \leqslant t\}$ is well-ordered by $\leqslant$. A *chain* in $\mathcal{T}$ is a linearly ordered subset of $\mathcal{T}$, a *branch* is a maximal chain and an *anti-chain* is a subset $\mathcal{S}$ such that, for all s, $t \in \mathcal{S}$, it is the case that neither $t \leqslant s$ nor $s \leqslant t$. The *height* of a tree, denoted $ht(\mathcal{T})$, is the supremum of the order-types of the sets $\{s \in \mathcal{T} : s \leqslant t\}$.
A *Souslin tree* is a tree of height ω_1 with no uncountable chains or anti-chains. There is a Souslin line if and only if there is a Souslin tree.

A *Lusin set* is an uncountable subset of $\mathbb{R}$ which meets every nowhere dense subset of $\mathbb{R}$ in at most countably many points. It is easily shown that Lusin sets can be constructed under CH. Kunen [K_2] generalised this notion to *Lusin spaces* which are uncountable regular spaces, having at most countably many isolated points and in which every nowhere dense subset is countable. He showed that under MA +

¬CH, there are no Lusin spaces. Also, every Souslin line contains a subset which is a Lusin space though it is consistent with ZFC that there is a Lusin space but no Souslin line.

In his thesis [vD$_1$], van Douwen defined a *nodec* space to be a crowded space whose nowhere dense subsets are all closed. A *maximal* topology on a set X is a crowded topology any refinement of which is not crowded. These are easy to find as given any crowded space X, take all crowded topologies refining τX, partially order it by inclusion and apply Zorn's Lemma to give a maximal topology. If X is Hausdorff then the maximal topology is also Hausdorff. Van Douwen has shown [vD$_3$] that all maximal topologies are nodec. Thus, given any crowded Hausdorff topology on a set there is a finer crowded Hausdorff topology which is nodec.

Finding a regular nodec space is not so straightforward. Once again, van Douwen provided a method for constructing these but, rather than using maximal topologies, he used *maximal regular* topologies. These are topologies which are maximal with respect to being crowded and regular. They were originally defined by Bourbaki as ultraspaces [Bo] and, just like maximal topologies, they can be found by applying Zorn's Lemma but this time to the lattice of regular crowded topologies refining the topology on any given regular crowded space.

Having obtained a countable maximal regular topology, say one refining $\tau\mathbb{Q}$, van Douwen gave a method for finding a dense subspace which is nodec. By Proposition 1.2.1, this is necessarily crowded and is clearly regular. In the case that the maximal regular space did refine $\mathbb{Q}$, take Θ to be the dense nodec subspace and X the subset of $\mathbb{Q}$ which underlies Θ. If $x \in X$ were isolated in X then $\{x\} \in \tau X \subseteq \Theta$. Hence X has no isolated points. As a crowded subspace of $\mathbb{Q}$, X is homoeomorphic to $\mathbb{Q}$. Thus, whenever we refer to *van Douwen's nodec space*, we mean a regular nodec space which refines $\mathbb{Q}$.

Apart from this, the salient properties of van Douwen's nodec space are that every nowhere dense subset of it is discrete and that it is not monotonically normal. I am very grateful to Ian Stares for his extremely useful exposition of van Douwen's construction [St$_1$].

Chapter 2

Boundary properties

The boundaries of open sets are fundamental objects in a topological space. However, very few classes of spaces have been studied solely for the properties of their boundaries - the most notable exceptions being dimension theory, Lusin sets and nodec spaces. In this chapter, we consider several topological properties $\mathcal{P}$ and consider those spaces in which the boundary of every non-empty open set is $\mathcal{P}$. We refer to this property as boundary-$\mathcal{P}$. We pay particular attention to the relationship between a space being $\mathcal{P}$ and it being boundary-$\mathcal{P}$.

The first section introduces the notion of boundary-$\mathcal{P}$ and summarises the work which has already been done where $\mathcal{P}$ is a covering property. In the second section, boundary-metrizable spaces are analysed and their metrizability is given in terms of the existence of Lusin spaces. In the third section, boundary-metrizability is considered in LOTS, not only for its intrinsic interest, but also as it provides a source of important examples. The exact relationship between boundary-separability and separability is examined in the fourth section. The next discusses boundary-scattered spaces where we obtain a result which will be useful later. Finally, we summarise the main results of the chapter and raise some relevant questions.

2.1 Boundary-$\mathcal{P}$ spaces

In order to be as flexible as possible, we make the following very general definition:

Definition 2.1.1 For a topological property $\mathcal{P}$, a space X is said to have the property *boundary-$\mathcal{P}$* if the boundary of every non-empty open set has property $\mathcal{P}$.

There are some immediate consequences of this definition which are worth noting.

Proposition 2.1.2 *For a space X:*

1. *X is boundary-$\mathcal{P}$ if and only if every closed nowhere dense subset of X is $\mathcal{P}$*
2. *If X is boundary-$\mathcal{P}$ then so too is every closed subspace of X*
3. *For a property $\mathcal{P}$ which is hereditary with respect to closed sets, if X is $\mathcal{P}$ it is also boundary-$\mathcal{P}$*
4. *If X is boundary-(hereditarily $\mathcal{P}$) then it is hereditarily boundary-$\mathcal{P}$*
5. *If X is boundary-$\mathcal{P}$ then it is rim-$\mathcal{P}$*

Proof (1) holds because a subset of X is nowhere dense and closed if and only if it is the boundary of some non-empty open set. (2) now follows since a closed nowhere dense subset of a closed subset of X is also a closed nowhere dense subset of X. If $\mathcal{P}$ is hereditary with respect to closed sets then every closed nowhere dense set is $\mathcal{P}$ hence (3).

To see (4), suppose that X is boundary-(hereditarily $\mathcal{P}$). If $Y \subseteq X$ and C is closed and nowhere dense in Y then $D = \overline{C}^X$ is closed and nowhere dense in X. Thus, D is hereditarily $\mathcal{P}$ which means that C, as a subset of D, is $\mathcal{P}$. That is, Y is boundary-$\mathcal{P}$.

(5) follows trivially from the definition of rim-$\mathcal{P}$.

□

Remark In the ensuing work, (1) is particularly useful as it provides a way of discussing boundaries without referring to specific open sets. Henceforth, it will normally be used without explicit reference.

Proposition 2.1.3 *For a space X and $Y \subseteq X$, if*

1. *$\mathcal{P}$ is a property which is hereditary with respect to closed sets*
2. *Y is $\mathcal{P}$*
3. *$X \setminus Y$ is a collection of isolated points of X*

then X is boundary-$\mathcal{P}$.

Proof Suppose C is a closed and nowhere dense subset of X. As C is nowhere dense, it contains no isolated points of X and must therefore be a subset of Y. Thus C is a closed subset of Y and C is $\mathcal{P}$. Hence X is boundary-$\mathcal{P}$ by Proposition 2.1.2 (1).

□

This gives an easy method for constructing a number of examples of spaces with certain boundary properties. Simply start with a space with the required property and throw in as many isolated points as required.

Examples 2.1.4 The Alexandroff duplicate is a compact boundary-metrizable space which is not metrizable.

Any space with all but one point isolated has every boundary property that the one point space has!

□

Thus, the real substance of boundary properties only emerges when the spaces considered have no (or few) isolated points. For this reason, in the remainder of the chapter, all spaces considered are crowded.

We are, in fact, already familiar with certain boundary properties though they may not have been viewed this way before. For instance, boundary-(non-empty) simply means connected and boundary-discrete is equivalent to nodec. Lusin spaces are those boundary-countable spaces which are regular, uncountable and have at most countably many isolated points.

Boundary-$\mathcal{P}$ for a covering property $\mathcal{P}$ was considered as early as 1947 by Katětov [Ka] who studied boundary-compactness albeit not by that name. He showed that (in crowded spaces) boundary-compactness was the same as compactness. This result was then generalised to $[\kappa, \lambda]$-compactness by Mills and Wattel [MW]. (A space X is said to be $[\kappa, \lambda]$-compact if every open cover of X with cardinality at most λ has a sub-cover of cardinality strictly less than κ. This is a natural generalisation of compactness which incorporates both countable compactness and the Lindelöf property as $[\omega, \omega]$-compactness and $[\omega_1, \infty]$-compactness respectively.) Blair [Bl] gave a much simpler proof of the Mills and Wattel result and also considered realcompactness and pseudocompactness. To summarise:

Theorem 2.1.5 *For a space X:*

1. [**Ka**] *If X is boundary-compact then X is compact*
2. [**MW**] *If X is boundary-($[\kappa, \lambda]$-compact) then X is $[\kappa, \lambda]$-compact*
3. [**Bl**] *If X is boundary-pseudocompact then X is pseudocompact*
4. [**Bl**] *If X is boundary-realcompact and every closed screenable subset has Ulam-non-measurable cardinality then X is realcompact*

We will use these results only in the following corollary:

Corollary 2.1.6 *If X is boundary-Lindelöf then it is Lindelöf.*

2.2 Boundary-metrizable spaces

Metrizable spaces have the most friendly properties of all topological spaces but boundary-metrizable spaces can exhibit some quite unruly behaviour even in the absence of isolated points!

Examples 2.2.1 The nodec space of van Douwen is a regular, countable space with no isolated points such that every nowhere dense set is discrete. However, it is far from being metrizable.

Take $p \in \beta\mathbb{R}$ which is a remote point, that is, for all $C \subseteq \mathbb{R}$ which are nowhere dense, $p \notin \overline{C}^{\beta\mathbb{R}}$. The space $\mathbb{R} \cup \{p\}$ is connected, boundary-metrizable and not metrizable.

□

Remark The second example was given by van Douwen in his review of [MW] in Mathematical Reviews, 82a:54045. However, finding remote points in $\mathbb{R}$ is a non-trivial task but fortunately (for our purposes) they do exist in ZFC. The details can be found in [Ha] on p.338.

The boundaries in both these examples are particularly well-behaved - they are countable and discrete in the former case and subsets of $\mathbb{R}$ in the latter. To achieve better behaviour, the boundaries must be further restricted to be compact as well. From Katětov's result, this simply means that we need to consider the class of compact boundary-metrizable spaces.

Proposition 2.2.2 *If X is a compact boundary-metrizable space then it is boundary-(hereditarily Lindelöf) and boundary-(hereditarily separable).*

Proof This follows immediately from the fact that compact metrizable spaces are hereditarily Lindelöf and hereditarily separable.

□

From Proposition 2.1.2 (4), X being boundary-(hereditarily Lindelöf) is actually telling us that X is hereditarily boundary-Lindelöf from which Corollary 2.1.6 gives:

Corollary 2.2.3 *If X is compact and boundary-metrizable then X is hereditarily Lindelöf.*

Therefore:

Corollary 2.2.4 *If X is compact and boundary-metrizable then X is perfect and first countable.*

Proof It is well-known (see [E], p.194) that hereditarily Lindelöf spaces are perfect. Since points are closed in X, each point has countable pseudo-character in X which together with compactness means that X is first countable.

$\square$

However, there is a much more direct proof of this which greatly illuminates the relationship between the boundary properties and the global properties of a space. First, we need a lemma.

Lemma 2.2.5 *If X is a boundary-ccc space then $s(X) = \omega$. In particular, if X is compact and boundary-metrizable then it satisfies the countable chain condition.*

Proof Suppose D is a discrete subset of X so that $\overline{D}$ is nowhere dense in X and hence ccc. By Proposition 1.2.1, $\{\{d\} : d \in D\}$ is a collection of isolated points in $\overline{D}$, in particular, a disjoint collection of non-empty open subsets of $\overline{D}$. Thus there are at most countably many sets in the collection. That is, D is countable.

The second part is a direct consequence of the fact that compact boundary-metrizable spaces are boundary-ccc and that $c(X) \leqslant s(X)$.

$\square$

Proof of Corollary 2.2.4 In order to show that X is perfect, consider first $C \subseteq X$ which is closed and nowhere dense. We now construct a maximal disjoint family of open sets whose closures do not intersect C.

Suppose that α is an ordinal and that for all $\beta < \alpha$, $U_\beta \in \tau X$ has been defined such that $\overline{U_\beta} \cap C = \emptyset$ and for $\beta' < \beta < \alpha$, $U_{\beta'} \cap U_\beta = \emptyset$. If $X \setminus \overline{\bigcup_{\beta<\alpha} U_\beta}$ is empty then stop and take λ to be α. $\bigcup_{\beta<\lambda} U_\beta$ is thus dense in X and $\{U_\beta : \beta < \lambda\}$ is the required maximal family.

If $X \setminus \overline{\bigcup_{\beta<\alpha} U_\beta}$ is non-empty, as it is also open and C is nowhere dense, there exists a point $x \in X \setminus (\overline{\bigcup_{\beta<\alpha} U_\beta} \cup C)$. By regularity of X, take U_α such that $x \in U_\alpha \subseteq \overline{U_\alpha} \subseteq X \setminus (\overline{\bigcup_{\beta<\alpha} U_\beta} \cup C)$. This means that $\{U_\beta : \beta < \alpha + 1\}$ is a collection of opens sets whose closures are disjoint and $U_\beta \cap C = \emptyset$ for all $\beta < \alpha + 1$. This completes the inductive construction.

Define $F = X \setminus \bigcup_{\beta<\lambda} U_\beta$ and $G = X \setminus \bigcup_{\beta<\lambda} \overline{U_\beta} = \bigcap_{\beta<\lambda}(X \setminus \overline{U_\beta})$. By Lemma 1.2.6, F is nowhere dense and hence metrizable. G is a G_δ-set in X since Lemma 2.2.5 implies that λ is countable. It is clear that $C \subseteq G \subseteq F$, so C is a closed subset of the metrizable G_δ-set G. Hence C is a G_δ-set in G and as a G_δ-set of a G_δ-set, C is G_δ-set in X.

Now suppose A is any closed subset of X. $A = \mathrm{int}A \cup \mathrm{bd}A$. $\mathrm{bd}A$ is closed and nowhere dense so is a G_δ-set in X. Take $\mathrm{bd}A = \bigcap_{n\in\omega} U_n$ for some $U_n \in \tau X$. Thus $A = \bigcap_{n\in\omega}(U_n \cup \mathrm{int}A)$ and A is a G_δ-set in X. That is, X is perfect.

$\square$

We are now in a position to consider when compact boundary-metrizable spaces are metrizable. The outcome is somewhat surprising.

Theorem 2.2.6 *If there are no Lusin spaces then every compact boundary-metrizable space is metrizable.*

This is proven by constructing in any compact boundary-metrizable space which is not metrizable a subspace which is a Lusin space. The exposition of the proof is simplified by:

Definition 2.2.7 For $Y \subseteq X$, $\mathcal{B} \subseteq \tau X$ *is a base for Y in X* if, whenever $y \in Y$ and $y \in U \in \tau X$, there exists $B \in \mathcal{B}$ such that $y \in B \subseteq U$. *Y is second countable in X,* if there is a countable base for Y in X.

Lemma 2.2.8 *If $C \subseteq D \subseteq X$ and D is second countable in X then C is second countable in X.*

Proof For all $c \in C$ and $U \in \tau X$ for which $c \in U$, since $c \in D$, there exists $B \in \mathcal{B}$ such that $c \in B \subseteq U$. Therefore $\mathcal{B}$ is a countable base for C in X.

$\square$

Lemma 2.2.9 *If $\mathcal{B}$ is a base for Y in X and $C \subseteq Y$ is second countable in X then there exists $\mathcal{B}' \in [\mathcal{B}]^\omega$ such that $\mathcal{B}'$ is a base for C in X.*

Proof Suppose $\mathcal{A}$ is a countable base for C in X. For a pair A_1 and $A_2 \in \mathcal{A}$, define $B(A_1, A_2)$ to be some element of $\mathcal{B}$ such that

$$A_1 \subseteq B(A_1, A_2) \subseteq A_2$$

whenever such an element exists, and to be X otherwise. Take $\mathcal{B}' = \{B(A_1, A_2) : A_1,\ A_2 \in \mathcal{A}\}$.

Consider $c \in C$, where $c \in U$ for some $U \in \tau X$. $\mathcal{A}$ is a base for C in X so there exists $A_2 \in \mathcal{A}$ such that $c \in A_2 \subseteq U$. But A_2 is open in X so there is a $B \in \mathcal{B}$ such that $c \in B \subseteq A_2$. And B is also open in X so there is an $A_1 \in \mathcal{A}$ for which $c \in A_1 \subseteq B$. Thus, there is an element of $\mathcal{B}$ sitting between A_1 and A_2 and hence, $B(A_1, A_2)$ is well-defined for A_1 and A_2 giving

$$c \in A_1 \subseteq B(A_1, A_2) \subseteq A_2 \subseteq U$$

More concisely, there exists $B \in \mathcal{B}'$ such that $c \in B \subseteq U$. Not only that, $\mathcal{B}'$ is countable as it is indexed by pairs from the countable set $\mathcal{A}$. Therefore $\mathcal{B}'$ is our required base.

$\square$

Lemma 2.2.10 *If X is a compact and perfect space and D is a closed metrizable subspace of X then D is second countable in X.*

Proof Take $\mathcal{B}$ to be a countable base for D, that is, $\mathcal{B} \subseteq \tau D$. D is closed so, for all $B \in \mathcal{B}$, $\overline{B}^D = \overline{B}^X$ (which means all closures may be taken in X) and, by perfectness of X, $\overline{B}$ is a G_δ-set in X. Therefore, there exists a sequence of sets open in X, $\{U_n(B)\}_{n\in\omega}$, for which

$$\overline{U_{n+1}(B)} \subseteq U_n(B) \text{ and } \overline{B} = \bigcap_{n\in\omega} \overline{U_n(B)}$$

Define $\mathcal{C} = \{U_n(B) : B \in \mathcal{B}, n \in \omega\}$. Clearly, $\mathcal{C} \in [\tau X]^\omega$. If $d \in D$ and $d \in U \in \tau X$ then there exist $V \in \tau X$ such that $d \in V \subseteq \overline{V} \subseteq U$ and $B \in \mathcal{B}$ such that $d \in B \subseteq V \cap D$. As $B \subseteq V$, $\overline{B} \subseteq \overline{V} \subseteq U$. Hence,

$$\bigcap_{n\in\omega} \overline{U_n(B)} \subseteq U$$

By compactness, for some $n \in \omega$, $\overline{U_{n+1}(B)} \subseteq U$ and then $d \in \overline{B} \subseteq \overline{U_{n+1}(B)} \subseteq U_n(B) \subseteq U$. This implies that $\mathcal{C}$ is a countable base for D in X.

$\square$

We now have all the machinery necessary to prove the theorem.

Proof of Theorem 2.2.6 Suppose X is a compact boundary-metrizable space which is not metrizable. A subset of X which is a Lusin space is constructed by an induction of length ω_1.

Assume that for a given $\alpha < \omega_1$ and for all $\beta < \alpha$, $Y_\beta \in [X]^\omega$ and $\mathcal{B}_\beta \in [\tau X]^\omega$ have been defined such that:

1. $\mathcal{B}_\beta$ is a countable base for Y_β in X

2. for all $\gamma < \beta$, $Y_\gamma \subseteq Y_\beta$, $\mathcal{B}_\gamma \subseteq \mathcal{B}_\beta$

3. if $x \in Y_\beta \setminus Y_\gamma$ then $\mathcal{B}_\gamma$ does not contain a local base for x in X

Take $Z = \bigcup_{\beta<\alpha} Y_\beta$, $\mathcal{C} = \bigcup_{\beta<\alpha} \mathcal{B}_\beta$. Z is countable and it is not hard to see that $\mathcal{C}$ is a countable base for Z in X. But X is not second countable, so there exists $x_\alpha \in X$ such that $\mathcal{C}$ does not contain a local base for x_α in X. It must be that $x_\alpha \notin Z$. Moreover, X is first countable by Corollary 2.2.4, so there is a countable local base, $\mathcal{B}(x_\alpha)$, for x_α in X. Define $Y_\alpha = Z \cup \{x_\alpha\}$ and $\mathcal{B}_\alpha = \mathcal{C} \cup \mathcal{B}(x_\alpha)$. By this definition, Y_α and $\mathcal{B}_\alpha$ must satisfy the inductive hypotheses.

Take $Y_{\omega_1} = \bigcup_{\alpha<\omega_1} Y_\alpha$ and $\mathcal{B}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$. From the construction, Y_{ω_1} is uncountable and $\mathcal{B}_{\omega_1}$ is a base for Y_{ω_1} in X. Y_{ω_1} will be the promised Lusin set and so it is necessary to show that every nowhere dense subset of Y_{ω_1} is countable and that Y_{ω_1} has at most countably many isolated points. The latter follows easily, though, from the fact that X has countable spread (Lemma 2.2.5) and the set of isolated points of a subset of X is a discrete set.

Consider $C \subseteq Y_{\omega_1}$ which is nowhere dense in Y_{ω_1}. $D = \overline{C}$ is nowhere dense in X hence metrizable and compact. By Lemma 2.2.10, since X is perfect, D is second countable in X. By Lemma 2.2.8, C is second countable in X. And by Lemma 2.2.9, there exists $\mathcal{B}' \in [\mathcal{B}_{\omega_1}]^\omega$ such that $\mathcal{B}'$ is a countable base for C in X. As $\mathcal{B}'$ is countable, there is some $\alpha < \omega_1$ such that $\mathcal{B}' \subseteq \mathcal{B}_\alpha$. But then, if C is uncountable, there exists $\beta > \alpha$ such that $x_\beta \in C$. However, $\mathcal{B}_\alpha$ does not contain a local base for x_β in X and thus neither does $\mathcal{B}'$. This is a contradiction and so C must be countable. That is, Y_{ω_1} is a Lusin space.

□

In order to provide an example of a boundary-metrizable compactum which is not metrizable, it is tempting simply to take a compact non-metrizable Lusin space. However, Corollary 2.5.5 rules out this option. Somewhat surprisingly though, boundary-metrizability in LOTS is much more tractable than in general and we can find some characterisations (and hence some examples) of boundary-metrizable, non-metrizable LOTS.

2.3 Boundary-metrizability in LOTS

Proposition 2.2.5 of the last section tells us that every boundary-separable space is ccc . It seems quite reasonable therefore that in LOTS the converse is true.

Proposition 2.3.1 *Every ccc LOTS is boundary-separable.*

Proof First, take X to be a compact LOTS which satisfies the countable chain condition. (At this point, it is worth noting that we could drop the assumption that there are no isolated points as the countable chain condition ensures that there are at most countably many and so they can easily be taken care of in the proof). Suppose C is a closed and nowhere dense subset of X.

$X \setminus C$ is open and can be divided into disjoint, maximally convex sets called the components of $X \setminus C$. If $\mathcal{U}$ denotes the family of components of $X \setminus C$ then, as X is ccc, $\mathcal{U}$ is countable and $\mathcal{U} = \{\mathcal{U}_\setminus : \setminus \in \omega\}$ say. Moreover, since X is compact, each component has a supremum and an infimum. This allows us to assert that each U_n can be written in the form (a_n, b_n) where a_n, $b_n \in C$ for all $n \in \omega$. Define $D = \{a_n : n \in \omega\} \cup \{b_n : n \in \omega\}$. D is a countable subset of C and will be shown to be dense in C as well.

Since X is compact so too is C and, as a compact subspace of a LOTS, C is also a LOTS. Thus its basic open neighbourhoods are $(s, t) \cap C$ where s, $t \in C$ and $s < t$. Consider such a non-empty basic open neighbourhood in C and take $x \in (s, t) \cap C$. As x is not an isolated point of X, either (s, x) or (x, t) is not empty. Without loss of generality, we may assume the former.

C is nowhere dense so $(s,x) \nsubseteq C$ and some component of $X \setminus C$, U_n say, must meet (s,x). Moreover, as it is convex, U_n must sit entirely inside (s,x). That is, there exists $n \in \omega$ such that $U_n = (a_n, b_n) \subseteq (s,x)$. So certainly $b_n \leqslant x$ and hence $(s,t) \cap D$ is non-empty. Therefore C is separable and X is boundary-separable.

For the more general case where X is not compact, take Y to be the Dedekind completion of X. Y is a compact LOTS which has a dense subspace satisfying the countable chain condition. It is easy to see that Y must also satisfy the countable chain condition and hence, by the previous case, be boundary-separable.

Suppose B is a closed nowhere dense subset of X. Define $C = \overline{B}^Y$. C is also nowhere dense in Y and so is separable. Moreover, C is a GO-space which means that, since C is separable, it must also be hereditarily separable. That is, B is separable and X is boundary-separable.

□

This gives an indication of where we can find our first example of a boundary-metrizable, non-metrizable compactum.

Example 2.3.2 A compact, connected Souslin line is a boundary-metrizable, non-metrizable continuum.

Proof Take X to be a compact, connected Souslin line. Thus X has the countable chain condition and is not metrizable. By the previous proposition, X is boundary-separable.

Consider C a closed, nowhere dense subset of X. C is therefore a separable, compact GO-space, hence a separable LOTS. Moreover, if $(a,b)_C$ is a jump in C, either $(a,b)_X$ is a jump or a component of $X \setminus C$. X is connected so has no jumps and has the countable chain condition so there are at most countably many components of $X \setminus C$. Thus, C has at most countably many jumps. By Proposition 1.2.11, C is metrizable and X is boundary-metrizable.

□

Of course, Souslin lines need not exist. In which case, since compact boundary-metrizable LOTS are then separable, we need only consider separable LOTS. These have a great deal of structure which significantly simplifies their study. To demonstrate this, first we need a definition:

Definition 2.3.3 For a LOTS X and $Y \subseteq X$, *the double arrow of X over Y*, denoted $DA(X;Y)$, is the LOTS formed by $Z = (X \times \{0\}) \cup (Y \times \{1\})$ with the lexicographic order. Take $\pi : DA(X;Y) \to X$ to be the natural projection map. π is clearly continuous.

Proposition 2.3.4 *If L is a separable LOTS then there exists $M \subseteq \mathbb{R}$, a LOTS without jumps, and $A \subseteq M$ such that $L = DA(M;A)$.*

Proof The proof of this is well-known and is an essentially straightforward technical exercise.

□

Having obtained such a nice characterisation, we can now express features of a separable LOTS in terms of its double arrow structure. For the rest of this section, the notation is as defined in Proposition 2.3.4.

Lemma 2.3.5 *$B \subseteq L$ is nowhere dense in L if and only if $\pi(B)$ is nowhere dense in M.*

Proof If $B \subseteq L$ is not nowhere dense then B contains some non-empty basic open interval of L in its closure. That is, there are m_1 and $m_2 \in M$ and $i,\ j \in \{0,1\}$ such that $\langle m_1, i\rangle <_L \langle m_2, j\rangle$ and $\emptyset \neq (\langle m_1, i\rangle, \langle m_2, j\rangle)_L \subseteq \overline{B}$. Take I to be the interval $(\langle m_1, i\rangle, \langle m_2, j\rangle)_L$. If $m_1 = m_2$ then it must be that $i = 0$ and $j = 1$ and then I is empty - a contradiction. Thus, $m_1 <_M m_2$ and it easily follows that $J = (\langle m_1, 1\rangle, \langle m_2, 0\rangle)_L \subseteq \overline{B}$. Now, if J were empty, since I is non-empty, it must be the case that either

$i \neq 1$ or $j \neq 0$ and either case would imply that L has an isolated point. However, we have the running assumption that this is false. Hence, J is non-empty and there exist $m_3 \in M$ and $k \in \{0, 1\}$ such that $\langle m_3, k \rangle \in J$.

Therefore, we have: $\pi(J) \subseteq \pi(\overline{B})$ which is a subset of $\overline{\pi(B)}$ by continuity of π. But $\pi(J) = (m_1, m_2)_M$ and $m_3 \in \pi(J)$ so $\pi(J)$ is a non-empty open subset of $\overline{\pi(B)}$ and $\pi(B)$ is somewhere dense.

Suppose now that $\pi(B)$ is somewhere dense for some $B \subseteq L$. Thus, $\pi(B)$ is dense in some interval $(m_1, m_2)_M$ where $m_1 <_M m_2$. We will show that B is dense in the interval $(\langle m_1, 1\rangle, \langle m_2, 0\rangle)_L$ from which it immediately follows that B is not nowhere dense.

Because L has no isolated points, any non-empty basic open interval in L contains a non-empty interval of the form $(\langle a, 1\rangle, \langle b, 0\rangle)_L$ where a, $b \in M$ and $a <_M b$. Consider such an interval contained inside $(\langle m_1, 1\rangle, \langle m_2, 0\rangle)_L$ so that $m_1 \leqslant_M a <_M b \leqslant_M m_2$. Since $\pi(B)$ is dense in $(m_1, m_2)_M$, there exists a $c \in \pi(B)$ such that $a <_M c <_M b$. Thus, $\langle c, i\rangle \in B$ for some $i \in \{0, 1\}$ and, by simply applying the definition of the lexicographic order, it is clear that $\langle c, i\rangle \in (\langle a, 1\rangle, \langle b, 0\rangle)_L$. This means that every basic open interval in $(\langle m_1, 1\rangle, \langle m_2, 0\rangle)_L$ picks up some $\langle c, i\rangle \in B$. That is, B is dense in $(\langle m_1, 1\rangle, \langle m_2, 0\rangle)_L$. □

Lemma 2.3.6 *A LOTS subspace C of L is metrizable if and only if $\pi(C) \cap A$ is countable.*

Proof From Proposition 1.2.11, $C \subseteq L$ is metrizable if and only if it has countably many jumps. Thus, C is metrizable if and only if $\{c \in C : c \text{ is an element of a jump }\}$ is countable. Since M has no jumps, $c \in C$ is an element of a jump in C if and only if $\pi(c) \in A$. This gives: $C \subseteq L$ is metrizable if and only if $\{c \in C : \pi(c) \in A\}$ is countable if and only if $C \cap \pi^{-1}(A)$ is countable if and only if $\pi(C) \cap A$ is countable (since π has finite fibres).

□

Given the previous lemmas, to construct a boundary-metrizable separable LOTS which is not metrizable requires that, first, in the double arrow construction, A must be uncountable to kill off metrizability of L. Secondly, <u>all</u> nowhere dense subsets of L must meet $\pi^{-1}(A)$ in only countably many points. This suggests that taking A to be a Lusin set would be the right place to look for such an example.

Theorem 2.3.7 *Every boundary-metrizable separable LOTS is metrizable if and only if there are no Lusin sets.*

Remark This theorem, in one direction at least, seems to be a consequence of Theorem 2.2.6. However, the hypothesis that there are no Lusin sets is not as sweeping as the hypothesis that there are no Lusin spaces.

Proof Suppose there is a Lusin set A in the closed unit interval. Define $X = DA(\overline{A}; A)$. X is clearly a compact LOTS. If D is a countable, dense subset of A then it is easily shown that $D \times \{0, 1\}$ is a countable dense subset of X. X is not metrizable as A is uncountable.

Take C to be a closed, nowhere dense subset of X. By Lemma 2.3.5, $\pi(C)$ is nowhere dense in $\overline{A}$ and closed as C is compact. Suppose for some $U \in \tau A$, $U \subseteq \pi(C) \cap A$. There exists $V \in \tau\overline{A}$ such that $U = V \cap A$ and since A is dense in $\overline{A}$, U is dense in V. Thus, $\overline{U} = \overline{V}$. As $\pi(C)$ is closed, $\overline{U} \subseteq \pi(C)$ which implies that $V \subseteq \pi(C)$. However, $\pi(C)$ is nowhere dense so V, and hence U, must be empty and $\pi(C) \cap A$ is nowhere dense in A. Because A is a Lusin set, $\pi(C) \cap A$ is countable and then Lemma 2.3.6 tells us that C is metrizable.

In summary, X is a compact, separable, boundary-metrizable LOTS which is not metrizable.

For the converse, assume that X is a boundary-metrizable, separable LOTS which is not metrizable. Then $X = DA(M; A)$ for some $M \subseteq \mathbb{R}$ and $A \subseteq M$. By Proposition 1.2.11, A must be uncountable as X is not metrizable and M is free from jumps. Consider $C \subseteq A$ which is nowhere dense in A. By Lemma

2.3.5, $\pi^{-1}(C)$ is nowhere dense in X because $\pi(\pi^{-1}(C)) = C$. Thus, $\pi(\pi^{-1}(C)) \cap A$ is countable since $\pi^{-1}(C)$ is metrizable and Lemma 2.3.6 holds. This implies that C is countable and A is a Lusin set.

□

Thus, the examples we required of boundary-metrizable non-metrizable compacta can all be found to be LOTS as well. Of course, our examples require set-theoretic hypotheses but we know from Theorem 2.2.6 that we can not eliminate this.

Remark Boundary-metrizability in LOTS has already been briefly considered before by M. E. Rudin in [R_2]. She asserted that:

($\Diamond$) For LOTS, X and Y where every nowhere dense subset is second countable, $X \times Y$ is ccc if and only if $X \times Y$ is hereditarily Lindelöf.

Phrased in another way, ($\Diamond$) simply says that if the product of two boundary-(second countable) LOTS is ccc then it is hereditarily Lindelöf. This however is not true in general as Pursich pointed out in [Pur]. He proved that statement ($\Diamond$) was equivalent to the non-existence of Lusin sets. In fact, his example showing that ($\Diamond$) is false is the same one given in Theorem 2.3.7.

Now, since the Dedekind completion of a boundary-(second countable) LOTS is a boundary-metrizable compactum (as in the proofs of Theorem 2.3.1 and Example 2.3.2), if there are no Lusin sets, Theorem 2.2.6 tells us that every boundary-(second countable) LOTS is metrizable and ($\Diamond$) holds. However, ($\Diamond$) holding does not imply that every boundary-metrizable compact LOTS is metrizable hence it cannot imply the non-existence of all Lusin spaces.

2.4 Comparing separability and boundary-separability

Boundary-separability appeared in the last section as a useful notion for analysing Example 2.3.2 and so we now turn our attention to that. The two key questions in relating boundary-separable spaces to separable spaces are: "When are separable spaces boundary-separable?" and "When are boundary-separable spaces separable?" The answer to the first question is straightforward and was given by Malykhin [Ml].

Proposition 2.4.1 *If X is boundary-separable then $d(X) = hcld(X)$.*

Proof Take D to be a subset of X which is dense in X and $|D| = d(X)$. For a closed subset Y of X, define $D_1 = D \cap \mathrm{int}Y$. Since Y is closed, $Y \setminus \mathrm{int}Y$ is nowhere dense and closed in X hence separable. Take D_2 to be a countable dense subset of $Y \setminus \mathrm{int}Y$. It is clear that $D_1 \cup D_2$ is dense in Y and has cardinality no greater than $d(X)$. Hence, $d(Y) \leqslant d(X)$ and $d(X) = hcld(X)$.

□

Corollary 2.4.2 *Any separable space is boundary-separable if and only if every closed subspace is separable.*

It is not possible to improve on this result as was also shown in [Ml] where, under CH, Malykhin produced a separable Lusin space which is not hereditarily separable. Another such example was given by Todorčević [T_2] under the weaker set-theoretic assumption that $\mathfrak{b} = \omega_1$ (recall the definition of $\mathfrak{b}$ from the introduction). We provide this example not only for the sake of completeness but also to provide some of the details which Todorčević omitted from his proof. As far as possible, the notation is the same as that used in Section 0 of [T_1] and Section 3 of [T_2] bar a few minor modifications in order to improve clarity. The proof makes use of an elementary submodel and so it may be useful to read Chapter 6 before going through the construction.

Theorem 2.4.3 *There is a completely regular space X such that $hd(X) = \mathfrak{b}$ but $d(F) < \mathfrak{b}$ for every closed subset F of X.*

Proof The proof will fall into three parts: the first is a definition of the space X and the proof that it is completely regular; the second shows that $hd(X) = \mathfrak{b}$; the last proves that for all closed subsets F of X, $d(F) < \mathfrak{b}$.

1. Take A to be an unbounded subset of monotone increasing functions in ω^ω which is well-ordered by $<^*$ in order type $\mathfrak{b}$. Such a set is shown to exist in [vD$_2$] Theorem 3.3. Define D to be the set of all those $d \in (\omega+1)^\omega$ which are monotone increasing and such that, for some $n \in \omega$, $d|_n \in \omega^n$ and for all $i \geqslant n$, $d(i) = \omega$. Z is taken to be $A \cup D$ with the topology inherited from $(\omega+1)^\omega$.

Now refine τZ by declaring $\{g \in Z : g \geqslant f\}$ to be open for all $f \in A$. Take X to be the same underlying set as Z with this new topology (in Todorčević's notation, $X = Z[A, \geqslant]$). Since τX is a refinement of the Hausdorff topology τZ, X is Hausdorff.

We will now show that X is zero-dimensional (has a basis of clopen sets) from which it follows that X is completely regular ([E] p.360).

Consider $X[\geqslant f] = \{g \in X : g \geqslant f\}$. If $g \notin X$ then there exists $n \in \omega$ for which $g(n) < f(n)$. Take $U_g = \{h \in Z : h(n) = g(n)\}$. U_g is a basic open set in Z which clearly contains g yet misses $X[\geqslant f]$. Thus $X[\geqslant f]$ is closed in Z. As τX refines τZ, $X[\geqslant f]$ must also be closed in X. Moreover, as it is declared open in X, $X[\geqslant f]$ is a clopen subset of X for all $f \in A$.

Z is zero-dimensional because it is a subset of the zero-dimensional space $(\omega+1)^\omega$. Hence we can find a basis, $\mathcal{B}$ say, consisting of clopen sets in Z. Moreover, we may assume that $\mathcal{B}$ is countable and consists of canonical basic open sets induced by the Tychonoff topology. If $B \in \mathcal{B}$, B must also be clopen in X. But note, $\{B \cap X[\geqslant f] : B \in \mathcal{B},\, f \in A\}$ forms a basis for X every element of which is clopen in X. Thus X is zero-dimensional and completely regular.

2. Consider A as a subset of X. Because A is well-ordered by $<^*$, every $B \subseteq A$ has a $<^*$-minimum element and it is not hard to see that this element must also be $\leqslant$-minimal as well. Hence $(A, \leqslant)$ is well-founded and there is some well-ordering $\preccurlyeq$ on A which extends $\leqslant$ on A. This well-ordering need not coincide with $<^*$. If $f, g \in A$ and $f \prec g$ then it cannot be the case that $f \in X[\geqslant g]$ since $\preccurlyeq$ extends $\leqslant$. This means $\{X[\geqslant f] \cap A : f \in A\}$ is a collection in τA witnessing that A is left-separated in type $\mathfrak{b}$ (see [Ro] p.301). Hence $d(A) \geqslant \mathfrak{b}$ and, as the cardinality of X is $\mathfrak{b}$, this implies that $hd(X) = \mathfrak{b}$.

3. Before proceeeding with the last section of this proof, it is worth remarking that, since $\mathfrak{b}$ is regular and A is well-ordered by $<^*$ in order type $\mathfrak{b}$, any family in A of size $\mathfrak{b}$ is cofinal in A and hence is also an unbounded collection in ω^ω. In addition, any family in A of cardinality less than $\mathfrak{b}$ has a $<^*$-upper bound.

Suppose now that F is a closed subset of $X[A, \geqslant]$. Define $Y_0 = F \setminus \overline{F \cap D}$. As D is countable, in order to show that $d(F) < \mathfrak{b}$, it would suffice to show that $|Y_0| < \mathfrak{b}$. Thus, assume for contradiction that $|Y_0| = \mathfrak{b}$. By shrinking, we can find $Y_1 \subseteq Y_0$ also of size $\mathfrak{b}$ and for which there exists $m \in \omega$ such that:

(a) $f|_m = g|_m$ for all $f, g \in Y_1$
(b) $f|_m \neq g|_m$ for all $f \in Y_1$ and $g \in F \cap D$

Choose a suitable countable elementary submodel $\mathcal{M}$ which contains A, X, $\mathfrak{b}$, F, Y_0, Y_1, D, $\mathcal{B}$ and $\leqslant$.

$\{f \in \mathcal{M} : \{ \in \mathcal{A}\}$ is necessarily countable so by the earlier remark there exists $h \in A$ such that $f <^* h$ for all $f \in A \cap \mathcal{M}$. And, since Y_1 is well-ordered by $<^*$, $|\{g \in Y_1 : h <^* g\}| = \mathfrak{b}$. By the pigeon-hole principle and the definition of $<^*$, there exists $p \in \omega$ for which $Y_2 = \{g \in Y_1 : h \leqslant_p g\}$ is also of size $\mathfrak{b}$ and hence unbounded in ω^ω.

If, for each $n \in \omega$, $R_n = \{g(n) : n \in \omega,\, g \in Y_2\}$ is bounded in ω, define $f \in \omega^\omega$ by $f(n) = \max R_n$ for all $n \in \omega$. But then, for all $g \in Y_2$, $g \leqslant f$ which contradicts the unboundedness of Y_2. Thus there exists $n \in \omega$ for which R_n is unbounded in ω. Choose n to be minimal (though it is necessarily greater than m) whence R_k is bounded for each $k < n$. This implies that $\{g|_n : g \in Y_2\}$ is finite and, again by the pigeonhole principle, we may choose $t \in \omega^n$ such that $\{g(n) : g \in Y_2 \text{ and } t \subseteq g\}$ is unbounded in ω. Given this, it is easy to find a sequence in Y_2, $\{g_i\}_{i \in \omega}$ say, such that for all $i \in \omega$:

(c) $t \subseteq g_i$
(d) $g_i(n) < g_{i+1}(n)$

Define $d \in D$ by $d|_n = t$ and, for all $i \geqslant n$, $d(i) = \omega$. D is a countable element of $\mathcal{M}$ hence is a subset of $\mathcal{M}$ (see Proposition 6.2.4) and $d \in \mathcal{M}$. By (b), $d \notin F$ so in particular $d \notin \overline{F \cap A}$. Thus there exist $f \in A$ and $B \in \mathcal{B}$ such that $d \in B \cap X[\geqslant f]$ and

$$(B \cap X[\geqslant f]) \cap (F \cap A) = \emptyset$$

As B is a basic open set induced by τZ, we may suppose that B has the form:

$$B = \{g \in X : g|_n = t\} \cap \{g \in X : g(j) \geqslant r \text{ for } j = n, n+1, \ldots, n+s\}$$

where r, $s \in \omega$.

Take $\Phi(f)$ to be the statement $\big((B \cap X[\geqslant f]) \cap (F \cap A) = \emptyset\big) \wedge \big(d \in (B \cap X[\geqslant f])\big)$ and so we have that $V \models \exists f \in A(\Phi(f))$. Since $\mathcal{B} \in \mathcal{M}$ and $\mathcal{B}$ is countable, by Proposition 6.2.4, $B \in \mathcal{M}$ and we have already assumed that all the other objects mentioned in Φ are in $\mathcal{M}$ hence by elementarity $\mathcal{M} \models \exists f \in A(\Phi(f))$. Take $f_t \in A \cap \mathcal{M}$ which witnesses the truth of this statement so that $\mathcal{M} \models \Phi(f_t)$. Again by elementarity, $V \models \Phi(f_t)$. Or more plainly,

(e) $f_t \nleqslant f$ for every $f \in F \cap A$ which extends t and for which $f(j) \geqslant r$ for $j = n, n+1, \ldots, n+s$

Now $f_t \in \mathcal{M}$ so $f_t <^* h$. Fix $k \geqslant p$ such that $f_t \leqslant_k h$. By (d), $\{g_i(n)\}_{i \in \omega}$ is unbounded in ω and there exists $i \in \omega$ such that

(f) $f_t(k) \leqslant h(k) \leqslant g_i(n)$ and $g_i(n) \geqslant r$

$d \in B \cap X[\geqslant f_t]$ thus $d|_n = t \geqslant f_t|_n$. But $g_i|_n = t$, hence $g_i|_n \geqslant f_t|_n$. As $f_t \leqslant_k h \leqslant_k g_i$, if $j \geqslant k$ then $f_t(j) \leqslant g_i(j)$. For $n \leqslant j < k$, f_t and g_i are monotone so $f_t(j) \leqslant f_t(k) \leqslant g_i(n) \leqslant g_i(j)$. So overall, we have that $f_t \leqslant g_i$.

However, $g_i \in Y_2 \subseteq F \cap A$, g_i extends t by its definition and, since $g_i(n) \geqslant r$ and g_i is monotone increasing, $g_i(j) \geqslant r$ for $j = n, n+1, \ldots, n+s$. This contradicts (e). Hence $|Y_0| < \mathfrak{b}$ and $d(F) < \mathfrak{b}$ as required. □

Example 2.4.4 **[Todorčević]** If $\mathfrak{b} = \omega_1$ then there exists a separable boundary-separable space which is not hereditarily separable.

Proof Take the space X of Theorem 2.4.3. If $\mathfrak{b} = \omega_1$ then $hd(X) = \omega_1$ but $d(F) = \omega$ for every closed subspace F of X. Therefore, $hcld(X) = \omega$. Thus X is a separable, boundary-separable (by Proposition 2.4.1) and not hereditarily separable. □

Ideally, it would be better if the set-theoretic hypothesis could be removed from the example. However, Todorčević remarked that all examples, X, where $hcld(X) < hd(X)$ must contain a subspace, Y, for which $hl(Y) < hd(Y)$. In our situation, this would be an L-space - one which is hereditarily Lindelöf but not hereditarily separable. It may yet be the case that there are L-spaces in ZFC. However, it is possible that all L-spaces with the extra properties which we require in this context dualise to S-spaces. This would mean that there are no such examples in ZFC as it is consistent with ZFC that there are no S-spaces. Either way, Todorčević's remark implies:

Proposition 2.4.5 *If there are no L-spaces then every separable boundary-separable space is hereditarily separable.*

We now move on to the second question of when boundary-separable spaces are separable. This is actually quite complex and it is easier to consider first when boundary-(hereditarily separable) spaces are separable. Even in this case, though, the answer is somewhat remarkable.

Theorem 2.4.6 *Every boundary-(hereditarily separable) space is separable if and only if there are no non-separable Lusin spaces.*

Proof First, suppose there is a non-separable Lusin space. It is boundary-countable hence is a boundary-(hereditarily separable) non-separable space, as required.

Now suppose X is a non-separable boundary-separable space. The proof works by showing that inside X there is a subset which is a non-separable Lusin space. The construction of the Lusin subspace proceeds by an induction of length ω_1.

For $\alpha < \omega_1$, assume $\{x_\beta : \beta < \alpha\}$ has been defined such that for all $\beta < \alpha$, $x_\beta \notin \overline{\{x_\gamma : \gamma < \beta\}}$. Now $\{x_\beta : \beta < \alpha\}$ is not dense in X as X is not separable. Thus there exists $x_\alpha \in X \setminus \overline{\{x_\beta : \beta < \alpha\}}$. Define $Y = \{x_\alpha : \alpha < \omega_1\}$.

If I is the set of isolated points of Y, I is discrete. X is boundary-separable hence boundary-ccc and so, by Lemma 2.2.5, I is countable. Y is clearly uncountable as the x_α are all distinct by their definition. Also, Y is non-separable since if D were a countable subset of Y then for some $\alpha \in \omega_1$, $D \subseteq \{x_\beta : \beta \leqslant \alpha\}$ from which it follows that $x_{\alpha+1} \notin \overline{D}$.

It remains to show that every nowhere dense subset of Y is countable. Thus take C to be a nowhere dense subset of Y. C is nowhere dense in X hence separable and there exists $D \in [C]^\omega$ such that $C = \overline{D}^C$. Now $D \subseteq Y$ and D is countable so there exists $\alpha < \omega_1$ such that $D \subseteq \{x_\beta : \beta < \alpha\}$. If C is uncountable, there is a $\gamma < \omega_1$ such that $\gamma > \alpha$ and $x_\gamma \in C$. But $x_\gamma \notin \overline{\{x_\beta : \beta < \gamma\}}^X \supseteq \overline{\{x_\beta : \beta < \alpha\}}^X \supseteq \overline{D}^C$. This is a contradiction.

Hence C must be countable and Y is a non-separable Lusin subspace of X.

□

Remark Example 2.3.2 together with Theorem 2.4.6 provide an alternative to Kunen's method in [K$_2$] for constructing a Lusin subspace of a Souslin line.

Whilst the above theorem may seem to avoid dealing with boundary-separability, it is actually crucial in determining when boundary-separable spaces are separable as can be seen in the next proof.

Theorem 2.4.7 *If there are no L-spaces then every boundary-separable space is separable.*

Proof Suppose X is boundary-separable. Consider $C \subseteq X$ which is closed and nowhere dense in X. Thus C is separable and boundary-separable by Proposition 2.1.2 (2). Now, by Proposition 2.4.5, C is hereditarily separable and hence X is boundary-(hereditarily separable). But note that any non-separable Lusin space is an L-space as it is obviously not hereditarily separable and it is hereditarily Lindelöf by the fact that it is boundary-countable and Corollary 2.1.6. So the assumption that there are no L-spaces also kills off non-separable Lusin spaces and then Theorem 2.4.6 tells us that X is separable.

□

In certain classes of spaces, the situation is much less complicated. In particular, in compact spaces we have the following result which was proven independently by both Šapirovskiĭ and Arhangel'skiĭ.

Theorem 2.4.8 [Š] [$\mathbf{A}_1$] *If X is compact then $hcld(X) = hd(X)$.*

This does for us what the assumption of no L-spaces did for us in the first part of the proof of Theorem 2.4.7. Hence, in the same way, we have:

Proposition 2.4.9 *If X is a boundary-separable compactum then X is boundary-(hereditarily separable).*

Taking this together with Theorem 2.4.6 gives:

Corollary 2.4.10 *If there are no non-separable Lusin spaces then every boundary-separable compactum is separable.*

However, we cannot greatly improve upon this since compactness does not kill off potential Lusin subspaces as can be seen in Example 2.3.2.

The assumption that there are no Lusin spaces is stronger than the assumption that there are no Souslin lines but we know that, in general, Lusin spaces are necessary for the existence of boundary-(hereditarily separable) spaces which are not separable. However, this can be weakened to Souslin's Hypothesis in certain classes of spaces.

Theorem 2.4.11 *Souslin's Hypothesis holds if and only if every boundary-separable locally connected space is separable.*

Proof By Example 2.3.2, it is enough to show that if X is boundary-separable and locally connected but non-separable then there is a Souslin line. In fact, it will be shown that there is a Souslin tree made up of open sets in X and ordered by reverse inclusion.

Take $\mathcal{T}_0$ to be an infinite, maximal family of disjoint open sets. Suppose that for a given ordinal α, for all $\beta < \alpha$, $\mathcal{T}_\beta$ has been defined. If $\alpha = \beta + 1$ then take $\mathcal{T}_\alpha$ to be a maximal disjoint family of open subsets of $\bigcup \mathcal{T}_\beta$ such that, for every $U \in \mathcal{T}_\alpha$, there exists $V(U) \in \mathcal{T}_\beta$ such that

$$\overline{U} \subseteq V(U) \text{ and } V(U) \setminus \overline{U} \neq \emptyset$$

If α is a limit ordinal then take

$$\mathcal{T}_\alpha = \{\mathrm{int} \bigcap \mathcal{C} : \mathcal{C} \text{ is a branch in } \bigcup_{\beta<\alpha} \mathcal{T}_\beta\}$$

The process stops when $\mathcal{T}_\alpha = \{\emptyset\}$. Define $\mathcal{T} = \bigcup_{\beta<\alpha} \mathcal{T}_\beta$ so that $\mathcal{T}$ is a tree of subsets of X ordered by reverse inclusion and take λ to be the height of $\mathcal{T}$. That λ is a limit ordinal can easily be seen from the construction of $\mathcal{T}$.

If $\lambda < \omega_1$ define $\partial\mathcal{T}_\alpha = \overline{\bigcup \mathcal{T}_\alpha} \setminus \bigcup \mathcal{T}_\alpha$. $\partial\mathcal{T}_\alpha$ is a boundary hence there exists $D_\alpha \in [\partial\mathcal{T}_\alpha]^\omega$ which is dense in $\partial\mathcal{T}_\alpha$. Define $D = \bigcup_{\alpha<\lambda} D_\alpha$. As $\lambda < \omega_1$, D is countable and D will be shown to be dense in X which gives a contradiction.

Suppose that V is a non-empty open subset of X for which $V \cap D = \emptyset$. X is locally connected so there is some non-empty, open, connected subset, U say, of V which also misses D. Now $\bigcup \mathcal{T}_0$ is dense in X by its definition hence U meets some $T_0 \in \mathcal{T}_0$. However, $U \cap D = \emptyset$ implies that $U \cap D_0 = \emptyset$ which implies that $U \cap \partial\mathcal{T}_0 = \emptyset$. Clearly $\overline{T_0} \setminus T_0 \subseteq \partial\mathcal{T}_0$ whence $U \cap \mathrm{bd}T_0 = \emptyset$. This means that $\overline{T_0} \cap U = T_0 \cap U$ and $T_0 \cap U$ is a non-empty clopen subset of U. U is connected so it must be the case that $T_0 \cap U = U$ and $U \subseteq T_0$.

Suppose for $\alpha < \lambda$ and every $\beta < \alpha$ that there exists a $T_\beta \in \mathcal{T}_\beta$ for which $U \subseteq T_\beta$. If $\alpha = \beta + 1$ then, similarly to when $\alpha = 0$, there exists $T_\alpha \in \mathcal{T}_\alpha$ for which $U \subseteq T_\alpha$. If α is a limit ordinal then it is easily shown that, for all $\gamma < \beta < \alpha$, $U \subseteq T_\beta \subseteq T_\gamma$. Therefore, $\mathcal{C} = \{T_\beta : \beta < \alpha\}$ is a chain in $\bigcup_{\beta<\alpha} \mathcal{T}_\beta$, indeed, a branch as $\mathcal{C} \cap \mathcal{T}_\beta \neq \emptyset$ for all $\beta < \alpha$, and $U \subseteq \bigcap \mathcal{C}$. More specifically, $U \subseteq \operatorname{int} \bigcap \mathcal{C} \in \mathcal{T}_\alpha$. So in both cases, there is a $T_\alpha \in \mathcal{T}_\alpha$ such that $U \subseteq T_\alpha$.

Thus for all $\alpha < \lambda$, there exists $T_\alpha \in \mathcal{T}_\alpha$ for which $U \subseteq T_\alpha$. But then $\mathcal{C} = \{T_\alpha : \alpha < \lambda\}$ is a branch in $\mathcal{T}$ such that $U \subseteq \operatorname{int} \bigcap \mathcal{C}$. This contradicts $\mathcal{T}_\lambda = \{\emptyset\}$.

Hence it must be that U meets D which implies that D is a countable dense subset of X. This contradicts the hypothesis on X.

Thus $\lambda \geqslant \omega_1$. However, since X is boundary-separable, it is boundary-ccc and, by Lemma 2.2.5, X itself satisfies the countable chain condition. It is well known that this implies that all chains and anti-chains of open sets (when ordered by inclusion) are countable. In particular, there can be no chains of sets as long as ω_1 and $\mathcal{T}_{\omega_1}$ is empty. It follows that $\mathcal{T}$ is an ω_1-tree without countable chains or anti-chains. That is, $\mathcal{T}$ is a Souslin tree.

□

2.5 Boundary-scattered spaces

Scattered spaces have a great deal of structure due to the possibility of layering the space via its scattered length. But like boundary-metrizable spaces, boundary-scattered spaces need not have especially nice structures. Once again the nodec space witnesses this - it is a regular, countable crowded space which is not only boundary-scattered but boundary-discrete. Moreover, the nodec space can be used as a building block to show that specifying the scattered length of the boundaries does not prevent scatteredness. To clarify what is meant by "specifying", we make a definition.

Definition 2.5.1 For a boundary-scattered space X, define the *boundary-(scattered length) of* X, denoted bdy-sl(X), to be the supremum of the scattered lengths of the boundaries of X.

Examples 2.5.2 For every ordinal α, there is a boundary-scattered, crowded, completely regular space X_α such that bdy-sl$(X_\alpha) = \alpha$.

Remark The construction of these examples uses resolutions as described in [W] and Chapter 5.

Proof Fix an ordinal α and choose some scattered space, Y_α say, for which $\mathrm{sl}(Y_\alpha) = \alpha$. Taking Θ to be a regular nodec space, fix some point $y_0 \in \Theta$. Now resolve βY_α over the set of isolated points into $\beta\Theta$ by constant mappings to y_0. Take the subspace of the resolved space $X_\alpha = Y_\alpha^d \cup \bigcup\{\{x\} \times \Theta : x \in Y_\alpha \setminus Y_\alpha^d\}$.

X_α is completely regular as it is a subspace of a compact space. Note that Y_α is homeomorphic to $Y_\alpha^d \cup \{\langle x, y_0\rangle : x \in Y_\alpha \setminus Y_\alpha^d\}$ so Y_α is identified with this set in X_α. Also, for each $x \in Y_\alpha \setminus Y_\alpha^d$ and $V \in \tau\Theta$, $\{x\} \times V$ is open in X_α.

Suppose C is a closed and nowhere dense subset of X_α. $C \cap (\{x\} \times \Theta)$ is nowhere dense in $\{x\} \times \Theta$ otherwise $\{x\} \times V$ is a subset of C for some $V \in \tau\Theta \setminus \{\emptyset\}$ which contradicts C being nowhere dense. Thus, $C \cap (\{x\} \times \Theta)$ is discrete and it is not too hard to see that $C \cap \bigcup\{\{x\} \times \Theta : x \in Y_\alpha \setminus Y_\alpha^d\}$ must be a collection of isolated points of C. Therefore, $C^d \subseteq Y_\alpha^d \subseteq X_\alpha$ and C^d is scattered with $\mathrm{sl}(C^d) \leqslant \mathrm{sl}(Y^d)$. Hence, C is scattered, $\mathrm{sl}(C) \leqslant \mathrm{sl}(Y_\alpha)$ and X is scattered with bdy-sl$(X) \leqslant \alpha$.

Consider $U \in \tau X_\alpha$ which meets Y_α. From the definition of resolutions, $U = (V \cap Y_\alpha^d) \cup \bigcup\{\{x\} \times \Theta : x \in V \setminus Y_\alpha^d\}$ for some $V \in \tau Y_\alpha$, and $U \cap \bigcup\{\{x\} \times \Theta : x \in Y_\alpha \setminus Y_\alpha^d\} \neq \emptyset$ since Y_α^d is nowhere dense in Y_α hence in X_α. But then for some $x \in Y_\alpha \setminus Y_\alpha^d$, $\{x\} \times \Theta$ is a subset of U and U cannot be a subset of Y_α. Moreover, Y_α is closed as it is the complement of the open set $\bigcup\{\{x\} \times (\Theta \setminus \{y_0\}) : x \in Y_\alpha \setminus Y_\alpha^d\}$. Hence Y_α is a closed nowhere dense subset of X_α and $\mathrm{sl}(Y_\alpha) = \alpha$. Therefore, bdy-sl$(X_\alpha) = \alpha$.

□

Like the metrizable situation, compactness comes to the rescue. I am indebted to Robin Knight for providing me with the proof of this next result.

Theorem 2.5.3 *Every crowded compact space has a nowhere dense subset which is also crowded.*

Proof Suppose X is a crowded compact space. We will essentially mimic the construction of a Cantor set in $\mathbb{R}$ in order to produce a nowhere dense crowded subspace of X. However, as X need not have all the structure of $\mathbb{R}$, we must considerably strengthen the analysis of the construction.

For each $f \in 3^{<\omega}$, inductively define $U_f \in \tau X$ as follows:

$U_\emptyset = X$. Suppose that $n \in \omega$ and that for all $f \in 3^{\leqslant n}$, $U_f \in \tau X \setminus \{\emptyset\}$ has been defined in such a way that if $g \in 3^{\leqslant n}$ properly extends f then $\overline{U_g} \subseteq U_f$.

Consider $f \in 3^n$. U_f is non-empty and X is crowded so U_f is infinite. Choose three points $x_0, x_1, x_2 \in U_f$ and find $V_i \in \tau X$ for which $x_i \in V_i \subseteq \overline{V_i} \subseteq U_f$ and $V_i \cap V_j = \emptyset$ when $i \neq j$ for $i, j \in \{1, 2, 3\}$. For each $g \in 3^{n+1}$ which extends f, define $U_g = V_i$ where $g(n) = i$. Then the U_g satsify the inductive hypothesis.

We now throw away the "middle thirds": for all $f \in 2^\omega$ define $C_f = \bigcap_{n\in\omega} U_{f|_n}$ (which means that $C_f = \bigcap_{n\in\omega} \overline{U_{f|_n}}$ as well) and also define $C = \bigcup_{f\in 2^\omega} C_f$. Note that

$$C = \bigcup_{f\in 2^\omega} \bigcap_{n\in\omega} \overline{U_{f|_n}} = \bigcap_{n\in\omega} \bigcup_{f\in 2^n} \overline{U_f}$$

Hence C is the intersection of closed sets and so is itself closed and compact.

C has many of the features of a Cantor set but it may not be nowhere dense. Therefore we define $Y = \mathrm{bd}C$ so that Y is clearly nowhere dense. Y however could still have some isolated points but we will show that it is at least not scattered by proving that $Y^{(\alpha)} \neq \emptyset$ for every ordinal α. The proof of this proceeds by induction but to make it work we actually need the stronger inductive hypothesis that for all ordinals α and for all $f \in 2^\omega$, $C_f^\alpha = C_f \cap Y^{(\alpha)} \neq \emptyset$.

For the base step in the induction, we must show that, for all $f \in 2^\omega$, $C_f^0 = C_f \cap Y \neq \emptyset$. Fix $f \in 2^\omega$, pick $x_n \in \overline{U_{f|_n \frown \{\langle n,2\rangle\}}}$ and, by compactness, find $x \in \overline{\{x_n : n \in \omega\}}$. By their definition, $x_n \notin C_g$ for all $g \in 2^\omega$ hence $x_n \in X \setminus C$ for all $n \in \omega$ and $x \in \overline{X \setminus C}$. However, $x \in C_f$ since $x \in \overline{\{x_n : n \geqslant k\}}$ for all $k \in \omega$. But $\overline{\{x_n : n \geqslant k\}} \subseteq \overline{U_{f|_k}}$ for all $k \in \omega$. Therefore, $x \in \bigcap_{k\in\omega} \overline{U_{f|_k}} = C_f$. This means that

$$x \in C_f \cap C \cap \overline{X \setminus C} = C_f^0$$

which completes the base step.

Consider a limit ordinal λ such that for all $\alpha < \lambda$, $C_f^\alpha \neq \emptyset$. $C_f^\lambda = C_f \cap Y^{(\lambda)} = C_f \cap \bigcap_{\alpha<\lambda} Y^{(\alpha)} = \bigcap_{\alpha<\lambda}(C_f \cap Y^{(\alpha)}) = \bigcap_{\alpha<\lambda} C_\lambda^\alpha$. Thus C_f^λ is the intersection of a strictly decreasing sequence of non-empty closed sets. Because X is compact, this means that C_f^λ is non-empty for all $f \in 2^\omega$.

This leaves the successor step. Suppose that, for an ordinal α and for all $f \in 2^\omega$, C_f^α is non-empty. Fix $f \in 2^\omega$ and a non-trivial sequence, $\{f_n\}_{n\in\omega}$ which converges to f in 2^ω. Choose $x_n \in C_{f_n}^\alpha$ and, again by compactness, find $x \in \overline{\{x_n : n \in \omega\}}$. $Y^{(\alpha)}$ is closed in X and x is an accumulation point of a sequence in $Y^{(\alpha)}$ so $x \in Y^{(\alpha+1)}$. Moreover, since $\{f_n\}$ converegs to f in 2^ω, given any $k \in \omega$, there exists $N \in \omega$ such that for all $n \geqslant N$, $f_n|_k = f|_k$. But then $C_{f_n}^\alpha \subseteq U_{f|_k}$ for all $n \geqslant N$ which implies that $\overline{\{x_n : n \geqslant N\}} \subseteq \overline{U_{f|_k}}$ and $x \in \overline{U_{f|_k}}$ for all $k \in \omega$. Hence $x \in C_f$ from which it follows that $x \in C_f^{(\alpha+1)}$ and $C_f^{(\alpha+1)} \neq \emptyset$ for all $f \in 2^\omega$.

This completes the transfinite induction. Hence Y is non-scattered and nowhere dense in X. Take $Z = Y^{(\alpha)}$ where α is such that $Y^{(\alpha)} = Y^{(\alpha+1)}$. Then Z is a nowhere dense, crowded subspace of X. $\square$

Corollary 2.5.4 *Every compact boundary-scattered space is scattered.*

Proof Suppose X were a compact boundary-scattered space. If X is not scattered then there is a subset Y of X which is crowded. By taking the closure of Y if necessary, Y can be assumed to be closed and, therefore, compact. By the previous result, Y would have a nowhere dense subset which was crowded. But Proposition 2.1.2 (2) implies that Y is also boundary-scattered. This is clearly a contradiction. □

Corollary 2.5.5 *There are no compact Lusin spaces.*

Proof Suppose X were a compact Lusin space. X is boundary-countable and, since countable compact spaces are scattered, X is also boundary-scattered. By Corollary 2.5.4, X is scattered so X^d is nowhere dense in X hence countable. By the definition of a Lusin space, X has only countably many isolated points. This means that X in total can only be countable which is a contradiction as Lusin spaces are assumed to be uncountable.

□

2.6 Summary and questions

The second and third sections of this chapter are concerned with characterising boundary-metrizable compacta. The second section deals with general spaces and we obtain the surprising result that if there are no Lusin spaces then every boundary-metrizable compactum is metrizable. By studying boundary-metrizability in LOTS, we obtain two important examples of boundary-metrizable non-metrizable compacta. These results are summarised here:

Theorem 2.6.1

1. *If there are no Lusin spaces then every boundary-metrizable compactum is metrizable*
2. *If there is a Souslin line then there is a boundary-metrizable, non-metrizable arc*
3. *If there is a Lusin set then there is a separable boundary-metrizable, non-metrizable compact LOTS*

From this, it is clear that there is a discrepancy between the hypotheses for an example and for a theorem. This gap could be filled by a positive answer to:

Question 2.1 *If there exists a Lusin space, is there a boundary-metrizable non-metrizable compactum?*

One way of solving this may arise by using (3) and answering:

Question 2.2 *If there is a Lusin space, is there also a Lusin subspace of* $\mathbb{R}$*?*

(2) also suggests a possible converse:

Question 2.3 *If there is a boundary-metrizable non-metrizable continuum, is there a Souslin line?*

The fourth section dealt with boundary-separability and gave two key results: first, if there are no L-spaces then every boundary-separable space is separable and, secondly, every boundary-(hereditarily separable) space is separable if and only if there are no non-separable Lusin spaces. However the hypothesis that there are no L-spaces may be inconsistent with ZFC so this would be improved if we could answer either of the following affirmatively:

Question 2.4 *If is there an L-space, is there a boundary-separable, non-separable space?*

Question 2.5 *If there are no S-spaces, is every boundary-separable space separable?*

We also saw that if there are no Lusin spaces then every boundary-separable compactum is separable. Does the converse hold?

Question 2.6 *If there is a Lusin space, is there are boundary-separable, non-separable compactum?*

The fifth section gives some ZFC results and we have that every boundary-scattered compactum is scattered. This also rules out the trivial answer of a compact Lusin set to Questions 2.1 and 2.6.

Chapter 3

Cohesion

The most well-known theory which is based on spaces defined by the properties of their boundaries is the theory of inductive dimension. The inductive dimension functions assert that certain open sets have boundaries of a lower dimension.

Following the flavour of this idea, in [A_2], Arhangel'skiĭ defined inductively a "function of the dimensional type" which he called *absolute dimension*: a space has absolute dimension n if every boundary has absolute dimension strictly less than n starting from the base case that a space has absolute dimension 0 if and only if it has small inductive dimension zero. This notion does not actually produce a dimension function in the usual sense - it is shown in the last section of this chapter that the absolute dimension of the unit square cannot be defined. However, absolute dimension is well-behaved on the real line - every subset of $\mathbb{R}$ has absolute dimension of at most 1.

In order to understand more closely what an inductive function of this nature says about a space, we modify the base case definition of absolute dimension to give the new notion of cohesion. In the first section of this chapter, cohesion and related terms are defined and we establish some basic properties. The relationship between cohesion and scattered length is fully investigated in the next section. Crowded cohesive spaces are considered in the third section and examples of finitely cohesive, regular, crowded spaces are constructed. It is also shown that there cannot be transfinitely cohesive, regular spaces. Using the results on boundary–scattered spaces, compact cohesive spaces are examined in the fourth section. The fifth gives some theorems on when cohesion is preserved under continuous maps and taking products. Finally, we show that absolute dimension is not defined on the unit square.

3.1 Definition and basic properties

Without further ado, we define cohesion.

Definition 3.1.1 For a topological space X, the *cohesion of X,* abbreviated to $\mathrm{coh}X$, is defined by transfinite recursion as follows:

$$\mathrm{coh}X = -1 \text{ if and only if } X = \emptyset$$

$$\text{for an ordinal } \alpha,\ \mathrm{coh}X \leqslant \alpha \text{ if for every nowhere dense subset } C \subseteq X,\ \mathrm{coh}C < \alpha$$

For a space X and an ordinal α, $\mathrm{coh}X = \alpha$ if $\mathrm{coh}X \leqslant \alpha$ and for every $\beta < \alpha$ it is not the case that $\mathrm{coh}X \leqslant \beta$. Finally, X is said to be *cohesive* if for some ordinal α, $\mathrm{coh}X = \alpha$, *finitely cohesive* if α is finite and *transfinitely cohesive* if α is infinite.

Remark Despite the fact that we will prove that there is no regular space of transfinite cohesion, we have given the definition in its full generality. This is for two reasons. First, in proving this fact, we wish to use certain lemmas which tell us about the structure of spaces with transfinite cohesion. Secondly, there may yet be some interesting Hausdorff spaces of transfinite cohesion.

Clearly, cohesion will have a similar feel to the boundary properties of the previous chapter. However, it is the inductive element of the definition which makes it very different in character as it forces the nowhere dense subsets into a rigid hierarchy. Because of the close relation between the nowhere dense subsets and the topology of a space, we are able to examine this hierarchy quite closely.

We now prove some basic properties of cohesion.

Proposition 3.1.2 *If X is a space such that, for some ordinal α, coh$X \leqslant \alpha$ and $Y \subseteq X$ then coh$Y \leqslant \alpha$.*

Proof This follows immediately on noting that a nowhere dense subset of Y is a nowhere dense subset of X.

□

Proposition 3.1.3 *A non-empty space X is discrete if and only if coh$X = 0$.*

Proof If X is discrete then every subset of X is open. This means that the only nowhere dense subset of X is the empty set so from the definition it follows that coh$X = 0$.

If coh$X = 0$ then every nowhere dense subset has a cohesion of -1. Thus, no non-empty subset is nowhere dense. Consider $\{x\}$. This is closed as X is T_1 but is not nowhere dense so contains a non-empty subset open in X. This must be $\{x\}$. Hence every point of X is open and X is discrete.

□

Remark By inducting up a step, Proposition 3.1.3 implies that a space has cohesion 1 if and only if it is boundary-discrete.

Proposition 3.1.4 *If X is a space such that, for some ordinal α, coh$X = \alpha$ then, for all $\beta < \alpha$, there exists a closed nowhere dense subset $C_\beta \subseteq X$ such that coh$C_\beta = \beta$.*

Proof If $\alpha = -1$ then there is literally nothing to prove! Assume the proposition has been proven for all spaces X such that coh$X = \beta$ where $\beta < \alpha$.

Consider the case where $\alpha = \gamma + 1$. If every nowhere dense subset of X has cohesion less than γ then, by definition, coh$X \leqslant \gamma$. Since this is not the case it must be that there is a nowhere dense subset A of X for which coh$A = \gamma$. Define $C_\gamma = \overline{A}$ so C_γ is nowhere dense in X. Hence coh$C_\gamma \leqslant \gamma$ and, since $A \subseteq C_\gamma$, Proposition 3.1.2 implies that coh$C_\gamma \geqslant$ coh$A = \gamma$. Therefore coh$C_\gamma = \gamma$.

Suppose $\beta < \alpha$. If $\beta = \gamma$ then C_β is already defined. If $\beta < \gamma$ then by the inductive hypothesis there exists a $C_\beta \subseteq C_\gamma$ closed and nowhere dense in C_γ such that coh$C_\beta = \beta$. But then C_β is also closed and nowhere dense in X and the hypothesis holds for α.

Consider now the case where α is a limit ordinal. For every $\beta < \alpha$, there exist $\gamma < \alpha$ and a nowhere dense subset A_γ of X such that $\beta < \gamma$ and coh$A_\gamma = \gamma$ (otherwise coh$X \leqslant \beta + 1$). As before, taking $C = \overline{A_\gamma}$, C is closed and nowhere dense in X with $\gamma \leqslant$ coh$C < \alpha$. Then coh$C > \beta$ and, by the inductive hypothesis, there exists a C_β closed and nowhere dense in C, and hence in X, such that coh$C_\beta = \beta$.

□

One other useful property is:

Proposition 3.1.5 *If $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of X such that, for some $n \in \omega$ and for all $\lambda \in \Lambda$, coh$U_\lambda \leqslant n$ then X is cohesive and coh$X \leqslant n$.*

Proof Suppose coh$U_\lambda \leqslant -1$ for all $\lambda \in \Lambda$, then each U_λ is empty but still form a cover of X so X must be empty and coh$X \leqslant -1$.

Assume now that for any space X and some $n \in \omega$ the proposition holds and consider the case where coh$U_\lambda \leqslant n+1$ for all $\lambda \in \Lambda$.

Suppose A is nowhere dense in X. Take $C = \overline{A}$ whence C is nowhere dense and closed in X. Consider $C \cap U_\lambda$ for some $\lambda \in \Lambda$. If $C \cap U_\lambda$ is not nowhere dense in U_λ, since $C \cap U_\lambda$ is closed in U_λ, there exists $V \in \tau U_\lambda$ such that $V \subseteq C \cap U_\lambda$. But $V \in \tau X$ as $U_\lambda \in \tau X$ and $V \subseteq C$ which means that C is not nowhere dense in X - a contradiction. Therefore $C \cap U_\lambda$ is nowhere dense in U_λ and then, by definition of cohesion, coh$(C \cap U_\lambda) \leqslant n$.

Taking $V_\lambda = C \cap U_\lambda$, $\{V_\lambda : \lambda \in \Lambda\}$ is an open cover for C such that coh$V_\lambda \leqslant n$ for all $\lambda \in \Lambda$. So by the inductive hypothesis, coh$C \leqslant n$ giving coh$A \leqslant n$ and hence coh$X \leqslant n+1$.

□

From the remark after Proposition 3.1.3, we know that any boundary-discrete space has cohesion defined on it. This gives us a source of spaces with cohesion 1 but it is informative to have some elementary examples of spaces which have higher cohesion.

Examples 3.1.6 For each $n \in \omega$, there exists $C_n \subseteq \mathbb{R}$ such that coh$C_n = n$.

Proof Take $C_1 \subseteq \mathbb{R}$ to be $C_1 = \{0\} \cup \{\frac{1}{n} : n \in \omega \setminus \{0\}\}$. This is clearly non-empty and not discrete. By Propositions 1.2.7 and 1.2.8, the only possible non-empty, nowhere dense subset of C_1 is $\{0\}$ which clearly has cohesion 0, therefore coh$C_1 = 1$.

Taking C_1 to be the base case, for each $n \in \omega$ inductively define scattered, closed subsets of $\mathbb{Q}$, call them C_n, such that coh$C_n = n$ as follows:

$$C_{n+1} = C_n \cup \{\frac{1}{k_1} + \ldots + \frac{1}{k_{n+1}} : k_i,\, k_{n+1} \in \omega \setminus \{0\},\, k_{i+1} \geqslant 2k_i(k_i - 1) \text{ for } i = 1, \ldots, n\}$$

This gives sequences of points which converge down to every point of C_n. Hence $C_{n+1}^d = C_n$ and since C_n is scattered so too is C_{n+1}. If A is nowhere dense in C_{n+1} then, by Proposition 1.2.8, $A \subseteq C_n$ and coh$A \leqslant n$. However, C_n is nowhere dense in C_{n+1}, by Proposition 1.2.7, and coh$C_n = n$ so by definition of cohesion coh$C_{n+1} = n+1$.

□

Since each of these spaces is a subset of $\mathbb{Q}$ this shows that if *coh*$\mathbb{Q}$ exists then it is transfinite. However, we in fact have:

Theorem 3.1.7 *coh*$\mathbb{Q}$ *is not defined.*

Proof Suppose for contradiction that coh$\mathbb{Q}$ is defined. $\mathbb{Q}$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ which contains $\{0\} \times \mathbb{Q}$ as a nowhere dense subset. Thus, by definition of cohesion, coh$(\{0\} \times \mathbb{Q}) <$ coh$(\mathbb{Q} \times \mathbb{Q})$. But as $\{0\} \times \mathbb{Q}$ is also homeomorphic to $\mathbb{Q}$ this gives us our required contradiction.

□

Remark This is actually a consequence of Theorem 3.2.1 but this proof is considerably shorter and more elegant and I am grateful to the referee of an earlier form of this work for suggesting it.

3.2 Cohesion and scattered spaces

The last theorem of the previous section was basically shown by finding a nowhere dense subspace of $\mathbb{Q}$ which was homeomorphic to $\mathbb{Q}$. The result then followed directly from the definition of cohesion. This is not in general possible but it is possible to find in certain spaces a subspace which contains a nowhere dense homeomorph of itself. It then follows, as for $\mathbb{Q}$, that such spaces cannot have cohesion defined on them.

The following theorem gives the details of how such subspaces can be constructed in a more general context.

Theorem 3.2.1 *If X is a Hausdorff, sequential, cohesive space then X is scattered.*

Proof Suppose X is not scattered. This means that there exists an $A \subseteq X$ which has no isolated points. Define $Y = \overline{A}^X$. Y is a closed subset of X so is also Hausdorff and sequential. Moreover, by Corollary 1.2.2, Y is crowded. Thus, for each $y \in Y$, $y \in \overline{Y \setminus \{y\}}^Y$ and hence $Y \setminus \{y\} \neq \overline{Y \setminus \{y\}}^Y$ or more simply $Y \setminus \{y\}$ is not closed in Y. Since Y is sequential, this implies that there exists a sequence in $Y \setminus \{y\}$ which converges to a point outside of $Y \setminus \{y\}$. There is only one possible point left in Y which this sequence could converge to and this is y. Denote such a sequence by $\{y_n\}_{n=0}^{\infty}$ and since Y is Hausdorff we can assume all elements of the sequence are distinct.

We need to separate the points of such sequences quite some way so we require the following:
Fact For all $n \in \omega$, there exists $U_n(y) \subseteq Y$ open in Y such that $y_n \in U_n(y)$, $y \notin U_n(y)$ and $U_n(y) \cap U_m(y) = \emptyset$ whenever n, $m \in \omega$ and $n \neq m$.
This can be proved using only that Y is Hausdorff.

We now show how, for a given $x \in Y$ contained in some open set U, there exist sets $I_n(x,U) \subseteq U$ for each $n \in \omega$ such that $(I_{n+1}(x,U))^d = I_n(x,U)$, $I_n^{(n)}(x,U) = \{x\}$ and for every $z \in I_{n+1}(x,U) \setminus I_n(x,U)$, there is a $U_z \subseteq X$ which is open in X with $U_z \cap I_{n+1}(x,U) = \{z\}$ and whenever $z \neq z'$, $U_z \cap U_{z'} = \emptyset$. These sets are equivalent to the C_n in Examples 3.1.6.

Take $I_0(x,U) = \{x\}$ and define $U_x = U$. This trivially satisfies the conditions.

Suppose then that for some $n \in \omega$, if $i \leqslant n$ the set $I_i(x,U)$ and the corresponding U_z's are defined. Consider a $z \in I_n(x,U) \setminus I_{n-1}(x,U)$ (taking $I_{-1}(x,U) = \emptyset$). Take $\{z_k\}$ to be the sequence contained in U_z converging to z whose existence is demonstrated at the beginning of this proof. Define

$$I_{n+1}(x,U) = I_n(x,U) \cup \{z_k : z \in I_n(x,U) \setminus I_{n-1}(x,U) \text{ and } k \in \omega\}$$

$$U_{z_k} = U_z \cap U_k(z)$$

where $U_k(z)$ is defined by the Fact.

Suppose $z, z' \in I_n(x,U) \setminus I_{n-1}(x,U)$. If $z \neq z'$ then for all $j,k \in \omega$, $U_{z_j} \cap U_{z'_k} \subseteq U_z \cap U_{z'} = \emptyset$. And if $z = z'$ then, for $j,k \in \omega$ with $j \neq k$, $U_{z_j} \cap U_{z'_k} \subseteq U_j(z) \cap U_k(z) = \emptyset$ by their definition. From this, U_{z_k} does not contain any z'_j whenever either $(j \neq k)$ or $(z \neq z')$. Moreover, from the Fact, $z \notin U_{z_k}$ and $U_{z_k} \cap I_{n+1} = \{z_k\}$. Thus the U_{z_k} are the open sets required in the definition of $I_{n+1}(x,U)$.

The U_{z_k} also show that if $z \in I_{n+1}(x,U) \setminus I_n(x,U)$ then z is an isolated point of $I_{n+1}(x,U)$. And if $z \in I_n$ then by its definition there is a sequence in I_{n+1} converging to z. These two statements together give

$$(I_{n+1}(x,U))^d = I_n(x,U)$$

from which it follows by part of the induction hypothesis that

$$(I_{n+1}(x,U))^{(n+1)} = \{x\}$$

Hence I_{n+1} is scattered.
We now take

$$Z = \bigcup_{n \in \omega} I_n(y_n, U_n(y))$$

If $z \in Z$ is isolated then, by the definition of the I_n, it cannot be the case that $z \in I_{n-1}(y_n, U_n(y))$ for any $n \in \omega$ and so it must be that z is isolated in some $I_n(y_n, U_n(y))$. In the opposite direction, if z is isolated in $I_n(y_n, U_n(y))$ for some $n \in \omega$ then $\{z\} = V \cap I_n(y_n, U_n(y))$ for some V open in Y. But then $\{z\} = Z \cap (V \cap U_n(y))$ as $I_n(y_n, U_n(y)) \subseteq U_n(y)$ and the $U_n(y)$ are pairwise disjoint. This means z is isolated in Z. Hence we have

$$Z^d = \bigcup_{n \in \omega} (I_n(y_n, U_n(y)))^d$$

$$Z^d = \bigcup_{n \in \omega} I_n(y_{n+1}, U_{n+1}(y))$$

which is clearly homeomorphic to Z. It is not hard to see that Z is scattered (with $\text{sl}(x) = \omega + 1$) giving that Z^d is nowhere dense in Z.

But if X is cohesive then so too are Z^d and Z. By the definition of cohesion, $\text{coh}Z^d < \text{coh}Z$ which is impossible since Z^d is homeomorphic to Z. Thus we have a contradiction.

Hence it must be the case that X is scattered.

□

In cohesive spaces which are scattered, we have two numbers attached to the space, the cohesion and the scattered length. The next two theorems give the relation between them in scattered spaces.

Theorem 3.2.2 *For X a scattered space and $n \in \omega$, $sl(X) = n$ if and only if $cohX = n - 1$.*

Proof Firstly suppose X is scattered with $\text{sl}(X) = 0$. Then $X = X^{(0)} = \emptyset$ and hence $\text{coh}X = -1$.

Assume for the purposes of induction that if $\text{sl}(X) = n$ then $\text{coh}X = n - 1$ and consider a space X of scattered length $n+1$. X^d is nowhere dense in X and clearly has scattered length n. Thus $\text{coh}X^d = n - 1$.

If C is a nowhere dense subset of X then, by Proposition 1.2.8, $C \subseteq X^d$. By Proposition 3.1.2, this implies that for every nowhere dense subset C of X, $\text{coh}C \leqslant n - 1$. From the definition of cohesion, $\text{coh}X \leqslant n$. However X^d is a nowhere dense subset of X of cohesion $n - 1$. Hence $\text{coh}X = n$ and induction gives the implication in one direction.

To do the reverse implication, if $\text{coh}X = -1$ then $X = \emptyset$ and hence $\text{sl}(X) = 0$. Assume now that if $\text{coh}X = n - 1$ then $\text{sl}(X) = n$. If X is a scattered space such that $\text{coh}X = n$, then X^d is nowhere dense in X and, since any nowhere dense subset of X is contained in X^d, this gives $\text{coh}X^d = n - 1$. But then by the inductive hypothesis, $\text{sl}(X^d) = n$ which clearly implies that $\text{sl}(X) = n + 1$. This completes the induction and the proof.

□

Theorem 3.2.3 *If X is scattered and cohesive then $sl(X)$ is finite.*

Proof Suppose $\text{sl}(X) = \kappa$ and $\text{coh}X = \mu$ for some ordinals κ , μ where κ is infinite. Define a function $f : \omega \to \mu$ by

$$f(n) = \text{coh}(X^{(n)}) \text{ for } n \in \omega$$

Since $X^{(n+1)} = (X^{(n)})^d$, $X^{(n+1)}$ is nowhere dense in $X^{(n)}$. Thus $\text{coh}X^{(n+1)} < \text{coh}X^{(n)}$. But then $\{f(n) : n \in \omega\}$ forms a strictly decreasing sequence in the ordinal μ which contradicts the well-ordering of μ. Hence κ cannot be infinite and we have $\text{sl}(X)$ is finite.

□

Putting these last three results together we have the following:

Corollary 3.2.4 *If X is a cohesive, sequential Hausdorff space then X is scattered and $sl(X) = n$ for some $n \in \omega$. Moreover, $cohX = n - 1$.*

3.3 Cohesion in non-scattered spaces

The results of the previous section fully characterise cohesion in scattered spaces. This, of course, leads us to inquire into the behaviour of more general cohesive spaces. In order to get cohesion in non-scattered spaces, we need look no further than the remark after Proposition 3.1.3 and consider our favourite boundary-discrete crowded space - the nodec space of van Douwen. This is a crowded space of cohesion 1. We now produce examples with higher cohesion using the nodec space as a building block.

The examples are based on products. However, the nowhere dense subsets of products can be very complicated and so we devise a different topology on a product which essentially adds only one new nowhere dense set.

Theorem 3.3.1 *If X and Y are topological spaces such that coh$X = n$, for some $n \in \omega$, and Y is a crowded boundary–discrete space then $(X \times Y, \mathcal{T})$ is a topological space such that*

$$coh(X \times Y) = n + 1$$

where $\mathcal{T}$ is the topology determined by the following basis:
fix some $y_0 \in Y$ and for $\langle x, y\rangle \in X \times Y$, a basic open neighbourhood of $\langle x, y\rangle$ is of the form:

1. *$\{x\} \times U$ when $y \neq y_0$ and where $U \in \tau Y$ with $y_0 \notin U$*
2. *$\bigcup\{\{a\} \times U_a : a \in V\}$ when $y = y_0$ and where $x \in V \in \tau X$ and, for all $a \in V$, $y_0 \in U_a \in \tau Y$*

Proof It is not too hard to check that the definition given does indeed define a topology on $X \times Y$.

First of all, we shall show that $\mathrm{coh}(X \times Y) \geqslant n + 1$.

The set $X \times \{y_0\}$ is a subset of $X \times Y$. It is closed since if $\langle x, y\rangle \notin X \times Y$ then $y \neq y_0$ and $\{x\} \times (Y \setminus \{y_0\})$ is an open neighbourhood of $\langle x, y\rangle$ which misses $X \times \{y_0\}$. Moreover, it is nowhere dense because any open set, say V, about $\langle x, y_0\rangle \in X \times Y$ contains $\{x\} \times U$ for some open neighbourhood U of y_0. But y_0 is not isolated so for some $y \in Y \setminus \{y_0\}$, $\langle x, y\rangle \in \{x\} \times U \subseteq V$. Thus, V cannot be a subset of $X \times \{y_0\}$. Clearly, $X \times \{y_0\}$ is homeomorphic to X so that $\mathrm{coh}(X \times \{y_0\}) = n$ and, by definition of cohesion, $coh(X \times Y) \geqslant n + 1$.

Secondly, we show that $\mathrm{coh}(X \times Y) \leqslant n + 1$ and then the proof is complete.

Suppose C is nowhere dense in $X \times Y$. Since, for all $x \in X$, $\{x\} \times (Y \setminus \{y_0\})$ is open in $X \times Y$ then $C_x = C \cap (\{x\} \times (Y \setminus \{y_0\}))$ is nowhere dense in $\{x\} \times (Y \setminus \{y_0\})$ and hence in $\{x\} \times Y$. Clearly $\mathrm{coh}(\{x\} \times Y) = 1$ giving us that $\mathrm{coh}C_x \leqslant 0$, that is, C_x is either empty or closed and discrete in $\{x\} \times Y$. (Note also that C_x is open in C.) But then there exists an open neighbourhood U_x of y_0 such that $(\{x\} \times U_x) \cap C_x = \emptyset$.

Take $V = \bigcup_{x \in X}(\{x\} \times U_x)$. By definition of $\mathcal{T}$, V is open in $X \times Y$ and by definition of the U_x's, $C \cap V \subseteq X \times \{y_0\}$ so that $\mathrm{coh}(C \cap V) \leqslant n$. But we now have that $\{C \cap V\} \cup \{C_x : x \in X\}$ is an open cover of C such that each element of the cover has cohesion at most n. So by Proposition 3.1.5, $\mathrm{coh}C \leqslant n$. Hence, by definition of cohesion, $\mathrm{coh}(X \times Y) \leqslant n + 1$.

□

We can now inductively construct our examples which, as they are all regular and countable have many nice properties such as hereditary Lindelöfness and hereditary separability.

Examples 3.3.2 For all $n \in \omega \setminus \{0\}$, there exists a space X_n which is countable, crowded, regular and $\mathrm{coh}X_n = n$.

Proof For $n = 1$, take X_1 to be van Douwen's nodec space. Assume that for some $n \in \omega$, X_n has been shown to exist. Now apply the previous theorem with $X = X_n$, Y also van Douwen's nodec space and y_0 some point of Y. Define X_{n+1} to be this new space.

It is clear to see that X_{n+1} is countable and that coh$X_{n+1} = n + 1$ by the previous result. That X_{n+1} is crowded follows since every open neighbourhood of a point $\langle x, y\rangle \in X_n \times Y$ contains a set of the form $\{x\} \times U$ where U is an open neighbourhood of y in Y. But no $y \in Y$ is isolated so U contains some point other than y and hence every neighbourhood of $\langle x, y\rangle$ contains some point other than $\langle x, y\rangle$.

We must show that X_{n+1} is also a T_1-space. Consider $\langle x, y\rangle \in X_{n+1}$. The set $U = (X_n \setminus \{x\}) \times Y$ is a basic open set as X_n is T_1. Also $V = Y \setminus \{y\}$ is open in Y as Y is T_1.

Case (1): If $y \neq y_0$ of the last theorem then $X_{n+1} \setminus \{\langle x, y\rangle\} = \bigcup\{\{a\} \times U_a : a \in X\}$ where $U_a = Y$ for $a \neq x$ and $U_x = V$. Hence the point $\langle x, y\rangle$ is closed.

Case (2): If $y = y_0$ then $\{x\} \times V$ is open in X_{n+1} and then complement of $\langle x, y\rangle$ is $U \cup V$ which is open and hence $\langle x, y\rangle$ is closed.

It remains to show that X_{n+1} is regular. Suppose U is an open neighbourhood of $\langle x, y\rangle$ in X_{n+1}. We need to find an open set $W \subseteq X_{n+1}$ such that $\langle x, y\rangle \in W \subseteq \overline{W}^{X_{n+1}} \subseteq U$.

Case (1): If $y \neq y_0$ then U contains an open set of the form $\{x\} \times U'$ for some U' open in Y. In this case there exists a $V \subseteq Y$ open such that $y \in V \subseteq \overline{V}^Y \subseteq U'$. It is not too hard to see that $\overline{\{x\} \times V}^{X_n \times Y} = \{x\} \times \overline{V}^Y$ and so $W = \{x\} \times V$ is our required open set.

Case (2): If $y = y_0$, then U contains an open set of the form $\bigcup\{\{a\} \times U_a : a \in V\}$ where V is an open neighbourhood of x in X_n and each U_a is an open neighbourhood of y_0 in Y. Take G to be an open set in X_n such that $x \in G \subseteq \overline{G}^{X_n} \subseteq V$ and, for all $a \in \overline{G}^{X_n}$, take an H_a open in Y such that $y_0 \in H_a \subseteq \overline{H_a}^Y \subseteq U_a$. Setting $W = \bigcup\{\{a\} \times H_a : a \in G\}$, it is clear to see that W is an open neighbourhood of $\langle x, y\rangle$ which is contained in the closed set $\bigcup\{\{a\} \times \overline{H_a}^Y : a \in \overline{G}^{X_n}\}$ which is in turn contained in U.

Hence X_{n+1} is regular and so by induction on the natural numbers the theorem is proven.

□

It would be ideal if this construction could be improved upon thereby allowing us to produce examples of crowded spaces with all possible cohesions. However, the next theorem shows that we cannot do this and keep regularity.

Theorem 3.3.3 *There is no regular, transfinitely cohesive space.*

We actually demonstrate that there is no regular space of cohesion ω. This suffices since Proposition 3.1.4 says that any regular space of transfinite cohesion contains a subset of cohesion ω which is necessarily regular.

The proof proceeds by demonstrating that if a space of cohesion ω exists then it contains a nowhere dense subset also of cohesion ω. This contradicts the definition of cohesion. To construct this nowhere dense subset, we need a couple of technical lemmas.

Lemma 3.3.4 *For n, $m \in \omega$, if A, $U \subseteq X$ and U is open with coh$(A \setminus U) \leqslant n$ and coh$U \leqslant m$ then coh$(A \cup U) \leqslant n + m + 1$.*

Proof Induct on m for a given n. Assume $m = -1$ so $U = \emptyset$ and coh$A \leqslant n$. Hence coh$(A \cup U) \leqslant n + -1 + 1 = n$ as required.

Thus suppose it has been proven for $m = k$ and assume $m = k+1$. If C is nowhere dense in $A \cup U$ then $C \cap U$ is nowhere dense in U as U is open in $A \cup U$. Thus, $\mathrm{coh}(C \cap U) \leqslant k$. But also $C \setminus U \subseteq A \setminus U$ so that $\mathrm{coh}(C \setminus U) \leqslant n$ by Proposition 3.1.2.

Thus, taking $C = X$ in the inductive hypothesis and noting that $C \cap U$ is open in C, $\mathrm{coh}C = \mathrm{coh}((C \setminus U) \cup (C \cap U)) \leqslant n + k + 1$. But this was for an arbitrary nowhere dense subset of $A \cup U$ hence

$$\mathrm{coh}(A \cup U) \leqslant n + k + 2 = n + (k+1) + 1$$

By induction the lemma holds for all m.

□

Lemma 3.3.5 *If X is regular and coh$X = \omega$ then for all $n \in \omega$, there exist $C \subseteq U \in \tau X$ such that C is nowhere dense in X, coh$C = n$ and coh$(X \setminus U) = \omega$.*

Proof By Proposition 3.1.4, for X as in the statement of the lemma and some $n \in \omega$, there exists $A \subseteq X$ which is closed and nowhere dense in X such that $\mathrm{coh}A = n$. If there exists a $U \in \tau X$ such that $A \subseteq U$ and $\mathrm{coh}(X \setminus U) = \omega$ then simply take $C = A$. Otherwise, assume that for all open sets U in X which contain A, $\mathrm{coh}(X \setminus U) < \omega$. Taking $X \setminus U$ to be A in Lemma 3.3.4, if $\mathrm{coh}U < \omega$ then $\mathrm{coh}X < \omega$. Hence $\mathrm{coh}U = \omega$ for all such U. Define $\mathcal{U}$ to be the collection of all open sets containing A and index this set by Λ.

Claim:

$$A = \bigcap_{\lambda \in \Lambda} \overline{U_\lambda}$$

Certainly $A \subseteq \bigcap_{\lambda \in \Lambda} \overline{U_\lambda}$ so consider $x \notin A$. By regularity, there exists a $\lambda \in \Lambda$ such that $A \subseteq U_\lambda \subseteq \overline{U_\lambda} \subseteq X \setminus \{x\}$. But then $x \notin \overline{U_\lambda}$ and moreover $x \notin \bigcap_{\lambda \in \Lambda} \overline{U_\lambda}$. Hence $\bigcap_{\lambda \in \Lambda} \overline{U_\lambda} \subseteq A$ and we have our claim.

Suppose now that, for all $\lambda \in \Lambda$ and some $M \in \omega, \mathrm{coh}(X \setminus U_\lambda) \leqslant M$.

The set $\{X \setminus \overline{U_\lambda} : \lambda \in \Lambda\}$ is an open cover for $X \setminus A$ by the Claim. Thus, by Proposition 3.1.5, $\mathrm{coh}(X \setminus A) \leqslant M$. But we now have that $\mathrm{coh}A = n$, $\mathrm{coh}(X \setminus A) \leqslant M$ and $X \setminus A$ is open in X. Hence, by Lemma 3.3.4, $\mathrm{coh}((X \setminus A) \cup A) \leqslant M + n + 1$ or in other words, $\mathrm{coh}X < \omega$ which is a contradiction.

Therefore, for the given n, there exists $\lambda \in \Lambda$ such that $\mathrm{coh}(X \setminus \overline{U_\lambda}) \geqslant n+1$. By Proposition 3.1.4, take C to be a subset of $X \setminus \overline{U_\lambda}$ which is nowhere dense in $X \setminus \overline{U_\lambda}$ and for which $\mathrm{coh}C = n$. This gives that

$$A \subseteq U_\lambda \subseteq \overline{U_\lambda} \subseteq X \setminus C$$

Define $U = X \setminus \overline{U_\lambda}$ so that $C \subseteq U$, $\mathrm{coh}C = n$ and $\omega \geqslant \mathrm{coh}(X \setminus U) = \mathrm{coh}\overline{U_\lambda} \geqslant \mathrm{coh}U_\lambda = \omega$. C is also nowhere dense in X and hence C and U are the sets which satisfy the lemma.

□

This last lemma allows us to find nowhere dense subsets of a space of cohesion ω of each finite cohesion, which are sufficiently well separated so that their union is still nowhere dense. But then, this nowhere dense subset has cohesion ω and this is the set we require for the contradiction. The details are as follows:

Proof of Theorem 3.3.3 Suppose X is a regular space of cohesion ω. First of all we construct nowhere dense subsets of X of each finite cohesion in a particularly nice way. By Lemma 3.3.5, we can find $C_0, U_0 \subseteq X$ where C_0 is nowhere dense in X, $\mathrm{coh}C_0 = 0$, U_0 is open in X, $C_0 \subseteq U_0$ and $\mathrm{coh}(X \setminus U_0) = \omega$.

We now define inductively $C_k, U_k \subseteq X$ such that:

1. C_k is nowhere dense in X
2. $\mathrm{coh}C_k = k$
3. U_k is open with $C_k \subseteq U_k$

4. $\mathrm{coh}(X \setminus U_k) = \omega$
5. $U_i \subseteq U_{i+1}$ for $i = 0, 1, 2, \ldots, k-1$
6. $C_{i+1} \subseteq X \setminus U_i$ for $i = 0, 1, 2, \ldots, k-1$

Assume that, for $i \leqslant n$, C_i and U_i have been defined satisfying the inductive assumptions. Define $C_{n+1}, V \subseteq X \setminus U_n$ by applying Lemma 3.3.5, so that C_{n+1} is nowhere dense subset of $X \setminus U_n$, and hence of X, contained in the set V open in $X \setminus U_n$ such that $\mathrm{coh}C_{n+1} = n+1$ and $\mathrm{coh}((X \setminus U_n) \setminus V) = \omega$.

Take V' to be a set open in X such that $V = V' \cap (X \setminus U_n)$. Take $U_{n+1} = V' \cup U_n$. It is easy to see from their definitions that C_{n+1} and U_{n+1} satisfy all the inductive conditions for $k = n+1$ except possibly (4). But note

$$X \setminus U_{n+1} = X \setminus (V' \cup U_n) = (X \setminus U_n) \setminus V' = (X \setminus U_n) \setminus V$$

Therefore,

$$\mathrm{coh}(X \setminus U_{n+1}) = \mathrm{coh}((X \setminus U_n) \setminus V) = \omega$$

Thus C_{n+1} and U_{n+1} are sets satisfying all of the inductive conditions for $k = n+1$ and this completes the induction.

Define $C = \bigcup_{n=0}^{\infty} C_n$. Clearly $\mathrm{coh}C \geqslant \mathrm{coh}C_n$ for all $n \in \omega$ and $C \subseteq X$ so $\mathrm{coh}C = \omega$.

It remains to show that C is nowhere dense in X and we have our contradiction. Suppose not then there is an open set U of X such that $U \subseteq \overline{C}$. Thus $U \cap C \neq \emptyset$ and therefore, for some $n \in \omega$, $U \cap C_n \neq \emptyset$. Since $C_n \subseteq U_n$, $V = U \cap U_n$ is a non-empty open set in X. Moreover, for all $i \geqslant n+1$, $C_{i+1} \subseteq X \setminus U_i \subseteq X \setminus U_n$ by (5) and (6) of the inductive assumptions. Thus $U_n \cap C_{i+1} = \emptyset$ for all $i \geqslant n$. That is, $U_n \cap \bigcup_{i=n+1}^{\infty} C_i = \emptyset$ and therefore, we have ($\diamondsuit$)

$$U_n \cap \overline{\bigcup_{i=n+1}^{\infty} C_i} = \emptyset$$

Now $U \subseteq \overline{C}$ hence $V \subseteq \overline{C}$ or, in other words,

$$V \subseteq \overline{C_0 \cup C_1 \cup \ldots \cup C_n} \cup \overline{\bigcup_{i=n+1}^{\infty} C_i}$$

But then ($\diamondsuit$) implies

$$V \subseteq \overline{C_0 \cup C_1 \cup \ldots \cup C_n}$$

This means that the closure of the union of the first n of the C_k contains a non-empty open set and hence the union of the first n of the C_k is not nowhere dense. This contradicts the fact that a finite union of nowhere dense sets is nowhere dense.

Thus C must be nowhere dense in X and we can conclude that there is no regular space of transfinite cohesion.

$\square$

Given this result, we may now feel justified in upgrading Lemma 3.3.4 to give a theorem very much like a sum theorem in dimension theory.

Theorem 3.3.6 *If A and B are subsets of some space X, at least one of which is closed, such that coh$A \leqslant n$, coh$B \leqslant m$ and $A \cup B = X$ then coh$X \leqslant n + m + 1$.*

3.4 The cohesion of compacta

In the previous chapter, it was shown that in the presence of compactness boundary properties are well-behaved and manageable. The same is true of cohesion. Arhangel'skiĭ asked whether every compact cohesive space is scattered and we provide here a positive answer. As the first stage in proving this, we have:

Lemma 3.4.1 *There is no compact crowded space of cohesion 1.*

Proof Suppose X is compact, crowded and $\mathrm{coh}X = 1$. If $C \subseteq X$ is closed and nowhere dense then C is compact and discrete therefore finite. However, X is crowded and Hausdorff so it is easy to find a countably infinite cellular family $\mathcal{U}$. Choose for each $U \in \mathcal{U}$, a point x_U of U. $\{x_U : U \in \mathcal{U}\}$ is infinite and nowhere dense, by Proposition 1.2.5. Thus $\overline{\{x_U : U \in \mathcal{U}\}}$ is finite which is contradictory. Thus there is no such X.

□

Applying Theorem 2.5.4 and using the previous Lemma as a base step, we inductively show:

Proposition 3.4.2 *Every cohesive compactum is scattered.*

Proof By Theorem 3.3.3, we need only prove that every finitely cohesive compactum is scattered. By Lemma 3.4.1, when $\mathrm{coh}X = 1$, X is scattered.

Assume that, for $k \in \omega$, if X is a compact space such that $\mathrm{coh}X \leqslant k$ then X is scattered. Consider a compactum X for which $\mathrm{coh}X = k + 1$. If $C \subseteq X$ is closed and nowhere dense then C is compact and $\mathrm{coh}C \leqslant k$. By the inductive hypothesis, C is scattered. Therefore, X is a compact boundary-scattered space and, by Corollary 2.5.4, X is itself scattered. This completes the induction and the proof.

□

Given this result, it is natural to ask how cohesion behaves in the presence of other covering properties. Examples 3.3.2 demonstrate that the Lindelöf property does not induce scatteredness. Generalising in a different direction, it is natural to consider local compactness. Using the Alexandroff compactification, this can be promoted to compactness and cohesion still behaves well as the following proposition shows. Because locally compact Hausdorff spaces are regular, we still only consider finitely cohesive spaces.

Proposition 3.4.3 *If X is a locally compact Hausdorff space with coh$X \leqslant n$, for some $n \in \omega$, and X^* denotes the one-point compactification of X then coh$X^* \leqslant n+1$.*

Proof For a space X define $X^* = X \cup \{\Omega\}$ for some $\Omega \notin X$ and a topology on X^* by

$$\tau X^* = \tau X \cup \{(X \setminus F) \cup \{\Omega\} : F \subseteq X \text{ and } F \text{ is compact}\}$$

It is well-known that if X is T_2 and is locally compact then X^* is compact with the given topology. Moreover, if X is not compact then X is embedded as a dense subset in X^*.

Assume that it has been shown for all locally compact Hausdorff spaces X with $\mathrm{coh}X \leqslant m$ where $m < n$ that $\mathrm{coh}X^* \leqslant m + 1$. Consider a locally compact Hausdorff space X such that $\mathrm{coh}X = n$. (In the cases where $\mathrm{coh}X < n$ the theorem follows by the induction hypothesis.)

Suppose $C \subseteq X^*$ is nowhere dense in X^*. Then $D = \overline{C}^{X^*}$ is also nowhere dense in X^*. Define $B = D \cap X$. As D is closed in X^* so B is closed in X. If there is a non-empty set $U \in \tau X$ such that $U \subseteq B$ then $U \in \tau X^*$ and $U \subseteq D$ which contradicts the fact that D is nowhere dense. Hence B is nowhere dense in X and $\mathrm{coh}B = m$ for some $m < n$.

If $\Omega \notin D$ then $C \subseteq D = B \subseteq X$ and, by Theorem 3.1.2, $\mathrm{coh}C \leqslant m < n$.

Thus suppose $\Omega \in C$. As B is closed in X, it is T_2 and locally compact. It is not too difficult to see that τB^*, the topology on $B^*(= D)$ coincides with the topology induced on D by τX^*.

If B is not compact then D is the one-point compactification of B and hence by the inductive hypothesis, as $\text{coh}B \leqslant m < n$ then $\text{coh}D \leqslant m + 1 \leqslant n$. If B is compact then from the definition of τB^* it is clear that Ω is an isolated point of D. But then any nowhere dense subset of D must not contain Ω and hence is a nowhere dense subset of B. Since $\text{coh}B = m$, the cohesion of any such subset is strictly less than m. This means that $\text{coh}D \leqslant m < n$.

Hence overall $\text{coh}D \leqslant n$ and $C \subseteq D$ so $\text{coh}C \leqslant n$. But then from the definition of cohesion it follows that $\text{coh}X^* \leqslant n + 1$.

□

This immediately gives:

Corollary 3.4.4 *Every locally compact cohesive space is scattered.*

Proof Suppose X is a locally compact cohesive space. By the previous result, the one-point compactification of X, X^*, is also cohesive. Proposition 3.4.2 implies that X^* is scattered and, as the subset of a scattered space, X is scattered.

□

3.5 Preserving cohesion

We have already seen a few methods for constructing cohesive spaces. These however are not amongst the more commonly used techniques for building topological spaces. We therefore examine in this section the behaviour of cohesion under the more familiar constructions of taking continuous images and products.

It is immediately clear that cohesion is not preserved under arbitrary continuous mappings.

Examples 3.5.1 Let $f : \omega \to \Theta$ be a denumeration of van Douwen's nodec space, then f is a continuous bijection. However, $\text{coh}\omega = 0$ and $\text{coh}\Theta = 1$; so, continuous maps in general do not lower cohesion.

Moreover, if $g : \omega \to \mathbb{Q}$ is a denumeration of the rationals, then it is a continuous bijection with domain having cohesion 0 but for which the image is not even cohesive!

□

In order to maintain cohesion under continuous maps, it is necessary to ensure that the nowhere dense subsets of the image are related to the nowhere dense subsets of the domain. This can be done by constraining in some way the behaviour of the images of the open sets under the mapping. Both open maps and perfect maps will do this and it transpires that they also constrain the behaviour of cohesion in the process.

Theorem 3.5.2 *If $f : X \to Y$ is an open, continuous surjection and coh$X \leqslant \alpha$, for some ordinal α, then coh$Y \leqslant \alpha$.*

Proof The proof is by transfinite induction.

If $\text{coh}X = -1$ then X is empty and f is surjective so it must be that Y is empty and hence $\text{coh}Y = -1$.

Thus assume that the theorem holds for all ordinals $\beta < \alpha$ and that $\text{coh}X = \alpha$. Consider $C \subseteq Y$ which is nowhere dense in Y. If $f^{-1}(C)$ is not nowhere dense in X then there exists $U \in \tau X$ such that $U \subseteq \overline{f^{-1}(C)}$. But f is continuous so $\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C})$. Hence $U \subseteq f^{-1}(\overline{C})$ and $f(U) \subseteq \overline{C}$. But f is open so $f(U)$ is open and non-empty in Y giving $\text{int}_Y \overline{C}^Y \neq \emptyset$, contradicting the fact that C is nowhere dense.

Therefore $f^{-1}(C)$ is nowhere dense in X and coh$f^{-1}(C) < \alpha$. Define $g = f\mid_{f^{-1}(C)}$ so that $g : f^{-1}(C) \to C$ is a continuous surjection. If $V \subseteq f^{-1}(C)$ is open in $f^{-1}(C)$ then $V = U \cap f^{-1}(C)$ for some U open in X. However,

$$g(V) = f(U \cap f^{-1}(C)) = f(U) \cap f(f^{-1}(C)) = f(U) \cap C$$

and $f(U)$ is open in Y so $g(V)$ is open in C.

Now, from the inductive hypothesis, coh$C < \alpha$. But this is for an arbitrary nowhere dense subset of Y hence coh$Y \leqslant \alpha$.

□

Theorem 3.5.3 *If $f : X \to Y$ is perfect and coh$X \leqslant \alpha$, for some ordinal α, then coh$Y \leqslant \alpha$.*

Proof Assume for the purposes of induction that, for all ordinals $\beta < \alpha$, the theorem is true and consider X such that coh$X = \alpha$.

Take $A \subseteq X$ and $g = f\mid_A : A \to Y$ as given in Proposition 1.2.10. If C is a nowhere dense subset of Y so too is $D = \overline{C}^Y$. If $g^{-1}(D)$ is not nowhere dense in A then there exists a non-empty open set $U \subseteq A$ such that $U \subseteq g^{-1}(D)(= \overline{g^{-1}(D)}^X$ as D is closed and g is continuous). However, by Proposition 1.2.9, $g^*(U)$ is non-empty and open in Y since g is closed and irreducible. Also $g^*(U) \subseteq g(U) \subseteq D$ which contradicts the fact that D is nowhere dense in Y. Therefore $g^{-1}(D)$ is nowhere dense in X and hence, for some $\beta < \alpha$, coh$(g^{-1}(D)) \leqslant \beta < \alpha$.

Define $h = g\mid_{g^{-1}(D)} : g^{-1}(D) \to D$. h is clearly a continuous surjection. As $g^{-1}(D)$ is closed it follows that h is perfect. Hence by the induction hypothesis coh$D \leqslant \beta < \alpha$. Since $C \subseteq D$, by Theorem 3.1.2, coh$C < \alpha$. C was an arbitrary nowhere dense subset of Y so this implies coh$Y \leqslant \alpha$.

□

Preserving cohesion in products of cohesive spaces is quite complex as the nowhere dense subsets of a product need have almost no relationship with the nowhere dense subsets of the factor spaces. This is clearly demonstrated in the next result.

Theorem 3.5.4 *If Θ is a crowded, boundary-discrete space refining the rationals then Θ^2 is not cohesive.*

Proof Assume for contradiction that Θ^2 is cohesive and hence we are able to discuss the cohesion of its subsets. If Θ is regular, it suffices to show that coh$\Theta^2 \geqslant \omega$ in order to obtain a contradiction. However, we will actually obtain a contradiction in the more general case but we must work a little harder to do this.

Throughout the proof, $\mathbb{Q}$ is the rationals with the usual topology. d is the usual metric on $\mathbb{Q}$. Θ is the same underlying set but with the finer nodec topology. Thus, all subsets of Θ^2 have the subspace topology induced by the nodec topology of Θ and the usual Tychonoff topology of products.

The first step is to construct inside Θ^2 subsets C_k for each $k \in \omega$ where coh$C_k \geqslant k$. These subsets are built up inductively from each other, however it is useful to be able to place them precisely where they are needed. This is gives rise to the following inductive hypotheses:

Fix $k \in \omega$, for every pair of open intervals U and V in $\mathbb{Q}$ and $y \in V$, there exist $C_i(U, V, y) \subseteq U \times V$, for $i = 1, 2, \ldots, k$, such that

1. $C_1(U, V, y) = U \times \{y\}$
2. $C_i(U, V, y)$ is closed in $U \times V$ for $i = 1, 2, \ldots, k$
3. $C_i(U, V, y)$ is a nowhere dense subset of $C_{i+1}(U, V, y)$ for $i = 1, 2, \ldots, k - 1$
4. $C_k(U, V, y)$ is nowhere dense in $U \times V$

5. $k \leqslant \mathrm{coh} C_k(U, V, y) \leqslant 2k - 1$

For all open intervals U and V in $\mathbb{Q}$ and any point $y \in V$, when $k = 1$ simply take (1) as the definition of $C_1(U, V, y)$.

Assuming we have found such C_i's for $i = 1, 2, \ldots, k$, we shall now construct the C_{k+1}'s. In order to simplify the construction, we shall only construct $C_{k+1}(U, V, y)$ in the case $U = V = \Theta$ and, moreover, we shall take $C_1 = \{0\} \times \Theta$. It is easy to see how to re-phrase this in order to produce C_{k+1}'s satisfying the inductive hypotheses.

Define $U_n = (\frac{-1}{n\sqrt{2}}, \frac{1}{n\sqrt{2}})$, a clopen interval in $\mathbb{Q}$, for each $n \in \omega \setminus \{0\}$ and $U_0 = \mathbb{Q}$. As Θ refines $\mathbb{Q}$, U_n is clopen in Θ for all $n \in \omega$. Now denumerate $\Theta = \{y_n : n \in \omega\}$ and for each $n \in \omega$, separate the first n points of Θ by V_{nj}, clopen intervals in $\mathbb{Q}$, such that $y_j \in V_{nj}$ and $V_{nj} \cap V_{ni} = \emptyset$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. Finally, define $C_1 = \{0\} \times \Theta$ and, for $j = 2, \ldots, k + 1$,

$$C_j = C_1 \cup \bigcup_{n \in \omega} \bigcup_{i=1}^{n} C_{j-1}(U_n \setminus U_{n+1}, V_{ni}, y_i)$$

For $j = 1, 2, \ldots, k$, $C_j \subseteq C_{j+1}$ by (3) of the inductive hypothesis.

C_1 is defined to have the form of (1) in the inductive hypothesis and is clearly closed in Θ^2.

For $j \in \{2, \ldots, k + 1\}$ and $\langle x, y \rangle \notin C_j$, as $x \neq 0$, there exists $n \in \omega$ such that $x \in U_n \setminus U_{n+1}$. Consider the two cases: when for some $i \leqslant n$, $\langle x, y \rangle \in (U_n \setminus U_{n+1}) \times V_{ni}$ and when there are no such n and i. In the first case, $\langle x, y \rangle \notin C_{j-1}(U_n \setminus U_{n+1}, V_{ni}, y_i)$ but this is closed in $(U_n \setminus U_{n+1}) \times V_{ni}$ so there exists W open in $(U_n \setminus U_{n+1}) \times V_{ni}$ containing $\langle x, y \rangle$ for which $W \cap C_{j-1}(U_n \setminus U_{n+1}, V_{ni}, y_i) = \emptyset$. But then W is open in Θ^2 and, since $((U_n \setminus U_{n+1}) \times V_{ni}) \cap ((U_m \setminus U_{m+1}) \times V_{mj}) = \emptyset$ whenever either $m \neq n$ or $i \neq j$, we have that $W \cap C_j = \emptyset$. In the second case, $\bigcup_{i=1}^{n} V_{ni}$ is closed as the finite union of closed sets so there exists $W \in \tau\Theta$ for which $y \in W$ and $W \cap \bigcup_{i=1}^{n} V_{ni} = \emptyset$. This means that $\langle x, y \rangle \in (U_n \setminus U_{n+1}) \times W$ and $((U_n \setminus U_{n+1}) \times W) \cap C_j = \emptyset$. Hence, C_j must be closed.

In order to show that C_j is nowhere dense in C_{j+1} for $j = 1, 2, \ldots, k$, since the C_j are closed, it is enough to show that if U, $V \in \tau\Theta$ and $(U \times V) \cap C_j \neq \emptyset$ then $(U \times V) \cap (C_{j+1} \setminus C_j) \neq \emptyset$. Consider first U and $V \in \tau\Theta$ for which $(U \times V) \cap C_1 \neq \emptyset$. Choose an $n \in \omega$ such that $\langle 0, y_n \rangle \in U \times V$. U is a neighbourhood of 0 as is U_n and 0 is not an isolated point of Θ hence there exists $x \in (U \cap U_n) \setminus \{0\}$. This implies that $\langle x, y_n \rangle \in (U \times V) \cap (C_2 \setminus C_1)$ and C_1 is nowhere dense in C_2.

For some $j \in \{2, \ldots, k\}$, assume that U, $V \in \tau\Theta$ are such that $U \times V$ meets C_j. It has already been shown that $(U \times V) \cap (C_2 \setminus C_1) \neq \emptyset$ which means that there are suitable i and $n \in \omega$ for which $(U \times V) \cap C_1(U_n \setminus U_{n-1}, V_{ni}, y_i) \neq \emptyset$. $C_1(U_n \setminus U_{n-1}, V_{ni}, y_i) \subseteq C_{j-1}(U_n \setminus U_{n-1}, V_{ni}, y_i)$ so $U \times V$ must meet $C_{j-1}(U_n \setminus U_{n-1}, V_{ni}, y_i)$. By the inductive hypothesis, $C_{j-1}(U_n \setminus U_{n-1}, V_{ni}, y_i)$ is nowhere dense in $C_j(U_n \setminus U_{n-1}, V_{ni}, y_i)$. Thus,

$$(U \times V) \cap (C_j(U_n \setminus U_{n-1}, V_{ni}, y_i) \setminus C_{j-1}(U_n \setminus U_{n-1}, V_{ni}, y_i)) \neq \emptyset$$

From the definition of C_j it follows that $(U \times V) \cap (C_{j+1} \setminus C_j) \neq \emptyset$ and C_j is nowhere dense in C_{j+1}.

Showing that C_{k+1} is nowhere dense in Θ^2 can be done in a way similar to that used to show C_j is nowhere dense in C_{j+1} but the following is a little more slick. Suppose U, $V \in \tau\Theta$ are such that $(U \times V) \cap C_1 \neq \emptyset$. Define $\mathfrak{d}(U) = \sup\{d(x, x') : x,\ x' \in U\}$ if the supremum exists and $\mathfrak{d}(U) = 1$ otherwise. V is infinite so there exists $n \in \omega$ for which $y_n \in V$ and $\frac{1}{n} < \mathfrak{d}(U)$. Thus there exists $x \in U \setminus U_n$ and $\langle x, y_n \rangle \in (U \times V) \setminus C_{k+1}$. If $(U \times V) \cap C_{k+1} \neq \emptyset$ but does not meet C_1 then for some n and $i \in \omega$, $(U \times V) \cap C_k(U_n \setminus U_{n+1}, V_{ni}, y_i) \neq \emptyset$. That $U \times V$ does not lie in C_{k+1} now follows from the inductive hypothesis that $C_k(U_n \setminus U_{n+1}, V_{ni}, y_i)$ is nowhere dense in $U_n \times V_{ni}$.

It remains to check the cohesion of C_{k+1}. Since the C_j's form a chain of sets nowhere dense in their successors and $\mathrm{coh} C_1 = 1$, a simple induction shows that $\mathrm{coh} C_k \geqslant k$. Moreover, $C_{k+1} \cap ((U_n \setminus U_{n+1}) \times V_{ni}) = C_k(U_n \setminus U_{n+1}, V_{ni}, y_i)$ for n, $i \in \omega$, hence

$$\{C_k(U_n \setminus U_{n+1}, V_{ni}, y_i) : n \in \omega,\ i = 1, 2, \ldots, n\}$$

is a cover of $C_{k+1} \setminus C_1$ by sets open in C_{k+1}. Proposition 3.1.5 and (5) of the inductive hypothesis together imply that $\mathrm{coh}(C_{k+1} \setminus C_1) \leqslant 2k-1$. By Theorem 3.3.6, $\mathrm{coh}C_{k+1} \leqslant \mathrm{coh}C_1 + \mathrm{coh}(C_{k+1} \setminus C_1) + 1$, that is $\mathrm{coh}C_{k+1} \leqslant 1 + (2k-1) + 1 = 2(k+1) - 1$. This completes the inductive construction.

Remark Thus far, we have found in Θ^2 subsets whose cohesions are finite but unbounded in ω so $\mathrm{coh}\Theta^2 \geqslant \omega$. If Θ were regular we could stop here. However, by carefully placing the C_k we can achieve the required contradiction without assuming regularity.

For each $n \in \omega$, define $W_n = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + \frac{1}{n})$ a clopen interval in $\mathbb{Q}$ and hence clopen in Θ. Choose y_n to be some fixed point in W_n. Now define

$$C = \bigcup_{n \in \omega} C_n(\Theta, W_n, y_n)$$

As the W_n are a closed discrete family in $\mathbb{Q}$, they are a closed discrete family in Θ and hence $\{\Theta \times W_n : n \in \omega\}$ is a closed discrete family in Θ^2. This implies that $C_n(\Theta, W_n, y_n)$ are a closed discrete family in Θ^2 and C is closed in Θ. Moreover, the $C_n(\Theta, W_n, y_n)$ are subsets of C whose cohesions are unbounded in ω hence $\mathrm{coh}C \geqslant \omega$. By Proposition 3.1.4, find $D \subseteq C$ which is closed and has cohesion precisely ω.

$\{D \cap C_n(\Theta, W_n, y_n) : n \in \omega\}$ form an open cover of D because $D \cap (\Theta \times W_n) = D \cap C_n(\Theta, W_n, y_n)$ and $\Theta \times W_n \in \tau\Theta^2$ for all $n \in \omega$. Also, the elements of the cover are finitely cohesive. This means that $\{\mathrm{coh}(D \cap C_n(\Theta, W_n, y_n)) : n \in \omega\}$ is unbounded in ω otherwise Proposition 3.1.5 gives that $\mathrm{coh}D < \omega$. Thus, for all $k \in \omega$, there exists $n_k \geqslant k$ for which $D \cap C_{n_k}(\Theta, W_{n_k}, y_{n_k}) > k$ and, again applying Proposition 3.1.4, find E_k which is closed and nowhere dense in $D \cap C_{n_k}(\Theta, W_{n_k}, y_{n_k})$ and for which $\mathrm{coh}E_k \geqslant k$.

Define $E = \bigcup_{k \in \omega} E_k$. As a discrete union of closed sets (in the same way that D is), E is closed. Suppose E were not nowhere dense then for some $U, V \in \tau\Theta$, $(U \times V) \cap D \subseteq E$ where $(U \times V) \cap E$ is non-empty. But this implies that for some $k \in \omega$, $(U \times V) \cap C_{n_k}(\Theta, W_{n_k}, y_{n_k}) \neq \emptyset$ and

$$(U \times V) \cap C_{n_k}(\Theta, W_{n_k}, y_{n_k}) \subseteq E \cap C_{n_k}(\Theta, W_{n_k}, y_{n_k}) = E_k$$

In other words, E_k is not nowhere dense in $C_{n_k}(\Theta, W_{n_k}, y_{n_k})$. This is a contradiction on the definition of E_k. Hence E is nowhere dense in D which implies that $\mathrm{coh}E < \omega$.

From the definition of the E_k's, $\{\mathrm{coh}E_k : k \in \omega\}$ is an unbounded set in ω and, therefore, $\mathrm{coh}E \geqslant \omega$. Thus, we have our contradiction and Θ^2 is not cohesive.

□

Remark The C_k generated in the above proof are finitely cohesive by the inductive assumption and they are clearly crowded. They do not, however, supercede Examples 3.3.2 as we cannot guarantee that, for all $n \in \omega$, there exists a $k \in \omega$ for which $\mathrm{coh}C_k = n$ as we can for the previous examples.

However, when one of the factors is scattered, the nowhere dense subsets can be specified more precisely than in general. Thus, we have:

Proposition 3.5.5 *If X and Y are finitely cohesive spaces for which Y is scattered then $X \times Y$ is cohesive and coh$(X \times Y) \leqslant ($coh$X + 1)($coh$Y) +$ coh$X$.*

Proof We induct on the cohesion of Y and, we assume that Y is finitely cohesive so the induction is only of length ω.

First, consider the case when $\mathrm{coh}Y = 0$ so that Y is discrete. Clearly, $\{X \times \{y\} : y \in Y\}$ is an open cover of $X \times Y$ every member of which has the same cohesion as X. Thus, by Proposition 3.1.5,

$$\mathrm{coh}(X \times Y) \leqslant \mathrm{coh}X = (\mathrm{coh}X + 1)(\mathrm{coh}Y) + \mathrm{coh}X$$

Suppose now that, for $n \in \omega$, if $\mathrm{coh}Y \leqslant n$ then $X \times Y$ is cohesive and $\mathrm{coh}(X \times Y) = (\mathrm{coh}X + 1)(\mathrm{coh}Y) + \mathrm{coh}X$ and assume that $\mathrm{coh}Y = n+1$. Divide $X \times Y$ into $A = X \times (Y \setminus Y^d)$ and $B = X \times Y^d$. B is clearly a closed subset of $X \times Y$. Since $\mathrm{sl}(Y^d) = \mathrm{sl}(Y) - 1$, Theorem 3.2.2 implies that $\mathrm{coh}Y^d = n$. Applying the inductive hypothesis, $\mathrm{coh}B \leqslant (\mathrm{coh}X + 1)n + \mathrm{coh}X$ and by the base step case $\mathrm{coh}A = \mathrm{coh}X$. Proposition 3.3.6 then gives

$$\mathrm{coh}(X \times Y) \leqslant \mathrm{coh}X + ((\mathrm{coh}X + 1)n + \mathrm{coh}X) + 1$$

from which it follows that $\mathrm{coh}(X \times Y) \leqslant (\mathrm{coh}X + 1)(n+1) + \mathrm{coh}X$. This proves the case when $\mathrm{coh}Y = n+1$ and completes the induction.

□

By considering the case when X and Y are both scattered cohesive spaces, it can be seen that the bound given in Proposition 3.5.5 is optimal. This follows by noting that $\mathrm{sl}(X \times Y) = \mathrm{sl}(X) \cdot \mathrm{sl}(Y)$ and then, by Theorem 3.2.2, this means that $\mathrm{coh}(X \times Y) + 1 = (\mathrm{coh}X + 1)(\mathrm{coh}Y + 1)$. Simplifying this expression, we find that $X \times Y$ attains the bound on its cohesion.

3.6 Absolute dimension

Arhangel'skiĭ defined absolute dimension as a tool by which to examine spaces which are cleavable over the reals.

Definition 3.6.1 [$\mathbf{A}_2$] The *absolute dimension* of a space X, denoted $adimX$ is defined inductively to be:

$$adimX = -1 \text{ if and only if } X = \emptyset$$

$$adimX = 0 \text{ if and only if } indX = 0$$

$$\text{for } n \in \omega \setminus \{0\},\ adimX \leqslant n \text{ if for every nowhere dense subset } C \text{ of } X,\ adimC < n$$

For a space X, $adimX = n$ means that $adimX \leqslant n$ but for any $k \in \omega$ such that $k < n$ it is not true that $adimX \leqslant k$.

Arhangel'skiĭ went on from this definition to show that a space which is cleavable over the reals has an absolute dimension of at most one. For our purposes it is sufficient to know that every subset of $\mathbb{R}$ is cleavable over the reals. It is straightforward to see that for a space X, if $adimX = n$ for some $n \in \omega$, then, for every $A \subseteq X$, $adimA \leqslant n$ holds.

The definition of cohesion is based on that of absolute dimension. So, it will come as no surprise that absolute dimension has rather different properties from the usual inductive dimension functions. In particular, absolute dimension does not agree with these functions on compact metric spaces. If I denotes the closed unit interval of $\mathbb{R}$, then we have:

Theorem 3.6.2 *$adimI^2$ is not defined.*

Proof Suppose $adimI^2$ is defined. We shall construct nowhere dense subsets C_n of I^2 for each $n \in \omega$ such that $adimC_n \geqslant n$. The definition of $adim$ then gives that $adimI^2 \geqslant n + 1$, for all $n \in \omega$, which obviously contradicts the fact that $adimI^2$ is defined.

Trivially $C_0 = \{\langle 0, 0\rangle\}$ satisfies the case when $n = 0$. Take $C_1 = I \times \{0\}$. C_1 is cleavable over the reals as it is embeddable in the real line and it is not empty or zero-dimensional so $adimC_1 = 1$. Clearly C_1 is closed and contains no open set in I^2 hence C_1 is nowhere dense in I^2.

Define $C_2 = I \times (\{0\} \cup \{\frac{1}{n} : n \in \omega \text{ and } n \geqslant 2\})$. This gives a sequence of lines converging down to C_1. As a product of two closed subsets of I, C_2 is closed in I^2 and clearly it cannot contain any open subset of I^2 so C_2 is nowhere dense in I^2. Any open set, U, in C_2 about a point $\langle x, 0\rangle \in I \times \{0\}$ contains an open ball of radius ε, for some $\varepsilon > 0$, so for all $n \in \omega$ such that $\frac{1}{n} \leqslant \varepsilon$, $\langle x, \frac{1}{n}\rangle \in U$. Thus C_1 contains no non-empty open subset of C_2 and is closed in C_2 so C_1 is nowhere dense in C_2. But $adimC_1 = 1$ hence $adimC_2 \geqslant 2$. ($adimC_2$ exists because of the assumption that $adimI^2$ exists.)

In general given C_k and noting that $\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$, define $C_{k+1} = C_k \cup \{\frac{1}{n_1} + \ldots + \frac{1}{n_k} : n_1 \geqslant 2, n_{i+1} \geqslant 2n_i(n_i - 1)$ for $i = 1, \ldots, k-1\}$. As before this gives a sequence of lines converging down to each line in C_k. It can be seen that C_{k+1} is closed and nowhere dense in I^2 (as a countable collection of horizontal lines, C_{k+1} cannot contain a non-empty open subset of I^2). As for C_1 in C_2, C_k is nowhere dense in C_{k+1} and hence $adimC_{k+1} \geqslant k+1$.

Therefore, for all $n \in \omega$, there is a $C_n \subseteq I^2$ such that $adimC_n \geqslant n$ which are the sets prophesied at the beginning of the proof and we are done.

□

3.7 Summary and questions

One of the important features of cohesion is its relationship to scattered spaces. The second section of the chapter shows that cohesive sequential spaces are scattered and the fourth that compact or even locally compact cohesive spaces are also scattered. Moreover, cohesive spaces which are scattered are only finitely cohesive as was seen in the second section. Whilst there are examples of crowded, finitely cohesive spaces which are regular and countable (and hence very well-behaved), it is also remarkable that there are no regular, transfinitely cohesive spaces. This raises the natural question:

Question 3.1 *Is there a transfinitely cohesive (Hausdorff) space?*

Such an example must not be regular, scattered or sequential and so may be quite a curiosity.

In the fifth section, it was seen that, provided one of the factors in a product is scattered a product of two cohesive spaces is cohesive and a bound for the cohesion of the product can be found. However, when both factors are crowded, the situation changes drastically and the square of a nodec space refining the rationals is not cohesive at all. Also, though not in general preserved by continuous maps, cohesion is preserved by open and perfect maps as they strongly control the behaviour of open sets under the mapping. It would therefore be interesting to find if other maps also preserve cohesion. For instance:

Question 3.2 *Is cohesion preserved under closed maps? quotient maps?*

The last theorem of the chapter proves that absolute dimension is not defined on the unit square. One of the important aspects of this result is that it uses techniques which were applied to cohesion. This demonstrates that varying the base case of cohesion might give useful notions (such as absolute dimension) but beyond the base case, the structure is very similar to that of cohesive spaces. Thus, minor alterations to the proofs and examples of the chapter would give corresponding results on any such variant.

Chapter 4

On compact monotonically normal spaces

Nikiel [N_1] has obtained a number of characterisations of arctic spaces and CICLOTS. However, these seem to bear little relation to the monotone normality structure of CICLOTS. Indeed, all that is known is that CICLOTS must be acyclic monotonically normal as LOTS are acyclic monotonically normal and this is preserved under closed maps. The Collins-Roscoe structuring mechanism, since its inception [CR], has been a powerful tool in the field of generalised metric spaces; in particular, in spaces related to monotonically normal spaces. The aim of this chapter is to analyse ordered spaces with respect to the structuring mechanism in order to find a structuring mechanism on CICLOTS which is as strong as possible. We hope that this will provide new insight into Nikiel's question.

The first section of this chapter defines the Collins-Roscoe structuring mechanism and gives a flavour of its strength and diversity by reviewing some key theorems. In the second section, we introduce the new property, linear chain (F), and show that it is held by all CICLOTS and all proto-metrizable spaces. We also observe that utter normality, which has recently been defined by Junnila, is implied by linear chain (F) and we extend Junnila's results. Finally, we summarise the chapter and raise some relevant questions.

4.1 The structuring mechanism and generalised metric spaces

Collins defined the structuring mechanism in order to abstract precisely the conditions used to show that separable metric spaces are second countable. He called his original condition (A) and, in [CR], (A) is shown to be equivalent to metrizability. There are many generalisations of (A) and the most general, in keeping with the notation of [CR], is called (F). A space X is said to satisfy condition (F) (or, more simply, is (F)) whenever there are an operator $V : X \times \tau X \to \tau X$ and, for every $x \in X$, families $\mathcal{W}(x)$ of subsets of X each containing x such that:

(F) for all $x \in X$ and $U \in \tau X$ such that $x \in U$, if $y \in V(x, U)$ then there exists $W \in \mathcal{W}(y)$ such that $x \in W \subseteq U$

(F) is so general that any space satisfies it! To see this, simply take $\mathcal{W}(y) = \{\{x, y\} : x \in X\}$ and $V(x, U) = U$. The strength of (F) comes only when further constraints are imposed on the $\mathcal{W}(x)$'s. There are three sorts of constraints: $\mathcal{W}(x)$ has a specified cardinality; $(\mathcal{W}(x), \supseteq)$ has a specified order structure, for example, being well-ordered; every $W \in \mathcal{W}(x)$ is of a certain type, for example, open. When the $\mathcal{W}(x)$ are taken to be countable, this is a special case of the structuring mechanism called (G) [CR]. If also each $\mathcal{W}(x) = \{W_n : n \in \omega\}$ and, for all $n \in \omega$, $W_{n+1} \subseteq W_n$ then the $\mathcal{W}(x)$ are said to be *decreasing*. The syntax of these conditions is: (ordering property of $(\mathcal{W}(x), \supseteq)$) (property of each $W \in \mathcal{W}(x)$) (F) or (G).

The structuring mechanism has been extensively studied, see for instance [CR], [CRRR], [MRRC] and [St$_2$] for some of the many important results in this area. We give here a sample of these results.

In the same spirit which gave rise to (A), we have:

Theorem 4.1.1 [**CRRR**] *If X is separable and open (G) then X is second countable.*

It is well known that X being open (G) is implied by X having a point countable base and that in many circumstances the converse holds, see [MRRC]. However, the following question remains open:

Question 4.1 *If X is open (G), does it have a point countable base?*

If open (G) is strengthened to be decreasing as well, we obtain an unusual metrization theorem:

Theorem 4.1.2 [**CRRR**] *A space is decreasing open (G) if and only if it is metrizable.*

Even without open-ness, decreasing (G) is an important condition:

Theorem 4.1.3 [**St$_2$**] *If a space is decreasing (G) then it has the Dugundji extension property.*

Despite the triviality of unrestricted (F), the addition of any constraints immediately gives useful notions.

Theorem 4.1.4 [**CRRR**] *If X is chain neighbourhood (F) or well-ordered (F) then X is hereditarily paracompact.*

In fact, chain (F) on its own implies acyclic monotone normality as the V operator in chain (F) is also an acyclic monotone normality operator. More surprisingly, the converse holds:

Theorem 4.1.5 [**MRRC**] *A space is chain (F) if and only if it is acyclic monotonically normal.*

A full discussion of the properties of chain (F) spaces is given in [MR]. It was also remarked there that the monotone normality operator defined in [HLZ] for GO-spaces is actually an acyclic monotone normality operator. Hence GO-spaces are chain (F). It is precisely this statement which provoked the next section.

4.2 The linear chain (F) condition

In the proof that a LOTS is (acyclic) monotonically normal, the Axiom of Choice is used to well-order the LOTS in question. In showing that acyclic monotone normality implies chain (F), Choice is again invoked, this time to extend a partial order induced by an acyclic monotone normality operator up to a total order. It is by combining these two instances of choice in showing that a LOTS is chain (F) that a much stronger version of chain (F) is defined. We call the new property *linear chain (F)* as it is derived from considering lines.

Definition 4.2.1 For a space X with an operator $V : X \times \tau X \to \tau X$ and, for each $x \in X$, a family $\mathcal{W}(x)$, X is *linear chain (F)* if the V and $\mathcal{W}$'s satisfy chain (F) in such a way that, for some $x \in U \in \tau X$, $y \in V(x, U)$, the $W \in \mathcal{W}(y)$ given by (F) also satisfies:

1. $x \in W \subseteq \overline{V(x,U)}$
2. $x \in \overline{\text{int}W}$ for $x \neq y$

Remarks In regular spaces, a simple argument shows that, for given $x \in U \in \tau X$, $\overline{V(x,U)}$ can be assumed to be contained in U. Thus, when checking that linear chain (F) holds in a regular space, it is sufficient to check condition (1) only as it then implies condition (F). For the remainder of this section, the notation of Definition 4.2.1 will be standard and, in any linear chain (F) space, V will be assumed to satisfy this regularity condition.

We now show that every ordered space is indeed linear chain (F). Because of the similarities with the proof that an acyclic monotonically normal space is chain (F), we try to follow as closely as possible the notation of [MRRC].

Theorem 4.2.2 *Every GO-space is linear chain (F).*

Proof Take $(X, <)$ to be a GO-space. Applying the Axiom of Choice, well-order X and denote the well-ordering by $\prec$. Any intervals are assumed to be the usual intervals in the natural GO-space ordering $<$. However, for $Y \subseteq X$, $\min Y$ denotes the least element of Y with respect to the well-ordering $\prec$.

For each $a \in X$, a further order is defined on X. This is used to construct the $\mathcal{W}(a)$ and hence guarantee that it is a chain. The order is given by:

$$x \lhd_a y \text{ if and only if } x \neq y \text{ and } \begin{cases} a \leqslant \min[x,y] & \text{for } x < y \\ a > \min[y,x] & \text{for } x > y \end{cases}$$

We need to check that this is indeed a total order on X, that is, an irreflexive, transitive relation with respect to which any pair of elements from X are comparable.

The irreflexivity of $\lhd_a$ follows immediately from its definition.

For transitivity, suppose x, y, $z \in X$ and that $(\star)$

$$x \lhd_a y \text{ and } y \lhd_a z$$

We must show that $x \lhd_a z$. There are six cases to consider:

1. $x < y < z$. Then $(\star)$ means that $a \leqslant \min[x,y]$ and that $a \leqslant \min[y,z]$. Clearly, $\min[x,z] = \min\{\min[x,y], \min[y,z]\}$ which implies that $a \leqslant \min[x,z]$. That is, $x \lhd_a z$.
2. $x < z < y$. In this case, $(\star)$ means that $a \leqslant \min[x,y]$ and $a > \min[z,y]$. But $a > \min[z,y] \geqslant \min[x,y] \geqslant a$ which is impossible. So this case does not occur.
3. $y < x < z$. Then $(\star)$ means that $a \leqslant \min[y,z]$. Either $\min[y,z] = \min[y,x]$ or $\min[x,z]$ and hence $a \leqslant \min[x,z]$. That is, $x \lhd_a z$.
4. $y < z < x$. This cannot happen for the same reasons as the second case.
5. $z < x < y$. Then $(\star)$ means that $a > \min[z,y]$ but $\min[z,x] \leqslant \min[z,y]$. Hence, $a > \min[z,x]$, that is, $x \lhd_a z$.
6. $z < y < x$. $x \lhd_a z$ follows similarly to the first case.

For totality, consider x, $y \in X$. If $x < y$ and $x \not\lhd_a y$ then $a \not\leqslant \min[x,y]$ so since $<$ is a total order, it must be that $a > \min[x,y]$ and thus $y \lhd_a x$. Similarly, if $x > y$ and $x \not\lhd_a y$ then $y \lhd_a x$. Hence, any two distinct elements of X are always comparable and $\lhd_a$ is a total order.

Now define $S_a(x) = \{y \in X : y \lhd_a x\} \cup \{x\}$. Explicitly,

$$S_a(x) = \{y < x : a \leqslant \min[y,x]\} \cup \{x\} \cup \{y > x : a > \min[x,y]\}$$

So if $a < x$, $S_a(x) = \{y < x : a \leqslant \min[y,x]\} \cup \{x\}$ and $[a,x] \subseteq S_a(x) \subseteq (\leftarrow, x]$. And if $a > x$, $S_a(x) = \{x\} \cup \{y > x : a > \min[x,y]\}$ giving that $[x,a] \subseteq S_a(x) \subseteq [x, \rightarrow)$. In both cases, $S_a(x)$ is a convex set.

Let $\mathcal{W}(a) = \{\overline{S_a(x)} : x \in X\}$. Given any $x, y \in X$, either $x \lhd_a y$ or $y \lhd_a x$ and so either $S_a(x) \subseteq S_a(y)$ or $S_a(y) \subseteq S_a(x)$ correspondingly. Since containment is preserved when taking closures, $\mathcal{W}(a)$ is a chain of closed sets. Also, obviously $a \lhd_a x$ for all $x \in X \setminus \{a\}$ therefore $a \in W$ for all $W \in \mathcal{W}(a)$. This gives the families of $\mathcal{W}(x)$'s. We must now define the corresponding V operator.

For $x \in U \in \tau X$, let U_x denote the convex component of U which contains x and $U_x^- = \{y \in U_x : y < x\}$ and $U_x^+ = \{y \in U_x : y > x\}$. Now define $V(x,U) \in \tau X$ by:

$$V(x,U) = \begin{cases} \{x\} & \text{if } x \text{ is isolated} \\ [x, \min U_x^+) & \text{if } x \in \overline{U_x^+} \setminus \overline{U_x^-} \\ (\min U_x^-, x] & \text{if } x \in \overline{U_x^-} \setminus \overline{U_x^+} \\ (\min U_x^-, \min U_x^+) & \text{if } x \in \overline{U_x^-} \cap \overline{U_x^+} \end{cases}$$

It is straightforward to check that $V(x,U)$ is indeed open in the GO-space topology.

Since $\min U_x^-$ and $\min U_x^+ \in U_x$ and $\overline{V(x,U)} \subseteq [\min U_x^-, \min U_x^+]$, it is clear that

$$x \in V(x,U) \subseteq \overline{V(x,U)} \subseteq U$$

Consider $a \in V(x,U)$. Take $W = \overline{S_a(x)} \in \mathcal{W}(a)$ and suppose first that $a < x$ so that $S_a(x) = \{y < x : a \leqslant \min[x,y]\} \cup \{x\}$. If $y < x$ but $y \notin \overline{V(x,U)}$ then, from the definition of $V(x,U)$, $y < \min U_x^-$. But this implies that $\min[y,x] \leqslant \min U_x < a$ since $a \in V(x,U)$. Therefore, $y \notin S_a(x)$. Hence, $S_a(x) \subseteq \overline{V(x,U)}$ and

$$x \in S_a(x) \subseteq W \subseteq \overline{V(x,U)} \subseteq U$$

Thus, conditions (F) and (1) hold for $a < x$.

Also, since $a < x$ and $a \in V(x,U)$, it must be the case that $x \in \overline{U_x^-}$. Thus, if $x \in T \in \tau X$, it follows that $T \cap U_x^- \neq \emptyset$ so $T \cap (a,x) \neq \emptyset$. But we know that $[a,x] \subseteq S_a(x)$, therefore $T \cap \operatorname{int} W \neq \emptyset$. That is, $x \in \overline{\operatorname{int} W}$.

If $x > a$ then the proof that W has the required properties follows in a similar fashion.

Finally, if $a = x$ then $W = S_x(x) = \{x\}$ satisfies the required properties. Thus, in all cases, conditions (1) and (2) hold and X is a linear chain (F) space.

□

Because of the strong interaction between the W's and the operator V, it is difficult for us to ascertain whether linear chain (F) is preserved under the usual topological constructions. First, with regards to hereditary properties, we do not know if a closed subspace of a linear chain (F) space is also linear chain (F). However we do have:

Proposition 4.2.3 *If X is linear chain (F) and U is a non-empty open subset of X then U is linear chain (F).*

Proof Define $V_U : U \times \tau U \to \tau U$ by $V_U(x,T) = V(x,T)$ whenever $x \in T \in \tau U$. This is valid as $T \in \tau U$ implies that $T \in \tau X$. For all $x \in U$, define $\mathcal{W}_U(x) = \{W \in \mathcal{W}(x) : \overline{W} \subseteq U\}$. These will witness that U is linear chain (F).

Clearly, for all $x \in U$, $\mathcal{W}_U(x) \subseteq \mathcal{W}(x)$ hence is a chain of sets containing x. Suppose now that $x \in T \in \tau U$ and that $y \in V_U(x,T)$. As $y \in V(x,T)$, there exists $W \in \mathcal{W}(y)$ such that $x \in W \subseteq \overline{V(x,T)}^X$ and, for

$x \neq y$, $x \in \overline{\mathrm{int}_X W}^X$. But $\overline{V(x,T)}^X \subseteq T \subseteq U$, by regularity, therefore $\overline{W} \subseteq U$. Thus, $W \in \mathcal{W}_U(x)$ and it is easy to see that

$$x \in W \subseteq \overline{V(x,T)}^X = \overline{V_U(x,T)}^U \subseteq T$$

that is, conditions (1) and (F) are satisfied.

Moreover, since $U \in \tau X$ and $\overline{W}^X \subseteq U$, $\overline{\mathrm{int}_X W}^X = \overline{\mathrm{int}_U W}^U$. Thus, condition (2) is also satisfied and U is linear chain (F).

□

Proposition 4.2.4 *If X is linear chain (F) and Y is a dense subspace of X then Y is linear chain (F).*

Proof If X is a linear chain (F) space, define the operator $V_Y : Y \times \tau Y \to \tau Y$ by: for all $y \in U \in \tau Y$, $V_Y(y,U) = V(y,U') \cap Y$ where U' is some open set in X such that $U = U' \cap Y$. Define also for each $y \in Y$, $\mathcal{W}_Y(y) = \{W \cap Y : W \in \mathcal{W}(y)\}$.

From this, it is clear that $\mathcal{W}_Y(y)$ is a chain of closed subsets of Y which contain y, that $V_Y(y,U) \in \tau Y$ and, by regularity, that $y \in V_Y(y,U) \subseteq \overline{V_Y(y,U)}^Y \subseteq U$. Now consider $z \in V_Y(y,U)$ where $y \in U \in \tau Y$. From the definition, $z \in V(y,U')$ so there exists a $W' \in \mathcal{W}(z)$ such that $y \in W' \subseteq \overline{V(y,U')}^X$ and, for $y \neq z$, $y \in \overline{\mathrm{int}_X W'}^X$.

Take $W = W' \cap Y$ so that $W \in \mathcal{W}_Y(z)$. Since Y is dense in X, $Y \cap \overline{V(y,U')}^X = \overline{Y \cap V(y,U')}^Y = \overline{V(y,U)}^Y$. Thus, $W = W' \cap Y \subseteq \overline{V(y,U')}^X \cap Y = \overline{V(y,U)}^Y$.

Moreover, if $T \in \tau Y$ such that $y \in T$, there exists $T' \in \tau X$ for which $T = T' \cap Y$ and $T' \cap \mathrm{int}_X W' \neq \emptyset$. Since Y is dense in X, this means that $T' \cap \mathrm{int}_X W' \cap Y \neq \emptyset$. That is, $T \cap \mathrm{int}_Y W \neq \emptyset$ and $y \in \overline{\mathrm{int}_Y W}^Y$. Hence, Y is linear chain (F).

□

With regard to taking continuous images, the structuring mechanism is not generally preserved by arbitrary maps. For an example of this, consider Theorem 4.1.2 and the fact that the continuous image of a metric space need not be metric. Certain mechanisms, though, are preserved under taking closed images and these are listed in [St$_2$]. Unfortunately, the method of proof given there does not obviously carry over to linear chain (F). However, we can preserve linear chain (F) under closed mappings if the mappings are also irreducible. The key to this is the next lemma.

Lemma 4.2.5 *If $f : X \to Y$ is an irreducible map, $V \in \tau X$ and $x \in \overline{V}^X$ then $f(x) \in \overline{f^*(V)}^Y$ and, hence, $f^*(\overline{V}^X) \subseteq \overline{f^*(V)}^Y$.*

Proof Suppose $x \in \overline{V}^X$ and that $f(x) \in U \in \tau Y$. Then $x \in f^{-1}(U)$ so that $f^{-1}(U) \cap V \neq \emptyset$. Let $T = f^{-1}(U) \cap V$. Since f is irreducible and T is non-empty and open in X, $f^*(T)$ is non-empty. Clearly, $f^*(T) \subseteq f^*(V) \cap f(f^{-1}(U)) = f^*(V) \cap U$. Hence, $U \cap f^*(V)$ is non-empty and

$$f(x) \in \overline{f^*(V)}^Y$$

Now if $y \in f^*(\overline{V}^X)$ then there exists $x \in f^{-1}(y) \cap \overline{V}^X$. The above gives that $y = f(x) \in \overline{f^*(V)}^Y$. Thus, $f^*(\overline{V}^X) \subseteq \overline{f^*(V)}^Y$.

□

Using Lemma 4.2.5, the proof of the next theorem follows the general form of proofs that certain structuring mechanisms are preserved under closed maps [St$_2$].

Theorem 4.2.6 *If X is linear chain (F) and $f : X \to Y$ is a closed and irreducible map then Y is linear chain (F).*

Proof The notation of Definition 4.2.1 will also be used for Y but no confusion should arise. For each $y \in Y$, choose some point $x_y \in f^{-1}(y)$. Define

$$\mathcal{W}(y) = \{\overline{f^*(W)}^Y \cup \{y\} : W \in \mathcal{W}(x_y)\}$$

Since $\mathcal{W}(x_y)$ is a chain of sets, $\mathcal{W}(y)$ is also a chain of sets which trivially contain y.

Consider some $y \in Y$ for which $y \in U \in \tau Y$. Define $V' = \bigcup_{x \in f^{-1}(y)} V(x, f^{-1}(U))$ so that, by regularity, $\overline{V'}^X \subseteq f^{-1}(U)$. Now define $V(y,U) = f^*(V')$. Since f is closed, $V(y,U)$ is open in Y. Moreover, from the definition of V', it is clear that $f^{-1}(y) \subseteq V' \subseteq \overline{V'}^X \subseteq f^{-1}(U)$ and, therefore, $y \in f^*(V') \subseteq f(\overline{V'}^X) \subseteq U$. But f being closed also implies $\overline{f^*(V')}^Y \subseteq f(\overline{V'}^X)$. Hence, overall, we have

$$y \in V(y,U) \subseteq \overline{V(y,U)}^Y \subseteq f(\overline{V'}^X) \subseteq U$$

Consider $z \in V(y,U) \setminus \{y\}$. The definition of V means that $f^{-1}(z) \subseteq V' = \bigcup_{x \in f^{-1}(y)} V(y, f^{-1}(U))$ and so, for some $x \in f^{-1}(y)$, $x_z \in V(x, f^{-1}(U))$. As X is linear chain (F) and $x_z \neq x$, there exists $W' \in \mathcal{W}(x_z)$ such that

1. $x \in W' \subseteq \overline{V(x,T)}^X$
2. $x \in \overline{\mathrm{int}W'}^X$

Take $W = \overline{f^*(W')}^Y \cup \{z\}$ which is an element of $\mathcal{W}(z)$. Now, $W' \subseteq \overline{V(x,T)}^X \subseteq \overline{V'}^X$, which, by Lemma 4.2.5, implies $f^*(W') \subseteq f^*(\overline{V'}^X) \subseteq \overline{f^*(V')}^Y$. That is, $f^*(W') \subseteq \overline{V(y,U)}^Y$. By taking closures and adding z this gives $W \subseteq \overline{V(y,U)}^Y$.

Since $x \in \overline{\mathrm{int}W'}^X$, again by Lemma 4.2.5, $f(x) \in \overline{f^*(\mathrm{int}W')}^Y \subseteq \overline{f^*(W')}^Y$. Thus $y \in \overline{f^*(\mathrm{int}W')}^Y \subseteq W$ and f being closed implies that $f^*(\mathrm{int}W') \in \tau Y$. Hence $y \in \overline{\mathrm{int}W}^Y$.

Finally, suppose that $z = y$ then $y \in W$ for all $W \in \mathcal{W}(y)$ from the definition. Find $W' \in \mathcal{W}(x_y)$ by the linear chain (F) property on X and let $W = \overline{f^*(W')}^Y \cup \{y\}$. Just as the previous case, this W satisfies condition (1). Hence Y is linear chain (F).

□

Theorem 4.2.6 now allows us to find some important classes of spaces which are linear chain (F). However, the result is applied indirectly through the following:

Corollary 4.2.7 *The perfect image of a GO-space is hereditarily linear chain (F).*

Proof Suppose X is a GO-space and that $f : X \to Y$ is a perfect map onto Y. Then there exists $A \subseteq X$ which is closed such that $f|_A : A \to Y$ is perfect and irreducible. As a subspace of a GO-space, A is also a GO-space and so is linear chain (F). Therefore, by the previous result, Y is linear chain (F).

Moreover, if $B \subseteq Y$, then $f|_B : f^{-1}(B) \to B$ is a perfect map from the GO-space $f^{-1}(B)$ to B. So B is linear chain (F). Hence, Y is hereditarily linear chain (F).

□

This gives us our first important class of spaces which are linear chain (F).

Proposition 4.2.8 *All CICLOTS are hereditarily linear chain (F).*

By Theorems 1.2.14 and 1.2.15 (3), every proto-metrizable spaces is the perfect image of a GO-space. Hence, we obtain a second large class of spaces which are linear chain (F).

Proposition 4.2.9 *Every proto-metrizable space is hereditarily linear chain (F).*

4.3 Utter normality

Linear chain (F) clearly implies chain (F) and hence linear chain (F) spaces are acyclic monotonically normal. However, linear chain (F) also implies another strengthening of monotone normality which has recently been defined by Junnila:

Definition 4.3.1 A regular space X is *utterly normal* if, for all $x \in X$, there is a neighbourhood base $\mathcal{B}_x$ of x such that

$$\text{for all } B_x \in \mathcal{B}_x \text{ and } B_y \in \mathcal{B}_y,\ B_x \cap B_y \neq \emptyset \text{ implies either } x \in \overline{B_y} \text{ or } y \in \overline{B_x}$$

Such a collection of $\mathcal{B}_x$'s is called an *utterly normal neighbourhood base assignment.*

Remark Junnnila identifies different types of utter normality according as to whether the utterly normal neighbourhood bases consist of open, closed or simply any neighbourhoods. We consider only the case where the neighbourhood bases consist entirely of open sets.

To see that this does indeed imply monotone normality, for $x \in U \in \tau X$, find $V \in \tau X$ such that $x \in V \subseteq \overline{V} \subseteq U$. As $\mathcal{B}_x$ is a neighbourhood base, there exists a $B \in \mathcal{B}_x$ such that $x \in B \subseteq V$. Define $H(x,U) = B$ so that $x \in H(x,U) \subseteq \overline{H(x,U)} \subseteq U$. If, for some $y \in W \in \tau X$, $H(x,U) \cap H(y,W) \neq \emptyset$ then, as $H(x,U) \in \mathcal{B}_x$ and $H(y,W) \in \mathcal{B}_y$, either $x \in \overline{H(y,W)} \subseteq W$ or $y \in \overline{H(x,U)} \subseteq U$. Hence, H is a monotone normality operator.

As yet, no details of utter normality have been published though some may be found in [Co]. However, we have that:

Theorem 4.3.2 *If X is linear chain (F) then X is utterly normal.*

Proof If V is a linear chain (F) operator on X, take $\mathcal{B}_x = \{V(x,U) : x \in U \in \tau X\}$. If $B_x \in \mathcal{B}_x$ and $B_{x'} \in \mathcal{B}_{x'}$ then there exist $U, U' \in \tau X$ such that $B_x = V(x,U)$ and $B_{x'} = V(x',U')$. If $z \in B_x \cap B_{x'}$ then there exist W and $W' \in \mathcal{W}(z)$ such that $x \in W$, $z \in W \subseteq \overline{V(x,U)}$ and $x' \in W'$, $z \in W' \subseteq \overline{V(x',U')}$. By the definition of linear chain (F), $\mathcal{W}(z)$ is a chain hence either $W \subseteq W'$ or $W' \subseteq W$. Without loss of generality, assume the former. This means that $x \in W \subseteq W' \subseteq \overline{V(x',U')}$. That is, $x \in \overline{B_{x'}}$. Hence X is utterly normal.

□

Using the results already obtained on linear chain (F), we can encompass many of the classes which Junnila has so far identified as utterly normal.

Corollary 4.3.3 *The following classes of spaces are (hereditarily) utterly normal:*

1. *GO-spaces*

2. [**Junnila**] *proto-metrizable spaces*
3. *CICLOTS*

The preservation of utter normality has the same complications as that of linear chain (F). Open subsets of an utterly normal space are utterly normal which is easily seen by considering the obvious restriction of the utterly normal neighbourhood base assignments. Also, in a similar fashion to Proposition 4.2.4, dense subspaces of an utterly normal space are utterly normal. With regards to mappings, we are unable to determine if utter normality is preserved under closed mappings but, just like linear chain (F), we do have:

Proposition 4.3.4 *If $f : X \to Y$ is closed and irreducible and X is utterly normal then Y is utterly normal.*

Proof Given $x \in U \in \tau X$, choose $B(x, U) \in \mathcal{B}_x$ such that $\overline{B(x,U)} \subseteq U$. For $y \in Y$, define

$$\mathcal{B}_y = \{f^*(\bigcup_{x \in f^{-1}(y)} B(x, f^{-1}(V))) : y \in V \in \tau X\}$$

Since f is closed and, for any $y \in V \in \tau X$, $C = \bigcup_{x \in f^{-1}(V)} B(x, f^{-1}(V))$ is open in X, $f^*(C)$ is open in Y. Moreover, it is clear from the definition of $B(,)$ that:

$$f^{-1}(y) \subseteq \bigcup_{x \in f^{-1}(y)} B(x, f^{-1}(V)) \subseteq V$$

Therefore, $y \in f^*(C) \subseteq V$. Thus, $\mathcal{B}_y$ is indeed a neighbourhood base for y in Y.

Suppose, for each $i \in \{1, 2\}$, $B_i \in \mathcal{B}_{y_i}$ where $B_i = f^*\big(\bigcup_{x \in f^{-1}(y_i)} B(x, f^{-1}(V_i))\big)$ for some $V_i \in \tau Y$. If $z \in B_1 \cap B_2$, from the definition of small image, $f^{-1}(z) \subseteq \bigcup_{x \in f^{-1}(y_1)} B(x, f^{-1}(V_1)) \cap \bigcup_{x \in f^{-1}(y_2)} B(x, f^{-1}(V_2))$. Hence, for $i \in \{1, 2\}$, there exist $x_i \in f^{-1}(y_i)$ such that $B(x_1, f^{-1}(V_1)) \cap B(x_2, f^{-1}(V_2)) \neq \emptyset$. Without loss of generality, the utter normality of X implies that

$$x_1 \in \overline{B(x_2, f^{-1}(V_2))}^X$$

By Lemma 4.2.5, $f(x_1) \in \overline{f^*(B(x_2, f^{-1}(V_2)))}^Y$. From this it follows that $y_1 \in \overline{B_{y_2}}^Y$.

That is, the $\mathcal{B}_y$ form an utterly normal neighbourhood base assignment and Y is utterly normal. □

Junnila asked if stratifiable spaces are utterly normal. A special subclass of the stratifiable spaces are the class of Lasnev spaces - those spaces which are closed images of metric spaces. Towards answering Junnila's question:

Proposition 4.3.5 *Every Lasnev space is utterly normal.*

Proof By Lemma 5.4 of [Gr], every Lasnev space is the closed irreducible image of a metric space. As metric spaces are utterly normal, Proposition 4.3.4 implies that every Lasnev space is utterly normal. □

4.4 Summary and questions

As hoped, by considering the structuring mechanism in GO-spaces, a new and strong version of condition (F), linear chain (F), was defined and shown to be held by all CICLOTS. This can be used to show that CICLOTS are acyclic monotonically normal, which was already known, and, moreover, that they are utterly normal. However, because of the difficulties in preserving linear chain (F), it is hard to see more precisely how it relates to monotone normality and CICLOTS. For example:

Question 4.2 *Is every compact monotonically normal space linear chain (F)?*

Question 4.3 *If X is compact and linear chain (F), is X a CICLOTS?*

Strengthening the results on the preservation of linear chain (F) could possibly help answer these questions.

Question 4.4 *Is linear chain (F) preserved under taking closed images? closed subspaces?*

The new notion of utter normality has a lot of potential uses. However, preservation is also a major difficulty here as well.

Question 4.5 *Is utter normality preserved under taking closed images? closed subspaces?*

The similarities between linear chain (F) and utter normality suggest a possible positive answer to the next question.

Question 4.6 *Is every utterly normal and acyclic monotonically normal space also linear chain (F)?*

Or even:

Question 4.7 *Is every compact utterly normal space linear chain (F)?*

Of course, as yet it is unclear that not all monotonically normal spaces are also utterly normal. We therefore re-iterate Junnila's questions in [Co].

Question 4.8 *Is every (compact) monotonically normal space utterly normal?*

Question 4.9 *Is every stratifiable space utterly normal?*

Remark It is easily seen that the local bases given by an utterly normal neighbourhood assignment are closure-preserving. Ito [I] proved that, for a stratifiable space X, if every point has a closure-preserving local base then X is M_1. Thus, a positive answer to this last question would provide a solution to the famous M_1-M_3 problem.

Chapter 5

A new resolution

One of the central problems in the study of monotonically normal spaces is Nikiel's famous question: is every compact monotonically normal space a CICLOTS? A major difficulty in answering this question is that there are almost no constructions which preserve monotonically normal spaces but which do not trivially preserve CICLOTS. In this chapter, we consider the preservation of monotone normality by taking resolutions.

The first example of a resolution was described by Fedorčuk [Fe] in order to construct a compact space with differing inductive and covering dimensions. Watson extracted from this example the general principle of resolutions. He has presented many important examples which have already been described but which are more easily and elegantly re-described using resolutions [W].

The first section is a description of resolutions and some of the key results of the general theory. We have changed the notation for the basic open sets of a resolution from that used in [W] so as to avoid certain ambiguities which the old notation engenders. We then define a new type of resolution which preserves monotone normality provided that the space which is being resolved over is a locally connected continuum. It is also shown that arctic spaces are preserved by this resolution but the proof of this relies on a deep result of Cornette [C] rather than any trivial observation.

Remark The resolution of monotonically normal spaces has already been considered by Nikiel and Treybig [NT] via the more general concept of fully closed maps. The outcome of their result is that if a resolved space is separable and monotonically normal then the resolution into spaces with cardinality greater than three was only made over countably many points of the original space. This however does not greatly effect our work as the resolved spaces we consider are generally non-separable.

5.1 Defining resolutions

For a space X, fix a family of spaces $\{Y_x : x \in X\}$. For each $x \in X$, take $f_x : X \setminus \{x\} \to Y_x$ to be a continuous function.

Definition 5.1.1 The *resolution of* X *at each* x *into* Y_x *by* f_x has the underlying set $Z = \bigcup_{x \in X}(\{x\} \times Y_x)$. For $x \in U \in \tau X$ and $V \in \tau Y_x$, define

$$\langle x, U, V\rangle = (\{x\} \times V) \cup \bigcup \{\{x'\} \times Y_{x'} : x' \in U \cap f_x^{-1}(V)\}$$

and then $\mathcal{B} = \{\langle x, U, V\rangle : x \in X \ , x \in U \in \tau X \ , V \in \tau Y_x\}$ is a basis for the topology on Z.

Remark If a space Y_x is not specified at every point of X then it is assumed that Y_x is the one-point space and f_x is just the constant map. If $C \subseteq X$ is the set of $x \in X$ for which Y_x and f_x are given then it is easy to see that $X \setminus C$ is embedded in Z as $Z \setminus \bigcup_{x \in C}(\{x\} \times Y_x)$. In the sequel, we simply identify these sets and say that $X \setminus C \subseteq Z$.

Theorem 5.1.2 (The fundamental theorem of resolutions [Fe], [W]) *If X is a compact Hausdorff space and, for all $x \in X$, Y_x is a compact Hausdorff space then Z is a compact Hausdorff space.*

This theorem is particularly useful since it means that when working with compact spaces we can guarantee that the resolution is normal. In general, this is not always possible. However, if the spaces involved are Tychonoff and the f_x are in some sense well-behaved, then by embedding the spaces in suitable compactifications and considering the required resolution as a subspace of the resolution of the compactifications, we can still assert that the space is Tychonoff.

In order to use resolutions, the following map is very convenient.

Definition 5.1.3 $\pi : Z \to X$ is the *projection from Z on to X* defined by $\pi(\langle x, y\rangle) = x$ for all $x \in X$ and $y \in Y_x$.

This allows us to abbreviate the description of the basic open sets. For suitable x, U and V we have that

$$\langle x, U, V\rangle = (\{x\} \times V) \cup \pi^{-1}(U \cap f_x^{-1}(V))$$

We now give a few basic properties of resolutions some of which are straightforward. Their proofs can be found in Watson's article [W].

Proposition 5.1.4 *π is a continuous surjection.*

Proposition 5.1.5 *For all $x \in X$, $\{x\} \times Y_x$, as a subset of Z, is homeomorphic to Y_x.*

The next result is a consequence of a theorem in [W]. However, the theorem there is much more general and so we give a simplified version for the special case.

Proposition 5.1.6 *If X is a continuum and for all $x \in X$, Y_x is also a continuum then Z is a continuum.*

Proof By the fundamental theorem of resolutions, it remains to show that Z is connected.

Suppose that A is a clopen subset of Z. If $\langle x, y\rangle \in A$ then since each Y_x is connected, $\{x\} \times Y_x$ is a subset of A. Thus, $A = \pi^{-1}(\pi(A))$. Since Z is compact and X is Hausdorff, π is closed which means that $\pi(A)$ is closed in X. Moreover, $Z \setminus A$ is also clopen so, just as for A, $Z \setminus A = \pi^{-1}(\pi(Z \setminus A))$. But then $\pi(A)$ and $\pi(Z \setminus A)$ form a closed partition of X. By the connectedness of X, one of them must be empty and, hence, either $A = Z$ or A is empty. That is, Z is connected.

$\square$

5.2 Constructing monotonically normal spaces

As a particular type of resolution, Watson defined resolutions by order mappings of a LOTS into other LOTS:

Definition 5.2.1 [W] If $(X, \leqslant)$ is a compact LOTS, $x \in X$ and Y_x is also a compact LOTS with $a_x = \min Y_x$ and $b_x = \max Y_x$ then an *order mapping* $f_x : X \setminus \{x\} \to Y$ is defined by:

$$f_x(x') = \begin{cases} a_x & \text{for all } x' < x \\ b_x & \text{for all } x' > x \end{cases}$$

Resolving by order mappings is equivalent to taking the LOTS topology induced on the resolved space by the lexicographic order. As the resolved space is still a LOTS, it is also monotonically normal. It is by adapting this construction that we are able to produce a resolution preserving monotone normality in more general spaces. For this reason, the resolution is said to be by order-like mappings.

Throughout this section, X is a continuum with the set of cut-points E and $\{Y_x : x \in E\}$ is a family of compact spaces with two distinguished points a_x and b_x. Other properties of X and the Y_x's will be specified as they are required.

Definition 5.2.2 For each $x \in E$, $X \setminus \{x\}$ is not connected so specify two open sets in X which witness this. These are denoted by X_x^+ and X_x^- so that $X \setminus \{x\} = X_x^- \cup X_x^+$ and $X_x^- \cap X_x^+ = \emptyset$. Define $f_x : X \setminus \{x\} \to Y_x$ by:

$$f_x(p) = \begin{cases} a_x & \text{if } p \in X_x^- \\ b_x & \text{if } p \in X_x^+ \end{cases}$$

These are called *order-like mappings* and are clearly continuous. The space Z formed by resolving X at each x into Y_x by these f_x is said to be the *resolution of X into the Y_x by order-like mappings.*

Remark The resolution of LOTS into other LOTS by order mappings can be obtained from order-like mappings by defining $a_x = \min Y_x$, $b_x = \max Y_x$, $X_x^- = (\leftarrow, x)$ and $X_x^+ = (x, \rightarrow)$.

The key property of order like mappings is given in this next theorem.

Theorem 5.2.3 *If X is locally connected and monotonically normal and, for all $x \in E$, Y_x is monotonically normal then Z is monotonically normal.*

Proof From the fundamental theorem of resolutions, we know that Z is T_1. We need to construct a monotone normality operator.

Suppose that $G : X \times \tau X \to \tau X$ is a monotone normality operator on X and, given that X is locally connected, we may assume that $G(x, U)$ is connected for all $x \in U \in \tau X$. For all $x \in X$, take $G_x : Y_x \times \tau Y_x \to Y_x$ to be monotone normality operator on Y_x. Now monotone normality operators need only be defined on a basis of a space so we define the operator $H : Z \times \mathcal{B} \to \tau Z$ by:

$$H(\langle x, y\rangle, \langle x, U, V\rangle) = \langle x, G(x, U), G_x(y, V)\rangle$$

where $x \in U \in \tau X$ and $y \in V \in \tau Y_x$.

To show that H is a monotone normality operator, we must show that for $\langle x, y\rangle \in \langle x, U, V\rangle \in \mathcal{B}$ and $\langle s, t\rangle \in \langle s, P, Q\rangle \in \mathcal{B}$, if $(*)$:

$$H(\langle x, y\rangle, \langle x, U, V\rangle) \cap H(\langle s, t\rangle, \langle s, P, Q\rangle) \neq \emptyset$$

then either $\langle x, y\rangle \in \langle s, P, Q\rangle$ or $\langle s, t\rangle \in \langle x, U, V\rangle$. By considering the definition of the basic open sets of Z, it becomes clear that if $(*)$ holds then there are four ways in which it may do so.

1. $(\{x\} \times G_x(y, V)) \cap (\{s\} \times G_s(t, Q)) \neq \emptyset$. In this case, $x = s$ and $G_x(y, V) \cap G_s(t, Q) \neq \emptyset$ so that either $y \in Q$ or $t \in V$. This gives the respective conclusions that either $\langle x, y\rangle \in \langle s, P, Q\rangle$ or that $\langle s, t\rangle \in \langle x, U, V\rangle$.
2. $(\{x\} \times G_x(y, V)) \cap \pi^{-1}\big(G(s, P) \cap f_s^{-1}(G_s(t, Q))\big) \neq \emptyset$. It must be that $x \in G(s, P) \cap f_s^{-1}(G_s(t, Q))$. Hence $\{x\} \times Y_x \subseteq \pi^{-1}\big(G(s, P) \cap f_s^{-1}(G_s(t, Q))\big) \subseteq H(\langle s, t\rangle, \langle s, P, Q\rangle)$ $\subseteq \langle s, P, Q\rangle$. This implies that $\langle x, y\rangle \in \langle s, P, Q\rangle$.
3. $(\{s\} \times G_s(t, Q)) \cap \pi^{-1}\big(G(x, U) \cap f_x^{-1}(G_x(y, V))\big) \neq \emptyset$. That $\langle s, t\rangle \in \langle x, U, V\rangle$ follows similarly to the previous case.

4. $\pi^{-1}\big(G(x,U) \cap f_x^{-1}(G_x(y,V))\big) \cap \pi^{-1}\big(G(s,P) \cap f_s^{-1}(G_s(t,Q))\big) \neq \emptyset$. (We may assume that $x \notin G(s,P)$ otherwise one of the above cases occurs.) Since this implies that $G(x,U) \cap G(s,P) \neq \emptyset$ then either $x \in P$ or $s \in U$. Without loss of generality, we may assume that $s \in U$. If $a_x = b_x$ then $\langle x,U,V\rangle = \pi^{-1}(U)$ and $\langle s,t\rangle \in \langle x,U,V\rangle$. Thus, also assume that $x \in E$ and $a_x \neq b_x$. Since $x \notin G(s,P)$, $G(s,P) \subseteq X_x^- \cup X_x^+$ but $G(s,P)$ is connected hence $G(s,P)$ is contained entirely in either X_x^- or X_x^+. Again without loss of generality, we may assume the latter. This means that $s \in G(s,P) \subseteq X_x^+$. For $\pi^{-1}\big(G(x,U) \cap f_x^{-1}(G_x(y,V))\big) \cap \pi^{-1}\big(G(s,P) \cap f_s^{-1}(G_s(t,Q))\big) \neq \emptyset$ to have occured, it must be that $b_x \in G_x(y,V)$ so that $b_x \in V$ and $X_x^+ \subseteq f_x^{-1}(V)$. Hence, $s \in f_x^{-1}(V) \cap U$ which implies that $\langle s,t\rangle \in \pi^{-1}(U \cap f_x^{-1}(V))$ and $\langle s,t\rangle \in \langle x,U,V\rangle$.

This shows that H is indeed a monotone normality operator.

□

In order to use this construction to build "interesting" monotonically normal spaces, it is best to start with a base space X with lots of cut-points. A particularly good example of this is a dendron since, for any two points in a dendron, there is a cut-point which separates them. Of course, we will need the dendron to be monotonically normal. This is implied by a theorem of Cornette:

Theorem 5.2.4 [C] *A Hausdorff locally connected continuum X is arctic if and only if every cyclic element of X is arctic.*

Corollary 5.2.5 *Every dendron is arctic.*

Proof Every cyclic element of a dendron is a singleton hence trivially the continuous image of an arc. Dendra are assumed to be Hausdorff and locally connected so Cornette's result applies.

□

Corollary 5.2.6 *If X is a dendron and, for all $x \in E$, Y_x is monotonically normal , then Z is monotonically normal .*

Proof Dendra are arctic hence monotonically normal and they are also locally connected. The result immediately follows from Theorem 5.2.3.

□

Resolving by order-like mappings also preserves other structure.

Proposition 5.2.7 *If X is locally connected and, for all $x \in E$, Y_x is both locally connected and connected then Z is locally connected.*

Proof Consider $\langle x,y\rangle \in Z$. If $\mathcal{B}_x$ is a neighbourhood base for x in X and $\mathcal{B}_y$ is a neighbourhood basis for y in Y_x, then it is easily seen that

$$\mathcal{B}_{\langle x,y\rangle} = \{\langle x,U,V\rangle : U \in \mathcal{B}_x \text{ and } V \in \mathcal{B}_y\}$$

is a neighbourhood basis for $\langle x,y\rangle$ in Z. Thus, to show that Z is locally connected, it is sufficient to show that if U is a connected neighbourhood of x and V is a connected neighbourhood of y then $\langle x,U,V\rangle$ is connected.

Assume that U and V are such neighbourhoods.

First, suppose that a_x, $b_x \notin V$. Then $\langle x,U,V\rangle = \{x\} \times V$. As this is homeomorphic to V, $\langle x,U,V\rangle$ is clearly connected.

Suppose now that $a_x \in V$ but $b_x \notin V$. (The case when $b_x \in V$ but $a_x \notin V$ is similar.) Consider a set C which is clopen in $\langle x, U, V\rangle$ but which does not contain $\langle x, a_x\rangle$. Since C is closed in $\langle x, U, V\rangle$, there exist $S \in \tau X$ and $T \in \tau Y_x$ such that $x \in S \subseteq U$, $a_x \in T \subseteq V$ and $\langle x, S, T\rangle \cap C = \emptyset$.

If $C \cap (\{x\} \times V) \neq \emptyset$ then this set would witness the fact that $\{x\} \times V$ is not connected. Thus, $C \subseteq \pi^{-1}(U \cap X_x^-)$. If for some $p \in X$, there is a $q \in Y_p$ for which $\langle p, q\rangle \in C$ then it must be that $Y_p \subseteq C$ otherwise $C \cap Y_p$ witnesses the fact that Y_p is not connected. Hence, $C = \pi^{-1}(\pi(C))$. Since $\pi|_{\pi^{-1}(U \cap X_x^-)}$ is a closed map, it is also a quotient map. Thus $\pi(C)$ is clopen in $U \cap X_x^-$. This means that $\pi(C)$ is in fact open in X and hence open in U.

If $\pi(C)$ were not closed in U then there would exist $p \in X$ for which $p \in \overline{\pi(C)}^U \setminus \pi(C)$. If $p \in X_x^- \cap U$ then $p \in \pi(C)$ since $\pi(C)$ is closed in the open set $X_x^- \cap U$. If $p \in X_x^+$ then $p \notin \overline{\pi(C)}^U$ since $\pi(C) \cap X_x^+ = \emptyset$. The only remaining possibility is that $p = x$. But $x \in S$ and $\langle x, S, T\rangle \cap C = \emptyset$. That is, $[\{x\} \times T \cup \pi^{-1}(S \cap X_x^-)] \cap \pi^{-1}(\pi(C)) = \emptyset$. From this, it is straightforward to check that $S \cap \pi(C) = \emptyset$. As S is an open neighbourhood of x, $x \notin \overline{\pi(C)}^U$. Therefore, there can be no such p and $\pi(C)$ is also closed in U.

However, U is connected and the complement of $\pi(C)$ in U contains x so $\pi(C)$ and hence C must be empty. This means that $\langle x, U, V\rangle$ is connected.

Finally if both a_x and b_x are both in $\langle x, U, V\rangle$ and C is some clopen set which contains a_x then it must also contain b_x since V is connected. But then $\langle x, U, V\rangle \setminus (C \cap \pi^{-1}(X_x^{\pm}))$ are both clopen subsets of $\langle x, U, V\rangle$ and, by the above reasoning, they must both be empty.

Thus, every point of Z has a basis of connected sets and Z is locally connected.

$\square$

The other properties which we consider deal only with continua. Thus for the remainder of this section assume that X and, for all $x \in E$, Y_x are connected. Thus, by Proposition 5.1.6, Z is also connected. We study the properties of dendra and continuous images of arcs via their cyclic elements. But first, we need to find the cut-points of Z.

Lemma 5.2.8 *For $z = \langle x, y\rangle \in Z$, z is a cut-point of Z if and only if $x \in E$ and either*

1. *y is a cut-point of Y_x or*
2. *$y = a_x$ or b_x*

Proof First, suppose $x \in E$. We consider the two cases for y individually.

1. We may assume that $y \notin \{a_x, b_x\}$ as this is covered in the second case. Suppose $Y_x \setminus \{y\} = G \cup H$ where G, $H \in \tau Y_x$ are disjoint and non-empty. If $\langle p, q\rangle \in Z \setminus \{z\}$ and $p = x$ then $q \in G$ or $q \in H$ so that $\langle p, q\rangle \in \langle x, X, G\rangle \cup \langle x, X, H\rangle$. If $p \neq x$ then $p \in f_x^{-1}(G \cup H)$ hence $\langle p, q\rangle \in \langle x, X, G\rangle \cup \langle x, X, H\rangle$ and $Z \setminus \{z\} = \langle x, X, G\rangle \cup \langle x, X, H\rangle$. Since G and H are open, disjoint and non-empty so too are $\langle x, X, G\rangle$ and $\langle x, X, H\rangle$ and z is a cut-point of Z.
2. If $y = a_x$ (the case $y = b_x$ is similar), it is easy to see that $Z \setminus \{z\} = \pi^{-1}(X_x^-) \cup [\langle x, X, Y_x \setminus \{a_x\}\rangle \cup \pi^{-1}(X_x^+)]$ and that these two sets are disjoint, open and non-empty. Thus z is a cut-point of Z.

Now suppose that $z = \langle x, y\rangle$ is a cut-point of Z. Thus $Z \setminus \{z\} = A \cup B$ where A, $B \in \tau Z$ are disjoint and non-empty. Since for all $x \in X$, Y_x is connected (for $x \notin E$, see the remark after Definition 5.1.1), if $\langle a, b\rangle \in A$ where $a \neq x$, $b \in Y_a$, then it must be that $\{a\} \times Y_a \subseteq A$ otherwise $A \cap (\{a\} \times Y_a)$ is a clopen subset of $\{a\} \times Y_a$. Hence, we have that

$$A = (A \cap (\{x\} \times Y_x)) \cup \pi^{-1}(\pi(A) \setminus \{x\})$$

and similarly

$$B = (B \cap (\{x\} \times Y_x)) \cup \pi^{-1}(\pi(B) \setminus \{x\})$$

If $x \notin E$ then $\{x\} \times Y_x = \{\langle x, y\rangle\}$ which, together with the above implies that $A = \pi^{-1}(\pi(A))$, $B = \pi^{-1}(\pi(B))$ and $X \setminus \{x\} = \pi(A) \cup \pi(B)$. It then follows that $\pi(A)$ and $\pi(B)$ are disjoint. Moreover, since they are both closed in $Z \setminus \{z\} = \pi^{-1}(X \setminus \{x\})$ and $\pi|_{\pi^{-1}(X\setminus\{x\})}$ is a closed map, then $\pi(A)$ and $\pi(B)$ are closed in $X \setminus \{x\}$. Thus $X \setminus \{x\}$ is not connected which contradicts $x \notin E$. Hence $x \in E$.

Suppose now that $y \notin \{a_x, b_x\}$. Since z is a cut-point of Z, $z \in \overline{A}^Z \cap \overline{B}^Z$. However, z has a neighbourhood base all of whose elements are contained in $\{x\} \times Y_x$. Thus, $A \cap (\{x\} \times Y_x)$ and $B \cap (\{x\} \times Y_x)$ are both non-empty. They are both open in $\{x\} \times Y_x$ as A, $B \in \tau Z$. But $(\{x\} \times Y_x) \setminus \{\langle x, y\rangle\} = (A \cap (\{x\} \times Y_x)) \cup (B \cap (\{x\} \times Y_x))$, so z is a cut-point of $\{x\} \times Y_x$. Since $\{x\} \times Y_x$ is canonically homeomorphic to Y_x, it follows that y is a cut-point of Y_x.

□

Having found the cut-points of Z, it is now easy to find the cyclic elements of Z.

Lemma 5.2.9 *If X is a continuum and, for all $x \in E$, Y_x is a continuum then a cyclic element Q of Z has one of the following two forms:*

1. *for some $x \in X$ and cyclic element Q' of Y_x, $Q = \{x\} \times Q'$*
2. *for some cyclic element, Q' of X, $Q = (Q' \setminus E) \cup \{\langle x, y_x\rangle : x \in Q' \cap E\}$ where Q' is a cyclic element of X and $y_x = a_x$ if Q' meets X_x^- and $y_x = b_x$ otherwise*

Proof It is clear that all the Q's of the above two forms are a closed cover of Z. If it is shown that these are indeed cyclic elements of Z then they must be all of them. Thus, it is enough to show that if Q has one of the above forms then it is cyclic and is maximal with respect to this property.

First, suppose $Q = \{x\} \times Q'$ where $x \in E$ and Q' is a cyclic element of Y_x. Also, assume that Q' is not equal to either $\{a_x\}$ or $\{b_x\}$. This case will be dealt with at the end. Since Q is homeomorphic to Q' then it is cyclic.

Suppose then that $R \subseteq Z$ is cyclic, connected and contains Q. If there exists $p \in Z$ such that for some $q \in Y_p$, $\langle p, q\rangle \in R$ and $p \neq x$ then, for the sake of argument, assume that $p \in X_x^+$. $\langle x, b_x\rangle$ is a cut-point of Z which separates any point in $\pi^{-1}(X_x^+)$ from any point in $\{x\} \times (Y_x \setminus \{b_x\})$. Hence, it separates $\langle p, q\rangle$ from any point in $Q \setminus \{b_x\}$ (note that this is a non-empty set). If $\langle x, b_x\rangle \notin R$ then, since $Q \subseteq R$, R is not connected. But if $\langle x, b_x\rangle \in R$ then R contains a cut-point of itself.

Thus, $R \subseteq \{x\} \times Y_x$ and R being homeomorphic to a subset R' of Y_x which is cyclic, connected and contains Q' implies $R' = Q'$ from which it follows that $R = Q$. Therefore, Q is maximal and any such Q is a cyclic element of Z.

Now suppose that $Q = Q' \setminus E \cup \{\langle x, y_x\rangle : x \in Q' \cap E\}$ where Q' and y_x are as in the statement of the lemma. $\pi|_Q : Q \to Q'$ is continuous and injective. Moreover, $\{x\} \times (Y_x \setminus \{y_x\})$ is open in $\pi^{-1}(Q')$ for each $x \in Q' \cap E$ and $\pi^{-1}(Q') \setminus Q = \bigcup_{x \in Q' \cap E} \{x\} \times (Y_x \setminus \{y_x\})$. Thus, Q is closed in $\pi^{-1}(Q')$ which is closed in Z and, therefore, Q is closed in Z. As the restriction of a closed map to a closed set, $\pi|_Q$ is closed and so is a homeomorphism. Because Q' is cyclic, so is Q.

Again, assume that Q' is not equal to a singleton containing a cut-point as this case will be dealt with at the end. Suppose $R \subseteq Z$ is cyclic, connected and contains Q. If there exists $r \in X$ and some $s \in Y_r$ for which $\langle r, s\rangle \in R \setminus Q$ and $r \notin \pi(Q)$ then there exists $p \in E$ which separates r from any point in $Q' \setminus \{p\}$. Assume, for the sake of argument, that $r \in X_p^-$. Then, from the proof of Lemma 5.2.8, $\langle p, a_p\rangle$ separates $\langle r, s\rangle$ from any point in $\pi^{-1}(Q') \supseteq Q$. Thus, as for the previous case, for R to be both cyclic and connected $\langle p, a_p\rangle$ can not be in R or its complement, respectively. Hence, there is no such point $\langle r, s\rangle$ and $R \subseteq \pi^{-1}(Q')$.

Now consider $x \in Q' \cap E$ and some point $y \in Y_x$ for which $y \neq y_x$ and $\langle x, y \rangle \in R$. By Lemma 5.2.8, $\langle x, y_x \rangle$ is a cut-point separating $\langle x, y \rangle$ from $Q \setminus \{\langle x, y_x \rangle\}$. Again, this cannot be the case if R is both cyclic and connected. Therefore $R = Q$ and Q is a cyclic element of Z.

The case which is not covered by the above arguments is when $x \in E$ is a cyclic element of X and a_x or b_x is a cyclic element of Y_x. For the sake of argument, suppose a_x is a cyclic element of Y_x. But in this case it is straightforward to see that $\langle x, a_x \rangle$ can be separated from any other point of Z by a third point. This means that $\{\langle x, a_x \rangle\}$ is a cyclic element of Z and as such satisfies both types of cyclic element given in the statement of the Lemma.

□

The Lemma immediately gives the following corollary.

Corollary 5.2.10 *For Z as in the statement of the previous lemma, any cyclic element of Z is homeomorphic to either a cyclic element of X or a cyclic element of Y_x for some $x \in E$.*

Given this, the next two results are straightforward.

Proposition 5.2.11 *If X is a dendron and, for all $x \in E$, Y_x is a dendron then Z is a dendron.*

Proof By Propositions 5.1.6 and 5.2.7, Z is a locally connected continuum. By Corollary 5.2.10, every cyclic element of Z is homeomorphic to a cyclic element of a dendron. Hence, every cyclic element of Z is trivial and Z is also a dendron.

□

Proposition 5.2.12 *If X is arctic as too are Y_x, for all $x \in E$, then Z is arctic.*

Proof By Cornette's result, since X and Y_x, for all $x \in E$, are arctic, so too are all their cyclic elements. Corollary 5.2.10 implies that every cyclic element of Z is arctic. Again, by Cornette's result, this implies that Z is also arctic.

□

5.3 Summary and further work

The main result of this chapter is that the resolution of monotonically normal, locally connected connected spaces into monotonically normal spaces by order-like mappings is monotonically normal. However, the last proposition tells us that this technique cannot be used on its own to construct a monotonically normal, locally connected continuum which is not arctic. The following, though, remains open:

Question 5.1 *Can resolving by order-like mappings produce a monotonically normal compactum which is not a CICLOTS?*

To help answer this question, the notion of order-like mappings can be expanded to allow more complicated constructions in two ways.

First, for $x \in E$, the number of distinguished points can be made equal to the number of components of $X \setminus \{x\}$. The map f_x then simply takes each component to a corresponding distinguished point. Local connectedness is needed here to ensure that each component is open and, hence, that the map f_x is continuous. By adapting the proofs given here, it is straightforward to show that such a construction preserves monotone normality, dendra and continuous images of arcs.

Secondly, in order to define a topology on the space Z, it is only necessary that, for each $x \in E$, f_x maps from some neighbourhood of x into Y_x. This would allow the resolution to occur at points which only cut one of their neighbourhoods and hence expand the class of spaces which can be used in the construction. For example, every point in the circle S^1 has this property but it has no cut-points.

We have not given the details of these generalisations as they are largely the same as the basic technique given but the description of the more general cases would have obscured the central idea. However, using these generalisations, it may yet be possible to give a positive answer to the question.

Chapter 6

On reflection

Elementary submodels have recently emerged as a powerful technique in general topology. They have been used to simplify considerably both proofs of theorems and the construction of counter-examples, see [Do], [W]. However, there is, as yet, no standard approach to applying them to problems.

The aim of this chapter is to give a new, general technique for using elementary submodels in the construction of topological spaces. The technique is illustrated by three quite diverse examples: Balogh's Q-set space [B_1], [W], a "small" normal but not collectionwise Hausdorff space [R_3], [Do], and Balogh's small Dowker space [B_2].

The first section of this chapter defines elementary submodels and raises some points on the practicalities of using them. The second section gives some straightforward results which are useful tools later on. Also, the proofs of the results serve to introduce reflection techniques. In the third section, the new method of construction is outlined and the three examples are described.

The work of this chapter was done jointly with my colleagues Chris Good and Will Pack.

6.1 Elementary submodels

A set $\mathcal{N}$, with some other structure, *models* a well-formed formula ϕ if, when the formula is interpreted in terms of the structure on $\mathcal{N}$, ϕ is true. This is denoted by $\mathcal{N} \models \phi$ and, informally, it is common to say that $\mathcal{N}$ thinks ϕ. $\mathcal{M} \subseteq \mathcal{N}$ is an *elementary submodel* of a model $\mathcal{N}$ if every formula ϕ is *absolute* for $\mathcal{M}$ and $\mathcal{N}$, that is, for any $x_1, \ldots, x_n \in \mathcal{M}$ which are the only objects mentioned in some logic formula ϕ,

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

Again informally, $\mathcal{M}$ thinks the same about its elements as $\mathcal{N}$ does, hence this property is called *elementarity.*

The existence of elementary submodels is given by:

Theorem 6.1.1 (Löwenheim-Skolem-Tarski Theorem) *For any model $\mathcal{N}$ which is a set and any subset X of $\mathcal{N}$, there is an elementary submodel $\mathcal{M}$ of $\mathcal{N}$ such that $X \subseteq \mathcal{M}$ and $|\mathcal{M}| \leqslant \max\{\omega, |X|\}$.*

See [K_1] p.156 for details of this Theorem.

In most applications, we want $\mathcal{N}$ to model ZF or ZFC. However, such a model cannot be a set so the Löwenheim-Skolem-Tarski Theorem is not applicable. Fortunately, any proof is finite so necessarily can use only finitely many instances of axioms and refer to only finitely many sets. By taking a model of those axioms used, which is a set containing those objects referred to, the proof is still valid in that model. Elementary submodels of that model are then known to exist. In practice, the exact set-model is unimportant and all elementary submodels are treated as submodels of V.

The real power of an elementary submodel comes from the ability to include in it any collection of objects which are being studied. What V thinks of such objects is also thought by the submodel but, because it is small, the objects are easier to manipulate there. For instance, if ω_1 is in some countable elementary submodel $\mathcal{M}$ then $\mathcal{M}$ thinks that ω_1 is uncountable. However, V knows that $\omega_1 \cap \mathcal{M}$ is countable. This allows us to do many things such as to find $\alpha \in \omega_1$ which is not in $\mathcal{M}$. It is because of this ability to find small objects which reflect the properties of large objects that such techniques are said to be reflective.

Before proceeding to some results, it is worth commenting on a few basic properties of elementary submodels. First, elements of an elementary submodel are not necessarily subsets as is demonstrated by considering, as above, ω_1 in a countable model. Secondly, subsets of a model are not necessarily elements of the model. To see this, take a countable elementary submodel of which ω is a subset. If all subsets of ω were in the model then $\mathcal{P}(\omega)$ is a subset of the countable model - obviously impossible. Finally, it is worth noting that elementarity refers only to logic formulae. This causes many difficulties as the language of mathematics is at a much higher level than the language of logic. Apparently simple statements may disguise references to objects which are not in an elementary submodel. Thus, these statements cannot be reflected down into the submodel. To avoid the confusion this may cause, all of our reflected statements are reduced to a form where it is clear that they are absolute.

6.2 Some introductory proofs

First, we consider when elements of an elementary submodel are subsets and vice-versa. Throughout this section, $\mathcal{M}$ denotes an elementary submodel of V, the standard model of ZFC.

Proposition 6.2.1 *If $A \subseteq \mathcal{M}$ and A is finite then $A \in \mathcal{M}$.*

Proof Suppose $A = \{a_1, \ldots, a_n\}$ for some $n \in \omega$. Then

$$V \models \exists x \forall y (y \in x \leftrightarrow (y = a_1 \vee y = a_2 \vee \ldots \vee y = a_n))$$

Namely, A is the x which V thinks satisfies this statement.
As the only objects which occur in the formula are the a_i and these are elements of $\mathcal{M}$, elementarity gives us

$$\mathcal{M} \models \exists x \forall y (y \in x \leftrightarrow (y = a_1 \vee \ldots \vee y = a_n))$$

Call the set which is asserted to exist in $\mathcal{M}$ by this sentence, B. Thus $B \in \mathcal{M}$ and the above tells us that

$$\mathcal{M} \models \forall y (y \in B \leftrightarrow (y = a_1 \vee \ldots \vee y = a_n))$$

But now elementarity gives

$$V \models \forall y (y \in B \leftrightarrow (y = a_1 \vee \ldots \vee y = a_n))$$

But this sentence defines A as well. Therefore, $B = A$ and since $B \in \mathcal{M}$ it must be that $A \in \mathcal{M}$.

$\square$

Proposition 6.2.2 *If $X \in \mathcal{M}$ and $|X| \leqslant \kappa$ where $\kappa + 1 \subseteq \mathcal{M}$ then $X \subseteq \mathcal{M}$.*

Proof $V \models \exists f : \kappa \to X \wedge f$ is surjective. In order to use elementarity, we need to put this into the language of logic. A more fundamental way of expressing this statement is, $V \models \exists f \Phi(f)$ where $\Phi(f)$ is the statement

$$(f \subseteq \kappa \times X) \wedge \forall \alpha \in \kappa \exists x \in X (\langle \alpha, x \rangle \in f \wedge \forall y (\langle \alpha, y \rangle \in f \to x = y)) \wedge \forall x \in X \exists \alpha \in \kappa (\langle \alpha, x \rangle \in f)$$

From this, it is clear to see that the only part which does not easily translate into a logic formula is $f \subseteq \kappa \times X$. This can be expressed as $\forall a(a \in f \leftrightarrow a \in \kappa \times X)$. In this form $\exists f \Phi(f)$ has only two objects in it, namely κ, and X, and these are elements of $\mathcal{M}$. So by elementarity $\mathcal{M} \models \exists f \Phi(f)$.

Let $f \in \mathcal{M}$ be such that $\Phi(f)$ holds. So $\mathcal{M} \models \Phi(f)$ and elementarity now gives us that $V \models \Phi(f)$. What we have achieved so far is that, given that V knows the cardinality of X, we have found a function in $\mathcal{M}$ which witnesses what the cardinality is.

Now suppose $x \in X$. Then there exists $\alpha \in \kappa$ such that $\langle \alpha, x \rangle \in f$ since f is surjective. Thus we have

$$V \models \exists y \in X(\langle \alpha, y \rangle \in f)$$

As $\kappa \subseteq \mathcal{M}$ and $\alpha, f \in \mathcal{M}$, elementarity tells us that

$$\mathcal{M} \models \exists y \in X(\langle \alpha, y \rangle \in f)$$

Hence for some $y \in X \cap \mathcal{M}$, $\mathcal{M} \models \langle \alpha, y \rangle \in f$. Again by elementarity, we now have that $V \models \langle \alpha, y \rangle \in f$. But given that V thinks that f is a function and $\langle \alpha, x \rangle \in f$, it must be that $x = y$ and so $x \in \mathcal{M}$. That is, $X \subseteq \mathcal{M}$.

$\square$

The next proposition tells us that we have some familiar, useful objects in any elementary submodel.

Proposition 6.2.3 $\omega + 1 \subseteq \mathcal{M}$

Proof First we show by induction that $\omega \subseteq \mathcal{M}$.
$V \models \exists x \forall y(y \notin x)$. The empty set is the set which is asserted to exist in this sentence. Elementarity now gives us that $\mathcal{M} \models \exists x \forall y(y \notin x)$. Let E be an element of $\mathcal{M}$ such that $\mathcal{M} \models \forall y(y \notin E)$. Elementarity in the other direction gives that $V \models \forall y(y \notin E)$. That is, $E = \emptyset$. Hence $\emptyset \in \mathcal{M}$.

Now consider any $x \in \mathcal{M}$. Define $x^+ = x \cup \{x\}$. Thus, $V \models \exists y \forall a(a \in y \leftrightarrow (a \in x \vee a = x))$ (namely, $y = x^+$) and so $\mathcal{M} \models \exists y \forall a(a \in y \leftrightarrow (a \in x \vee a = x))$. Let $y \in \mathcal{M}$ be a set which is asserted to exist by this sentence. Then, elementarity tells us that $V \models \forall a(a \in y \leftrightarrow (a \in x \vee a = x))$, which is a definition of x^+. So $y = x^+$ which means that $x^+ \in \mathcal{M}$.

Thus if for some $n \in \omega$, $n \in \mathcal{M}$ then we have that $n^+ \in \mathcal{M}$ and by induction, $\omega \subseteq \mathcal{M}$.

Now let $\Psi(x)$ be the formula $(\emptyset \in x) \wedge (\forall y \in x(y^+ \in x))$. It is clear from the above that Ψ is absolute for $\mathcal{M}$ and V. The axiom of infinity holds in V, so $V \models \exists x \Psi(x)$. Hence $\mathcal{M} \models \exists x \Psi(x)$. Let $N \in \mathcal{M}$ such that $\mathcal{M} \models \Psi(N) \wedge (\forall x(\Psi(x) \rightarrow N \subseteq x))$. ($N$ is constructed in the same way that ω is constructed in V from the axiom of infinity.) So N is the object that $\mathcal{M}$ thinks of as the natural numbers.

By elementarity, $V \models \Psi(N)$ so $\omega \subseteq N$ since ω is the smallest inductive set in V. Also by elementarity $V \models \forall x(\Psi(x) \rightarrow N \subseteq x)$. $\Psi(\omega)$ holds so $N \subseteq \omega$. Hence $N = \omega$ and $\omega \in \mathcal{M}$.

$\square$

These last two results in combination give:

Proposition 6.2.4 *If $X \in \mathcal{M}$ and X is countable then $X \subseteq \mathcal{M}$.*

We now present two results which do not tell us about elementary submodels directly but which do typify elementary submodel proofs. The first is a specific case of the Pressing Down Lemma, the second is the Δ-system lemma. Both have well-known combinatorial proofs (see [K_1] p.80 and p.49 respectively) but much of the combinatorics can be effortlessly subsumed into an elementary submodel.

Theorem 6.2.5 (Pressing Down Lemma) *If $f : \omega_1 \setminus \{0\} \rightarrow \omega_1$ is such that $f(\alpha) < \alpha$ for all $\alpha \in \omega_1$, then for some $\gamma \in \omega_1$, $f^{-1}(\gamma)$ is stationary.*

Proof Take $\mathcal{M}$ to be a countable elementary submodel which contains ω_1 and f. Define $\beta \in \omega_1$ to be the least ordinal such that $\beta \notin \mathcal{M}$. If $\gamma = f(\beta)$ then $\gamma \in \mathcal{M}$ since $f(\beta) < \beta$. Moreover, no ordinal greater than β is in $\mathcal{M}$ otherwise, as it is countable, it is a subset of $\mathcal{M}$ and this would imply that $\beta \in \mathcal{M}$.

Now define $\Phi(C)$ to be

$$(\forall \alpha \in \omega_1 \exists \beta \in C(\beta > \alpha)) \wedge (\forall \alpha \in \omega_1 \setminus C \exists \beta < \alpha \forall \gamma \in \omega_1 (\beta < \gamma \leqslant \alpha \rightarrow \gamma \notin C))$$

That is, $\Phi(C)$ means that C is a club set in ω_1 and Φ is clearly absolute between $\mathcal{M}$ and V.

For a given $\alpha < \beta$, $\alpha \in \mathcal{M}$. Moreover, for each club set C in $\mathcal{M}$, $V \models \exists \delta \in C(\alpha < \delta)$. Hence, $\mathcal{M} \models \exists \delta \in C(\alpha < \delta)$. The δ asserted to exist by this statement, tells us that $C \cap (\alpha, \beta] \neq \emptyset$. As this holds for any $\alpha < \beta$ and C is a club, $\beta \in C$.

Thus, for each $C \in \mathcal{M}$, $V \models \Phi(C) \rightarrow \exists \alpha \in C(f(\alpha) = \gamma)$, namely β. Elementarity implies that, for all $C \in \mathcal{M}$, $\mathcal{M} \models \Phi(C) \rightarrow \exists \alpha \in C(f(\alpha) = \gamma)$. We therefore have that $\mathcal{M} \models \forall C(\Phi(C) \rightarrow \exists \alpha \in C(f(\alpha) = \gamma))$. Applying elementarity once more gives that $V \models \forall C(\Phi(C) \rightarrow \exists \alpha \in C(f(\alpha) = \gamma))$. In other words, $f^{-1}(\gamma)$ is stationary.

□

Theorem 6.2.6 (Δ-system Lemma) *Any family $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ of finite sets contains an uncountable Δ-system. That is, there exist an uncountable subset B of ω_1, an $n \in \omega$ and a finite set r such that, for all α, $\beta \in B$, $|A_\alpha| = n$ and $A_\alpha \cap A_\beta = r$.*

Proof Take a countable elementary submodel $\mathcal{M}$ containing $\mathcal{A}$, ω and ω_1. As $\mathcal{M}$ is countable, choose $\gamma \in \omega_1 \setminus \mathcal{M}$ and let $r = A_\gamma \cap \mathcal{M}$. As a finite subset of $\mathcal{M}$, r is an element of $\mathcal{M}$ as is the natural number $n = |A_\gamma|$. Let $\Phi(\alpha, B)$ be the statement:

$$(|A_\alpha| = n) \wedge \forall \beta \in B((\alpha \neq \beta) \rightarrow (A_\alpha \cap A_\beta = r))$$

The only objects in $\Phi(\alpha, B)$ are elements of $\mathcal{M}$. This means that $\Phi(\alpha, B)$ is absolute between $\mathcal{M}$ and V.

Now, $V \models \exists \alpha \in \omega_1((|A_\alpha| = n) \wedge (A_\alpha \supseteq r))$. In particular, γ witnesses the truth of this in V. Therefore, elementarity tells us that $\mathcal{M}_\beta \models \exists \alpha \in \omega_1(|A_\alpha| = n \wedge A_\alpha \supseteq r)$. Take $\alpha \in \mathcal{M}$ which is declared to exist by this expression. If $B_0 = \{\alpha\}$ then $\mathcal{M} \models \forall \alpha \in B_0(\Phi(\alpha, B_0))$. Using Zorn's Lemma, which holds inside $\mathcal{M}$, find a maximal such B, that is, a $B \in \mathcal{M}$ for which:

$$\mathcal{M} \models \forall \alpha \in B(\Phi(\alpha, B)) \wedge \forall \beta \in \omega_1(\Phi(\beta, B) \rightarrow \beta \in B)$$

Interpreting this sentence, this means that $\{A_\alpha : \beta \in B\}$ is thought to be a maximal Δ-system by $\mathcal{M}$. Applying elementarity gives

$$V \models \forall \alpha \in B(\Phi(\alpha, B)) \wedge \forall \beta \in \omega_1(\Phi(\beta, B) \rightarrow \beta \in B)$$

Suppose B were countable. As a countable element of $\mathcal{M}$, $B \subseteq M$ and so $\beta \in \mathcal{M}$ for all $\beta \in B$. Hence, for all $\beta \in B$, $A_\beta \in \mathcal{M}$ and as they are finite sets, $A_\beta \subseteq \mathcal{M}$. Thus, for any $\beta \in B$, $A_\gamma \cap A_\beta = A_\gamma \cap \mathcal{M} \cap A_\beta = r \cap A_\beta = r$. Note also $|A_\gamma| = n$ whence $V \models \Phi(\gamma, B)$. Together with $V \models \forall \beta \in \omega_1(\Phi(\beta, B) \rightarrow \beta \in B)$, this gives $\gamma \in B$. But $B \subseteq \mathcal{M}$ so $\gamma \in \mathcal{M}$ - a contradiction.
Hence B is uncountable and $\{A_\alpha : \alpha \in B\}$ is a Δ-system.

□

6.3 Three examples

The results of the last section are fundamental to practical applications of elementary submodels. They are not, therefore, explicitly referred to in what follows.

We give a rough outline of our approach to using elementary submodels in constructing examples. It is based on Watson's construction of a Q-space in [W]. However, he omits many of the details from his proof. We have filled in many of these gaps and from it abstracted the general technique used in the other two examples. This approach is founded on the fact that there are only $\mathfrak{c}$ many essentially different countable elementary submodels. This is because any model is fully determined by its interpretation of $\in$ on the underlying set. Hence, up to isomorphism, there are as many countable elementary submodels as there are binary relations $\in$ on a countable set, that is, $\mathfrak{c}$ many.

The underlying set in each example is of size $\mathfrak{c}$ and all countable elementary submodels are listed as $\{\mathcal{M}_\beta : \beta \in \mathfrak{c}\}$. However, we often require that $\alpha \supseteq \mathcal{M}_\alpha \cap \mathfrak{c}$. This is not always possible, for instance, if $\alpha < \omega$. Indeed, because $\omega \in \mathcal{M}_\alpha$ for all $\alpha \in \mathfrak{c}$, many small ordinals, such as ω^ω, are contained in every countable elementary submodel. Thus, the listing is started from some fixed ordinal ϵ so that, for all $\alpha \geqslant \epsilon$, $\alpha \supseteq \mathcal{M}_\alpha \cap \mathfrak{c}$. The neighbourhoods of points in $\mathfrak{c}$ are defined almost entirely by finitely many subsets of $\mathfrak{c}$. Through expressing this functionally, it is possible to obtain all but one of the properties of each example from quite simple restraints on the functions involved. The remaining property is reduced to a combinatorial relationship between special functions on $\mathfrak{c}$ and $\mathcal{P}(\mathfrak{c})$. These functions are defined in an induction of length $\mathfrak{c}$ where the values of the functions involved at $\beta \in \mathfrak{c}$ are determined by the β^{th} elementary submodel in the list. This definition involves a diagonalisation procedure on families of pairwise-disjoint finite sets. That they satisfy the required combinatorics is shown by using something akin to a Δ-system which reduces all cases into one involving a pairwise-disjoint family. Because of the importance of the Δ-system, it is beneficial to be *au fait* with the proof of Theorem 6.2.6.

A Q-set space

A *Q-set* is an uncountable subset of the reals, every subset of which is a G_δ-set. Under MA + $\neg$CH, every uncountable subset of $\mathbb{R}$ of size less then $\mathfrak{c}$ is a Q-set and, under $2^\omega < 2^{\omega_1}$, there are no Q-sets. As well as this, much work has been done in showing whether Q-sets do and do not exist in a wide range of different models of ZFC. For a good summary of this work, see Balogh's article [B_1].

Their significance comes in that they provide an easy construction of a separable, normal, non-metrizable Moore space. The space in question is the subspace $(A \times \{0\}) \cup (\mathbb{R} \times (0, \rightarrow))$ of the usual tangent disc space where A is a Q-set.

The generalisation of a Q-set, a *Q-set space*, is one in which every subset is a G_δ-set but not for trivial reasons. That is, a Q-set space is also regular, zero-dimensional space but not σ-discrete (the countable union of discrete subspaces). Given that there are many models of ZFC in which Q-sets do not exist, it is conceivable that there are models in which there are no Q-set spaces. Balogh's example [B_1] shows that this is not the case as there is a Q-set space in ZFC.

Balogh's space, which we shall call X, has $\mathfrak{c}$ as its underlying set where the topology is defined in terms of the following functions:

For all $Y \subseteq \mathfrak{c}$, $G_Y : \mathfrak{c} \to \omega + 1$ and $G_{Y,n} : \mathfrak{c} \to \omega$ are defined such that $G_Y^{-1}(\omega) = Y$.

For each $Y \subseteq \mathfrak{c}$ and $n, k \in \omega$,

$$U(Y, n, k, 1) := \{\beta \in \mathfrak{c} : G_Y(\beta) \geqslant n, G_{Y,n}(\beta) = k\}$$

$$U(Y, n, k, 0) := \mathfrak{c} \setminus U(Y, n, k, 1)$$

The topology is then given by the sub-base $\mathcal{B} = \{\mathcal{U}(\mathcal{Y}, \backslash, \|, \rangle)\} : \mathcal{Y} \subseteq \mathfrak{c}, \backslash, \| \in \omega$ and $\rangle \in \{\prime, \infty\}\}$.

Clearly, this means that every element of the sub-base is clopen and hence that the space is both regular and zero-dimensional. Moreover,

$$G_Y^{-1}((n, \omega]) = \bigcup_{k \in \omega} U(Y, n, k, 1)$$

which means that for each $n \in \omega$, $G_Y^{-1}((n, \omega])$ is open. Also,

$$Y = \bigcap_{n \in \omega} G_Y^{-1}((n, \omega])$$

so that every subset of X is a G_δ. It follows from this that X is T_1.

Remark The purpose of the $G_{Y,n}$'s is simply to allow an easy proof of regularity by declaring the sub-base to consist of clopen sets. This could not be done using the G_Y's alone as this would mean that every subset is the intersection of closed sets hence closed. This makes the space discrete!!

As we have a space for all possible G_Y's and $G_{Y,n}$'s which we could define, the trick now is to carefully define them so as to avoid σ-discreteness.

Suppose the space were σ-discrete so that $X = \bigcup_{n\in\omega} A_n$ where the A_n are disjoint discrete subsets. Define $f : \mathfrak{c} \to \omega$ by $f(\alpha) = n$ if and only if $\alpha \in A_n$ and $h : \mathfrak{c} \to [\mathcal{B}]^{<\omega}$ so that $\{\bigcap h(\alpha) : \alpha \in f^{-1}(n)\}$ is a set of neighbourhoods witnessing that each A_n is discrete. That is, $\alpha \in \bigcap h(\alpha)$ and if $f(\alpha) = f(\beta)$ then $\alpha \notin \bigcap h(\beta)$. The G's are defined in such a way as to kill off all of these pairs.

List all countable elementary submodels up to isomorphism type as $\{\mathcal{M}_\beta : \beta \in \mathfrak{c} \setminus \epsilon\}$ in such a way that, for $\beta > \epsilon$, $\beta \supseteq \mathcal{M}_\beta \cap \mathfrak{c}$. The G's are now defined by induction. For all $\beta \leqslant \epsilon$, take $G_Y(\beta)$ and $G_{Y,n}(\beta)$ to be defined arbitrarily though still satisfying $G_Y^{-1}(\omega) = Y$ for all $Y \subseteq \mathfrak{c}$. Suppose that for $\beta \in \mathfrak{c}$, for all $\alpha < \beta$, $G_Y(\alpha)$ and $G_{Y,n}(\alpha)$ have been defined.

In order to prevent a pair f and h from witnessing σ-discreteness, we need to enlarge some of the neighbourhoods already defined by $G_Y(\alpha)$ and $G_{Y,n}(\alpha)$ for $\alpha < \beta$. However, not every such pair needs to be considered. As will become clear, any h and f can be reduced to a canonical one for which there exists $k \in \omega$ such that $\{\pi h(\alpha)\}_{\alpha\in f^{-1}(k)}$ contains an infinite pairwise-disjoint collection. We can simplify further by disguising the fibres of f as just some infinite subset of $\mathfrak{c}$.

As $\mathcal{M}_\beta$ is countable, we can list all $h : \mathfrak{c} \to [\mathcal{B}]^{<\omega}$ and infinite subsets A of $\mathfrak{c}$ which are in $\mathcal{M}_\beta$ by $\{h_i : i \in \omega\}$ and $\{A_j : j \in \omega\}$ respectively. Also define $\pi h(\alpha) = \{Y \subseteq \mathfrak{c} :$ there exist $n,\ k \in \omega$ and $i \in \{0, 1\}$ such that $U(Y, n, k, i) \in h(\alpha)\}$. Denumerate all pairs $\langle i, j\rangle \in \omega^2$ for which $\{\pi h_i(\alpha)\}_{\alpha\in A_j}$ is an infinite pairwise disjoint collection by $\{\langle i_n, j_n\rangle : n \in \omega\}$. Thus, for each $n \in \omega$,

$$V \models \{\pi h_{i_n}(\alpha)\}_{\alpha\in A_{j_n}} \text{ is an infinite pairwise disjoint collection}$$

It is not too hard to rephrase this as a logic formula and then use elementarity to show that

$$\mathcal{M}_\beta \models \{\pi h_{i_n}(\alpha)\}_{\alpha\in A_{j_n}} \text{ is an infinite pairwise disjoint collection}$$

Define α_1 to be an arbitrary ordinal in $A_{j_1} \cap \mathcal{M}_\beta$. Given $\alpha_1 \dots \alpha_{n-1}$, choose $\alpha_n \in A_{j_n} \cap \mathcal{M}_\beta$ such that $\alpha_n \notin \{\alpha_1, \dots, \alpha_{n-1}\}$ and

$$\pi h_{i_n}(\alpha_n) \cap \bigcup_{m<n} \pi h_{i_m}(\alpha_m) = \emptyset$$

This is possible as $\bigcup_{m<n} \pi h_{i_m}(\alpha_i)$ is a finite collection of Y's yet, by the previous comments, $\mathcal{M}_\beta$ thinks that $\{\pi h_{i_n}(\alpha)\}_{\alpha\in A_{j_n}}$ contains an infinite, disjoint collection of finite sets consisting of Y's. The definition of the α_n means that the following function is well-defined:

$$n_\Delta(Y) = \alpha_n \text{ if and only if } Y \in \pi h_n(\alpha_n)$$

Whenever $n_\Delta(Y) = \alpha_m$, define $G_Y(\beta) > \max\{n \in \omega : U(Y, n, k, i) \in h_{i_m}(\alpha_m)$ for some $k \in \omega,\ i \in \{0, 1\}\}$ and still satisfying $G_Y(\beta) = \omega$ if and only if $\beta \in Y$. As $h_{i_m}(\alpha_m)$ is a finite set, this is a good definition. For each $n \in \omega$, if for some $k \in \omega$, $\alpha_m \in U(Y, n, k, 1) \in h_{i_m}(\alpha_m)$ then let $G_{Y,n}(\beta) = k$. Such a k is the unique value of $G_{Y,n}(\alpha_m)$ hence this definition is also good. If there is no such k for the given n, then choose $G_{Y,n}(\beta)$ so that it is not equal to any k for which $\alpha_m \in U(Y, n, k, 0) \in h_{i_m}(\alpha_m)$. Again, this is a good definition since $h_{i_m}(\alpha_m)$ is finite. Otherwise, define $G_Y(\beta)$ and $G_{Y,n}(\beta)$ arbitrarily apart from ensuring that $G_Y(\beta) = \omega$ if and only if $\beta \in Y$.

The upshot of this definition is that if $\alpha_m \in U(Y, n, k, i) \in h_{i_m}(\alpha_m)$ for suitable n, k and i, then $\beta \in U(Y, n, k, i)$.

This completes the construction of the space X. We have that X is a Q-set space provided we can show that it is not σ-discrete. By all of the previous discussion, this follows from:

Theorem 6.3.1 *Given any $h : \mathfrak{c} \to [\mathcal{B}]^{<\omega}$ and $f : \mathfrak{c} \to \omega$ such that, for all $\alpha \in \mathfrak{c}$, $\alpha \in \bigcap h(\alpha)$, there exist $\alpha < \beta < \mathfrak{c}$ such that*

$$f(\alpha) = f(\beta) \text{ and } \beta \in \bigcap h(\alpha)$$

Proof Take a countable elementary submodel which contains f, h, $\{G_Y, G_{Y,n} : Y \subseteq \mathfrak{c}, n \in \omega\}$, $\mathfrak{c}$. This submodel is isomorphic to $\mathcal{M}_\beta$ for some $\beta \in \mathfrak{c}$. Note, we assume that $\beta \notin \mathcal{M}_\beta$. Let $f(\beta) = m \in \omega$ so $f(\beta) \in \mathcal{M}_\beta$ and, as a finite subset of $\mathcal{M}_\beta$, $h(\beta) \cap \mathcal{M}_\beta \in \mathcal{M}_\beta$. Define $\Psi(\alpha, A)$ to be the sentence:

$$(f(\alpha) = f(\beta)) \wedge \forall \alpha' \in A((\alpha \neq \alpha') \to (\pi h(\alpha) \cap \pi h(\alpha') = \pi h(\beta) \cap \mathcal{M}_\beta))$$

Since the only objects in $\Psi(\alpha, A)$ are elements of $\mathcal{M}_\beta$, $\Psi(\alpha, A)$ is absolute for V and $\mathcal{M}_\beta$. It is clear that $V \models \exists \alpha' \in \mathfrak{c}(\Psi(\alpha', \emptyset) \wedge (\pi h(\alpha') \supseteq \pi h(\beta) \cap \mathcal{M}_\beta))$, namely, β is the α' which V has in mind. By elementarity, $\mathcal{M}_\beta \models \exists \alpha' \in \mathfrak{c}(\Psi(\alpha', \emptyset) \wedge (\pi h(\alpha') \supseteq \pi h(\beta) \cap \mathcal{M}_\beta))$ Take $\alpha' \in \mathcal{M}_\beta$ whose existence is asserted by this statement and define $A_0 = \{\alpha'\}$. From the definition of α', it follows that $\mathcal{M}_\beta \models \forall \alpha \in A_0(\Psi(\alpha, A_0))$. Just as in the proof of Theorem 6.2.6, use Zorn's Lemma to find an $A \in \mathcal{M}_\beta$ such that

$$\mathcal{M}_\beta \models \forall \alpha \in A(\Psi(\alpha, A)) \wedge \forall \gamma \in \mathfrak{c}(\Psi(\gamma, A) \to \gamma \in A)$$

For any $\alpha \in A$, $V \models \exists \alpha' > \alpha(\Psi(\alpha', A))$, namely β, and elementarity gives that $\mathcal{M}_\beta \models \exists \alpha' > \alpha(\Psi(\alpha', A))$. Find an $\alpha' \in \mathcal{M}_\beta$ which witnesses this. By maximality of A in $\mathcal{M}_\beta$, $\alpha' \in A$. Hence, for any $\alpha \in A$, there is $\alpha' > \alpha$ which is also in A. Thus A is infinite and, for all $\alpha \in A$, $\Psi(\alpha, A)$. This implies that $\{\pi h(\alpha)\}_{\alpha \in A}$ is an infinite Δ-system with root $h(\beta) \cap \mathcal{M}_\beta$. Hence, define (in $\mathcal{M}_\beta$), $h' : \mathfrak{c} \to [\mathcal{B}]^{<\omega}$ by

$$h'(\alpha) = h(\alpha) \setminus (h(\beta) \cap \mathcal{M}_\beta)$$

Clearly, $\{\pi h'(\alpha)\}_{\alpha \in A}$ is an infinite pairwise-disjoint collection such that h', $A \in \mathcal{M}_\beta$. Thus, there is an $m \in \omega$ such that $h = h_{i_m}$ and $A = A_{j_m}$. Take $\alpha = \alpha_m$. These α and β will satisfy the theorem.

By definition, $\alpha \in A$ so $\Psi(\alpha, A)$ and $f(\alpha) = f(\beta)$. We must show that $\beta \in \bigcap h(\alpha)$. That is, we must show that for every $U(Y, n, k, i) \in h(\alpha)$ that $\beta \in U(Y, n, k, i)$.

Consider such a $U(Y, n, k, i) \in h(\alpha)$. By the way in which α was defined, $h(\alpha)$ is the disjoint union of $h'(\alpha)$ and $h(\beta) \cap \mathcal{M}_\beta$. Thus, there are two cases:

1. $U(Y, n, k, i) \in h(\beta) \cap \mathcal{M}_\beta$. In which case, as $\beta \in \bigcap h(\beta)$, $\beta \in U(Y, n, k, i)$.
2. $U(Y, n, k, i) \in h'(\alpha)$. As $h'(\alpha) = h_{i_m}(\alpha_m)$, from the definition of $G_Y(\beta)$ and $G_{Y,n}(\beta)$, if $\alpha \in U(Y, n, k, i)$ then $\beta \in U(Y, n, k, i)$. But we know $\alpha \in \bigcap h(\alpha)$ hence $\beta \in U(Y, n, k, i)$.

This completes the proof.

$\square$

A normal, not collectionwise Hausdorff space

In [R$_3$], Rudin described this space as an answer to a question of Dowker. However, the space also provides an example of a normal, not collectionwise Hausdorff space which only has cardinality $\mathfrak{c}$. Prior to this, the standard example of such a space was Bing's famous example (G) which has cardinality $2^{2^{\omega_1}}$ [Bi].

Rudin's original construction does not refer to elementary submodels but it clearly has all the combinatorial coding which they disguise. In [Do], Dow suggested a way to introduce elementary submodels into the proof. His method, though, is only sketched and, for those unfamiliar with reflection techniques, it is hard to decipher what he intends. We therefore give here a full presentation of Rudin's space using elementary submodels in exactly the same way as in the previous construction.

The set underlying the space is the continuum with all pairs of points from the continuum, that is, $X = \mathfrak{c} \cup [\mathfrak{c}]^2$. The topology is as follows. Each pair $\{\alpha, \beta\} \in [\mathfrak{c}]^2$ is isolated. A point $\alpha \in \mathfrak{c}$ has sub-basic neighbourhoods consisting of the point α and some subcollection of $[\mathfrak{c}]^2$ for which every pair in the subcollection contains α. More precisely, we shall define $f : \mathfrak{c} \times \mathcal{P}(\mathfrak{c}) \to \mathcal{P}(\mathfrak{c})$ and then let the sub-basic neighbourhoods be:

$$U(\alpha, Y, K) = \{\alpha\} \cup \{\{\alpha, \beta\} : \beta \in f(\alpha, Y) \setminus K\}$$

where K is some finite subset of $\mathfrak{c}$.

With this topology, $[\mathfrak{c}]^2$ is a collection of isolated points so that $\mathfrak{c}$ is a closed subset of X. Because any sub-basic neighbourhood of $\alpha \in \mathfrak{c}$ does not meet $\mathfrak{c}$ anywhere else, $\mathfrak{c}$ is a closed discrete subset of X. It is precisely this subset which will not be pointwise separated by a disjoint collection of open sets in X. However, it is interesting to note that the only way two neighbourhoods of distinct points α and β in $\mathfrak{c}$ can meet is if they both contain $\{\alpha, \beta\}$. Thus the possibility of X being collectionwise Hausdorff is destroyed by a single point!

The K in the definition of the sub-basic neighbourhoods is enough to guarantee that the space is T_1.

To make the space normal, we begin to put some restraints (albeit rather weak ones) on f. Consider two disjoint closed subsets Y and Z. Isolated points in Y and Z do not cause problems when it comes to separating Y and Z. Thus we may assume that Y and Z are subsets of $\mathfrak{c}$. But then, since $\mathfrak{c}$ is a closed discrete set, $\mathfrak{c} \setminus Y$ is closed and disjoint from Y and contains Z. Hence, it suffices to provide a separation of Y from its complement in $\mathfrak{c}$. Y is used to index the sub-basic neighbourhoods of the $\alpha \in \mathfrak{c}$ which would achieve this separation.

Consider the following condition which we shall call (†):

$$\text{for all } Y \subseteq \mathfrak{c},\ \alpha \in Y \text{ and } \beta \notin Y \text{ implies } \alpha \notin f(\beta, Y) \text{ or } \beta \notin f(\alpha, Y)$$

Given this, we can simply define $U = \bigcup_{\alpha \in Y} U(\alpha, Y, \emptyset)$ and $V = \bigcup_{\beta \notin Y} U(\beta, Y, \emptyset)$. Clearly they are open sets which contain Y and $\mathfrak{c} \setminus Y$ respectively. If they were not disjoint, then for some $\alpha \in Y$ and $\beta \in \mathfrak{c} \setminus Y$, $U(\alpha, Y, \emptyset)$ meets $U(\beta, Y, \emptyset)$ and that must occur at the point $\{\alpha, \beta\}$. But this would mean that $\alpha \in f(\beta, Y)$ and $\beta \in f(\alpha, Y)$ which contradicts (†). Hence U and V are the required separation.

We must now ensure that f satisfies (†). For each $\alpha \in \mathfrak{c}$ and $Y \subseteq \mathfrak{c}$, define $g_{\alpha,Y} : \mathfrak{c} \to 2$ to be any function and define:

$$\chi_\alpha(Y) = \begin{cases} Y & \alpha \in Y \\ \mathfrak{c} \setminus Y & \alpha \notin Y \end{cases}$$

Let

$$f(\alpha, Y) = \chi_\alpha(Y) \cup \{\beta > \alpha : g_{\alpha,Y}(\beta) = 1\} \cup \{\beta < \alpha : g_{\beta,Y}(\alpha) = 0\}$$

Lemma 6.3.2 *For any collection of $g_{\alpha,Y}$, the resulting f satisfies condition* (†).

Proof Suppose that $Y \subseteq \mathfrak{c}$, $\alpha \in Y$ and $\beta \notin Y$. Assume that $\alpha < \beta$. The other case is almost identical. Suppose too, for contradiction, that $\beta \in f(\alpha, Y)$ and $\alpha \in f(\beta, Y)$.
Since $\chi_\beta(Y) = \mathfrak{c} \setminus Y$, $\alpha \in f(\beta, Y)$ means that $\alpha \in \{\alpha < \beta : g_{\beta,Y}(\alpha) = 0\}$, that is, $g_{\beta,Y}(\alpha) = 0$. Similarly, $\beta \in f(\alpha, Y)$ means that $g_{\beta,Y}(\alpha) = 1$ which gives the contradiction.

□

So overall, given any collection of $g_{\alpha,Y}$'s and f defined as above, X is a normal, T_1 space. We can now use elementary submodels to construct the $g_{\alpha,Y}$ and prevent X from being collectionwise Hausdorff.

Suppose X were collectionwise Hausdorff. Then there exists $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ and $k : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$ such that $\{\bigcap_{Y \in h(\alpha)} U(\alpha, Y, k(\alpha)) : \alpha \in \mathfrak{c}\}$ is a disjoint family of open sets which separate all the points in $\mathfrak{c}$. Note, the basic open neighbourhoods $U(\alpha, Y, K)$ should have a different K for each Y but as the intersections are finite and the K's are finite they can be joined into a single $k(\alpha)$. To prevent such a separation, we need to build the $g_{\alpha,Y}$ such that for every possible candidate for h and k, there exist $\alpha < \beta < \mathfrak{c}$ such that $\bigcap_{Y\in h(\alpha)} U(\alpha, Y, k(\alpha))$ meets $\bigcap_{Y\in h(\beta)} U(\beta, Y, k(\beta))$. This would happen if $(*)$:

$$\alpha \notin k(\beta) \text{ and, for all } Y \in h(\beta),\ \alpha \in f(\beta, Y)$$

$$\beta \notin k(\alpha) \text{ and, for all } Y \in h(\alpha),\ \beta \in f(\alpha, Y)$$

List all the countable elementary submodels up to isomorphism type as $\{\mathcal{M}_\beta : \beta \in \mathfrak{c} \setminus \epsilon\}$ and assume that $\beta \supseteq \mathcal{M}_\beta \cap \mathfrak{c}$. For $\beta < \epsilon$ define $g_{\alpha,Y}(\beta)$ arbitrarily. We will define $g_{\alpha,Y}(\beta)$ for all $\alpha \in \mathfrak{c}$ and $Y \subseteq \mathfrak{c}$. List all $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ in $\mathcal{M}_\beta$ and all infinite subsets A of $\mathfrak{c}$ in $\mathcal{M}_\beta$ as $\{h_i : i \in \omega\}$ and $\{A_j : j \in \omega\}$ respectively. As in the Q-space construction, there are only certain h's and A's which we need worry about. Thus, let $\{\langle i_n, j_n\rangle : n \in \omega\}$ be an enumeration of the pairs $\langle i, j\rangle \in \omega^2$ for which $\{h_i(\alpha)\}_{\alpha\in A_j}$ is an infinite pairwise-disjoint collection. The enumeration is done in such a way that each pair is listed infinitely many times.

Now fix $\alpha_1 \in A_{i_1} \cap \mathcal{M}_\beta$. Given $\alpha_1, \ldots, \alpha_{n-1} \in \mathfrak{c}$, using elementarity just as in the Q-space, define $\alpha_n \in A_{j_n} \cap \mathcal{M}_\beta$ such that $\alpha_n \notin \{\alpha_1, \ldots, \alpha_{n-1}\}$ and

$$h_{i_n}(\alpha_n) \cap \bigcup_{m<n} h_{i_m}(\alpha_m) = \emptyset$$

This is possible since each of the $h_{i_m}(\alpha_m)$ is a finite set and $\mathcal{M}_\beta$ thinks that $\{h_{i_n}(\alpha)\}_{\alpha\in A_n}$ is an infinite pairwise-disjoint collection.

Define $n_\Delta(Y) = \alpha_n$ if and only if $Y \in h_{i_n}(\alpha_n)$. Otherwise, $n_\Delta(Y)$ is not defined. This is a good definition, since if for some $Y \subseteq \mathfrak{c}$, $Y \in h_{i_n}(\alpha_n)$, then, because the $h_{i_m}(\alpha_{i_m})$ form a disjoint collection, the α_n for which this occurs is unique.

Now let

$$g_{\alpha,Y}(\beta) = \begin{cases} 1 & \text{if } n_\Delta(Y) = \alpha \\ 0 & \text{otherwise} \end{cases}$$

X is then built as already described from these $g_{\alpha,Y}$ and as to be hoped:

Theorem 6.3.3 *Given any $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ and $k : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$, $(*)$ is satisfied.*

Proof Take a countable elementary submodel which contains h, k, $\mathfrak{c}$, $\mathcal{P}(\mathfrak{c})$, and $\{g_{\alpha,Y} : \alpha \in \mathfrak{c} \text{ and } Y \subseteq \mathfrak{c}\}$. This submodel is isomorphic to $\mathcal{M}_\beta$ for some $\beta \in \mathfrak{c}$ (remember that $\beta \supseteq \mathcal{M}_\beta \cap \mathfrak{c}$). Since $h(\beta)$ is a finite set, $h(\beta) \cap \mathcal{M}_\beta$ is a finite subset of $\mathcal{M}_\beta$ and hence is an element of $\mathcal{M}_\beta$. Thus define $\Sigma(\alpha, A)$ to be the formula:

$$((\chi_\alpha(Y) = \chi_\beta(Y)) \leftrightarrow (Y \in h(\beta) \cap \mathcal{M}_\beta)) \wedge \forall \alpha' \in A((\alpha \neq \alpha') \to (h(\alpha) \cap h(\alpha') = h(\beta) \cap \mathcal{M}_\beta))$$

$V \models \exists \alpha' \in \mathfrak{c}(\Sigma(\alpha', \emptyset) \wedge (h(\alpha') \supseteq h(\beta) \cap \mathcal{M}_\beta))$, namely V thinks β satisfies this statement. By elementarity, $\mathcal{M}_\beta \models \exists \alpha' \in \mathfrak{c}(\Sigma(\alpha', \emptyset) \wedge (h(\alpha') \supseteq h(\beta) \cap \mathcal{M}_\beta))$. Find an $\alpha' \in \mathcal{M}_\beta$ which witnesses the truth of this statement. If $A_0 = \{\alpha'\}$ then $\mathcal{M}_\beta \models \forall \alpha \in A_0(\Sigma(\alpha, A_0))$. Now, applying Zorn's Lemma, produce a maximal such $A \in \mathcal{M}_\beta$, that is:

$$\mathcal{M}_\beta \models \forall \alpha \in A(\Sigma(\alpha, A)) \wedge \forall \gamma \in \mathfrak{c}(\Sigma(\gamma, A) \to (\gamma \in A))$$

For any $\alpha \in A$, $V \models \exists \alpha' > \alpha(\Sigma(\alpha', A))$, namely β. Elementarity gives $\mathcal{M}_\beta \models \exists \alpha' > \alpha(\Sigma(\alpha', A))$. Find such an $\alpha' \in \mathcal{M}_\beta$ and maximality of A in $\mathcal{M}_\beta$ implies that $\alpha' \in A$. Hence, for any element of A there is a strictly greater one whence A is infinite. From the definition, for all $\alpha \in A$, $\Sigma(\alpha, A)$ and hence $\{h(\alpha)\}_{\alpha\in A}$ is a Δ-system with root $h(\beta) \cap \mathcal{M}_\beta$. Define $h' : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ by

$$h'(\alpha) = h(\alpha) \setminus (h(\beta) \cap \mathcal{M}_\beta) \text{ for all } \alpha \in \mathfrak{c}.$$

This means $\{h'(\alpha)\}_{\alpha\in A}$ is an infinite pairwise-disjoint collection where h' and $A \in \mathcal{M}_\beta$. Hence, there exists $m \in \omega$ such that $h' = h_{i_m}$ and $A = A_{j_m}$. Moreover, as a pair h' and A were listed infinitely often and $k(\beta)$ is finite, we may ensure that $\alpha_m \notin k(\beta)$.

We will now show that $\alpha = \alpha_m$ and β are those required in $(*)$.

By definition, $\alpha \in \mathcal{M}_\beta$ so $\alpha < \beta$. Also $\alpha \notin k(\beta)$. Since $k, \alpha \in \mathcal{M}_\beta$, it is not too hard to use elementarity to show that $k(\alpha) \in \mathcal{M}_\beta$. But then as a finite element of $\mathcal{M}_\beta$, $k(\alpha)$ is also a subset of $\mathcal{M}_\beta$ and hence $\beta \notin k(\alpha)$.

Now consider $Y \in h(\beta)$. Either $Y \in h(\beta) \cap \mathcal{M}_\beta$ or $Y \in h(\beta) \setminus [h(\beta) \cap \mathcal{M}_\beta]$. If the first case holds then, by the fact that $\alpha \in A$, $\Sigma(\alpha, A)$ holds and $\chi_\alpha(Y) = \chi_\beta(Y)$. Clearly, from the definition of χ_α, $\alpha \in \chi_\alpha(Y)$ and hence $\alpha \in \chi_\beta(Y)$. But $\chi_\beta(Y) \subseteq f(\beta, Y)$ so $\alpha \in f(\beta, Y)$.

If $Y \in h(\beta) \setminus [h(\beta) \cap \mathcal{M}_\beta]$, then $Y \notin \mathcal{M}_\beta$. But $h_{i_m} = h' \in \mathcal{M}_\beta$ and $\alpha = \alpha_m \in \mathcal{M}_\beta$, thus $h'(\alpha) \in \mathcal{M}_\beta$. Moreover, as a finite element of $\mathcal{M}_\beta$, $h'(\alpha)$ is a subset of $\mathcal{M}_\beta$. Hence, $Y \notin h'(\alpha)$ so $n_\Delta(Y) \neq \alpha$ and $g_{\alpha,Y}(\beta) = 0$. From the definition of f, this implies that $\alpha \in f(\beta, Y)$ and we have demonstrated one half of $(*)$.

Suppose that $Y \in h(\alpha)$. Then either $Y \in h(\beta) \cap \mathcal{M}_\beta$ or $Y \in h'(\alpha) = h(\alpha) \setminus [h(\beta) \cap \mathcal{M}_\beta]$. Just as for the above, $Y \in h(\beta) \cap \mathcal{M}_\beta$ means that $\beta \in f(\alpha, Y)$.

If $Y \in h'(\alpha)$, $n_\Delta(Y) = \alpha$. But then $g_{\alpha,Y}(\beta) = 1$ from which it follows that $\beta \in f(\alpha, Y)$.

Hence $(*)$ is satisfied.

$\square$

A small Dowker space

A Dowker space is a normal Hausdorff space the product of which with the unit interval is not normal. Such spaces are named after Dowker who showed that:

Theorem 6.3.4 **[D]** *For a normal space X, the following are equivalent:*

1. *$X \times I$ is normal*
2. *X is countably paracompact*
3. *X is countably metacompact*
4. *For every countable open cover $\mathcal{U} = \{U_n : n \in \omega\}$, there is an open cover $\mathcal{V} = \{V_n : n \in \omega\}$ such that $\overline{V_n} \subseteq U_n$ for all $n \in \omega$*
5. *For any increasing sequence of open sets $\{G_n : n \in \omega\}$ which cover X, there is an increasing sequence of closed sets $\{F_n : n \in \omega\}$ which also covers X and such that $F_n \subseteq G_n$ for all $n \in \omega$*

He then asked if all normal Hausdorff spaces are countably paracompact. Answering this question has provoked a great deal of exciting work and Dowker's characterisations (5) of countable paracompactness has been crucial in attacking the problem. The question was finally answered by Rudin [R$_1$] who produced a Dowker space in ZFC. However, the example is "big" in many senses, for example, it has weight and cardinality $(\omega_\omega)^\omega$. This has provoked the question of whether Dowker spaces could be smaller than this. Many excellent examples of small Dowker spaces have been given in various models of set-theory, see [R$_4$], but until recently Rudin's remained the quintessential Dowker space in ZFC.

In 1994, Balogh announced at the Spring Topology Conference that he had found a small Dowker space in ZFC - one which was hereditarily normal, σ-discrete and of cardinality $\mathfrak{c}$. His construction makes essential use of elementary submodels. Some have challenged that it is not truly "small" because it is not first countable but it is indisputably a truly new example of a Dowker space and, as such, is of great value. We describe here the space based on [B$_2$].

The space has its roots in the normal, not collectionwise Hausdorff space which has just been described. Whereas that space is made up of two "layers", $\mathfrak{c}$ and $[\mathfrak{c}]^2$, the Dowker space has countably many and the underlying set is $X = \mathfrak{c} \times \omega$. Take $X_n = \mathfrak{c} \times \{n\}$ and $G_n = \mathfrak{c} \times (n+1)$. Each X_n will be discrete and $\{G_n : n \in \omega\}$ will be the open cover witnessing that X is not countably paracompact. That is, if $F_n \subseteq G_n$ are closed sets for each $n \in \omega$ then $\bigcup_{n\in\omega} F_n \neq X$.

The basic open neighbourhoods are defined in terms of local network elements. More precisely, for a point $\langle \alpha, n\rangle \in X$, if $n = 0$ then $N(\langle \alpha, n\rangle, Y, K) = \{\langle \alpha, n\rangle\}$. For $n > 0$, we shall define $f : \mathfrak{c} \times \mathcal{P}(\mathfrak{c}) \to \mathcal{P}(\mathfrak{c})$ and take

$$N(\langle \alpha, n\rangle, Y, K) = \{\langle \alpha, n\rangle\} \cup \{\langle \beta, n-1\rangle : \beta \in f(\alpha, Y) \setminus K\}$$

where K is a finite subset of $\mathfrak{c}$. The presence of the K ensures that X is T_1. A set U is open if and only if, for every point $x \in U$, there exist $\mathcal{C} \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$ and $K \in [\mathfrak{c}]^{<\omega}$ such that

$$\bigcap_{Y\in\mathcal{C}} N(x, Y, K) \subseteq U$$

This definition immediately implies that G_n is open and X_n is discrete for all $n \in \omega$. In particular, X_0 is the set of all isolated points of X.

Normality will follow from a straightforward boot-strapping argument once it has been shown that any two disjoint closed sets in X_n can be separated by disjoint open sets. Because each X_n is discrete, we must consider <u>any</u> pair of disjoint sets in X_n. The proof of this proceeds by induction, the case for X_0 being trivial. Thus assume that for some $n \in \omega$, if B_0, $B_1 \subseteq X_{n-1}$ are disjoint then they can be separated by disjoint open sets. Consider X_n. As in the previous example, it suffices to show that for any $A \subseteq X_n$, A can be separated from its complement. To do this we place a constraint on the $f(\alpha, Y)$: define $g : \mathcal{P}(\mathfrak{c}) \to \mathcal{P}(\mathfrak{c})$ and set

$$f(\alpha, Y) = \begin{cases} g(Y) & \text{if } \alpha \in Y \\ \mathfrak{c} \setminus g(Y) & \text{if } \alpha \notin Y \end{cases}$$

If $Y = \{\alpha \in \mathfrak{c} : \langle \alpha, n\rangle \in A\}$, it is clear that $A \cup (g(Y) \times \{n-1\})$ contains $N(\langle \alpha, n\rangle, Y, \emptyset)$ for all $\langle \alpha, n\rangle \in A$ and $(X_n \setminus A) \cup (\mathfrak{c} \setminus g(Y) \times \{n-1\})$ contains $N(\langle \alpha, n\rangle, Y, \emptyset)$ for all $\langle \alpha, n\rangle \notin A$. $g(Y) \times \{n-1\}$ and $\mathfrak{c} \setminus g(Y) \times \{n-1\}$ are disjoint subsets of X_{n-1} and, by the inductive hypothesis, they can be separated by disjoint open sets U and V. (We may assume that $U \cup V \subseteq G_{n-1}$.) It then follows that $A \cup U$ and $(X_n \setminus A) \cup V$ are disjoint open sets separating A from its complement in X_n. Hence X is normal.

To show that X is not countably paracompact, Balogh introduces the notion of σ-decomposable. As this is always used negatively, we define rather the term *indecomposable.* A subset A of $\mathfrak{c}$ is indecomposable if for any $l : A \to \omega$, $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ and $k : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$, there exist α, $\beta \in \mathfrak{c}$ such that $l(\alpha) = l(\beta)$ and $\beta \in \bigcap\{f(\alpha, Y) \setminus k(\alpha) : Y \in h(\alpha)\}$. In other words, (‡)

$$l(\alpha) = l(\beta),\ \beta \notin k(\alpha) \text{ and, for all } Y \in h(\alpha),\ \beta \in g(Y) \text{ if and only if } \alpha \in Y$$

Because of the presence of l in this definition, if Y is indecomposable and the countable union of some sets then one of those sets must also be indecomposable. Moreover,

Lemma 6.3.5 *If $n \in \omega$ and $Y \subseteq \mathfrak{c}$ is indecomposable then $Y_1 = \{\alpha \in Y : \langle \alpha, n+1\rangle \in \overline{Y \times \{n\}}\}$ is indecomposable.*

Proof Define $Y_0 = Y \setminus Y_1$. If $\alpha \in Y_0$ then $\langle \alpha, n+1\rangle \notin \overline{Y \times \{n\}}$. Hence, there exist $Y_1, \ldots, Y_k \in \mathcal{P}(\mathfrak{c})$ and $K_1, \ldots, K_k \in [\mathfrak{c}]^{<\omega}$ such that

$$\bigcap_{i=1}^{k} N(\langle \alpha, n+1\rangle, Y_i, K_i) \cap Y = \emptyset$$

Therefore, $\bigcap_{i=1}^k (f(\alpha, Y_i) \setminus K_i) \cap Y = \emptyset$ and, for all $\beta \in Y$, $\beta \notin \bigcap_{i=1}^k f(\alpha, Y_i) \setminus K_i$. By defining $h(\alpha) = \{Y_i : i = 1, \ldots, k\}$, $k(\alpha) = \bigcup_{i=1}^k K_i$ and $l(\alpha) = 0$, it is easy to check that the resulting h, k and l witness that Y_0 is decomposable. But $Y = Y_0 \cup Y_1$ and Y is indecomposable hence Y_1 must also be indecomposable. $\square$

The key to the construction is proving that for some suitable g, $\mathfrak{c}$ is indecomposable. Given this, consider the open cover $\{G_m : m \in \omega\}$ of X. Suppose $\{F_m : m \in \omega\}$ is a sequence of closed sets which also cover X. Define $Y_m = \{\alpha \in \mathfrak{c} : \langle \alpha, 0 \rangle \in F_m\}$. It must be that $\mathfrak{c} = \bigcup_{m \in \omega} Y_m$ and since $\mathfrak{c}$ is indecomposable, for some $m_0 \in \omega$, Y_m is indecomposable. Now $F_{m_0} \supseteq \overline{Y_{m_0} \times \{0\}}$ and, by inducting up using the Lemma, for all $n \in \omega$, $X_n \cap F_{m_0} \neq \emptyset$. In particular, $F_{m_0} \not\subseteq G_{m_0}$. Hence, X is not countably paracompact.

This completes the description of the Dowker space. It remains to prove:

Theorem 6.3.6 *There is a $g : \mathcal{P}(\mathfrak{c}) \to \mathcal{P}(\mathfrak{c})$ which makes $\mathfrak{c}$ indecomposable.*

Remark A noticeable difference in this construction is the use of pairs of elementary submodels rather than only one at a time. The reason for this is that in the process of defining the $g(Y)$, there are two cases to be considered. The first case treats what happens on the root of some Δ-system and the second, what happens off the root. In the previous examples, what happens on the root has been quite trivial but for this example more care must be taken. However, it is impossible to predict beforehand what this root will be! What we can say though is that, from the proof of the Δ-system which we have given, we can ensure that the root will always lie within any suitable countable elementary submodel. Moreover, anything which is not in the root is not in that submodel.

Proof List all countable elementary submodels as $\{M_\beta : \beta \in \mathfrak{c} \setminus \epsilon\}$ and, for each $\beta \in \mathfrak{c}$, choose another countable elementary submodel $\mathcal{N}_\beta$ for which $\mathcal{M}_\beta \in \mathcal{N}_\beta$. Note that since $\mathcal{M}_\beta$ is a countable element of $\mathcal{N}_\beta$ it is also a subset and anything which is placed in $\mathcal{M}_\beta$ is automatically in $\mathcal{N}_\beta$. We may also assume that $\beta \supseteq \mathfrak{c} \cap \mathcal{N}_\beta$ whence $\beta \notin \mathcal{N}_\beta$.

For all $Y \in \mathcal{P}(\mathfrak{c})$, we inductively define whether $\beta \in \mathfrak{c}$ is an element of $g(Y)$ or not by considering the model $\mathcal{N}_\beta$. List all the functions $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c}) \setminus \mathcal{M}_\beta]^{<\omega}$ and infinite subsets A of $\mathfrak{c}$ in $\mathcal{N}_\beta$ as $\{h_i : i \in \omega\}$ and $\{A_j : j \in \omega\}$ respectively. It will become apparent that we will only need to deal with certain h and A. Thus, let $\{\langle i_n, j_n \rangle : n \in \omega\}$ be a denumeration of the pairs $\langle i, j \rangle \in \omega^2$ such that $\{h_i(\alpha)\}_{\alpha \in A_j}$ is an infinite disjoint collection.

Just as in the Q-space construction, use the elementarity of $\mathcal{N}_\beta$ to choose $\alpha_n \in A_{j_n} \cap \mathcal{N}_\beta$ such that $\alpha_n \notin \{\alpha_1, \ldots, \alpha_{n-1}\}$ and

$$h_{i_n}(\alpha_n) \cap \bigcup_{m<n} h_{i_m}(\alpha_m) = \emptyset$$

Because $\{h_{i_n}(\alpha_n)\}_{n \in \omega}$ forms a disjoint collection, setting $n_\Delta(Y) = \alpha$ if and only if $Y \in h_{i_n}(\alpha_n)$ is a good definition.

In order to define $g(Y)$, there are two case to consider as mentioned in the remark preceding the proof:

1. For $Y \in \mathcal{M}_\beta$, $\beta \in g(Y)$ if and only if $\beta \in Y$
2. For $n_\Delta(Y) = \alpha$, $\beta \in g(Y)$ if and only if $\alpha \in Y$.

Since the range of all the h_i's misses $\mathcal{M}_\beta$, the two cases are not conflicting. For $Y \in \mathcal{P}(\mathfrak{c})$, β is placed arbitrarily in $g(Y)$.

This completes the definition of the g and it remains to show that this does make $\mathfrak{c}$ indecomposable.Consider some $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$, $l : \mathfrak{c} \to \omega$ and $k : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$. We wish to find $\alpha \in \beta \in \mathfrak{c}$ satisfying (‡) above.

Define $\pi h(\alpha) = \{Y \in h(\alpha) : \alpha \in Y\}$. Take an elementary submodel containing g, h, l, k, $\mathfrak{c}$, $\mathcal{P}(\mathfrak{c})$ and $\{\pi h(\alpha)\}_{\alpha \in \mathfrak{c}}$. This submodel is isomorphic to $\mathcal{M}_\beta$ say. Recall, $\beta \notin \mathcal{N}_\beta$ and $\mathcal{M}_\beta \subseteq \mathcal{N}_\beta$. Since $h(\beta)$ is a finite set, $h(\beta) \cap \mathcal{M}_\beta$ and $\pi h(\beta) \cap \mathcal{M}_\beta$ are finite subsets of $\mathcal{M}_\beta$ and are hence elements of $\mathcal{M}_\beta$. $l(\beta)$ being a natural number is also an element of $\mathcal{M}_\beta$. Define $\Xi(\alpha, A)$ to be the statement:

$$(l(\alpha) = l(\beta)) \wedge (h(\alpha) \cap \mathcal{M}_\beta = h(\beta) \cap \mathcal{M}_\beta) \wedge (\pi h(\alpha) \cap h(\beta) \cap \mathcal{M}_\beta = \pi h(\beta) \cap \mathcal{M}_\beta) \wedge$$
$$\wedge \forall \alpha' \in A(\alpha \neq \alpha' \to h(\alpha) \cap h(\alpha') = h(\beta) \cap \mathcal{M}_\beta)$$

Once again β ensures $V \models \exists \alpha' \in \mathfrak{c}(\Xi(\alpha', \emptyset))$. By elementarity, $\mathcal{N}_\beta \models \exists \alpha' \in \mathfrak{c}(\Xi(\alpha', \emptyset))$. Find $\alpha' \in \mathcal{N}_\beta$ which is asserted to exist by this expression. If $A_0 = \{\alpha'\}$ then $\mathcal{N}_\beta \models \forall \alpha \in A_0(\Xi(\alpha, A_0))$. Just as in the previous examples, apply Zorn's Lemma to find a maximal such $A \in \mathcal{N}_\beta$ so that:

$$\mathcal{N}_\beta \models \forall \alpha \in A(\Xi(\alpha, A)) \wedge \forall \gamma \in \mathfrak{c}(\Xi(\gamma, A) \to \gamma \in A)$$

Also as previously, A is an infinite element of $\mathcal{N}_\beta$ and for all $\alpha \in A$, $\Xi(\alpha, A)$ holds. In particular, $\{h(\alpha)\}_{\alpha \in A}$ is a infinite Δ-system of sets which only meet $\mathcal{M}_\beta$ on the root $h(\beta) \cap \mathcal{M}_\beta$. Define $h' : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c}) \setminus \mathcal{M}_\beta]^{<\omega}$, by

$$h'(\alpha) = h(\alpha) \setminus (h(\beta) \cap \mathcal{M}_\beta)$$

Thus, $\{h'(\alpha)\}_{\alpha \in A}$ is an infinite pairwise-disjoint family and h' and A were listed so that, for some $n \in \omega$, $h' = h_{i_n}$ and $A = A_{j_n}$. Take $\alpha = \alpha_n$. α and β will satisfy (‡).

First, since $\alpha \in A \cap \mathcal{N}_\beta$ and $k \in \mathcal{N}_\beta$, $k(\alpha) \in \mathcal{N}_\beta$ whence $\beta \notin k(\alpha)$. As $\alpha \in A$, $\Xi(\alpha, A)$ holds . This means $l(\alpha) = l(\beta)$. Now, we know that $h(\alpha) = (h(\beta) \cap \mathcal{M}_\beta) \cup h'(\alpha)$ and that this union is disjoint. There are two cases to consider:

1. For $Y \in h(\beta) \cap \mathcal{M}_\beta$, $\alpha \in Y$ if and only if $Y \in \pi h(\alpha)$. $\Xi(\alpha, A)$ implies that $\pi h(\alpha) \cap h(\beta) \cap \mathcal{M}_\beta = \pi h(\beta) \cap \mathcal{M}_\beta$. Therefore, $Y \in \pi h(\alpha)$ if and only if $Y \in \pi h(\beta)$ if and only if $\beta \in Y$. From the definition of $g(Y)$, $\beta \in g(Y)$ if and only if $\beta \in Y$ if and only if $\alpha \in Y$.
2. For $Y \in h'(\alpha)$, $n_\Delta(Y) = \alpha$ and $\beta \in g(Y)$ if and only if $\alpha \in Y$.

Thus for all $Y \in h(\alpha)$, $\beta \in g(Y)$ if and only if $\alpha \in Y$. That is, α and β do indeed satisfy (‡) and $\mathfrak{c}$ is indecomposable.

□

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