The AllDifferent Constraint: Efficiency Measures

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AllDifferent

- A vector of variables must take distinct values
- Very widely used – very important
- Examples:
  - A class of students must have lectures at distinct times
  - In a sports schedule, the teams playing on a particular week are all distinct
  - No pair of golfers play together more than once
  - Sudoku

AllDifferent

- Van Hoeve surveys various strengths of inference
- In order of increasing strength:
  - Weak and fast pairwise decomposition (AC) -- O(r)
  - Bound consistency – find Hall intervals (as described by Toby) and prune bounds – O(r log r)
  - Range consistency – find Hall intervals and prune
  - Generalised arc consistency (GAC) – $O(k^{0.5}\ r\ d)$
- We focus on GAC algorithm by Régin

GAC AllDifferent

- One expensive pass achieves consistency
- Traditionally has large incremental, backtracked data structure
- Traditionally low priority
- Triggered on any domain change
  - But many changes are processed together
- *No paper that we are aware of comprehensively investigates implementation decisions*

Our approach

- Investigate optimizations in literature (tried to find everything!)
- Trigger only on relevant values
  - It is not necessary to trigger on all domain removals
  - Identify O(2r+d) trigger values
- Partition the constraint dynamically
  - Algorithm already identifies independent sub-constraints
  - Store and re-use this partition
  - Run expensive algorithm only on sub-constraint

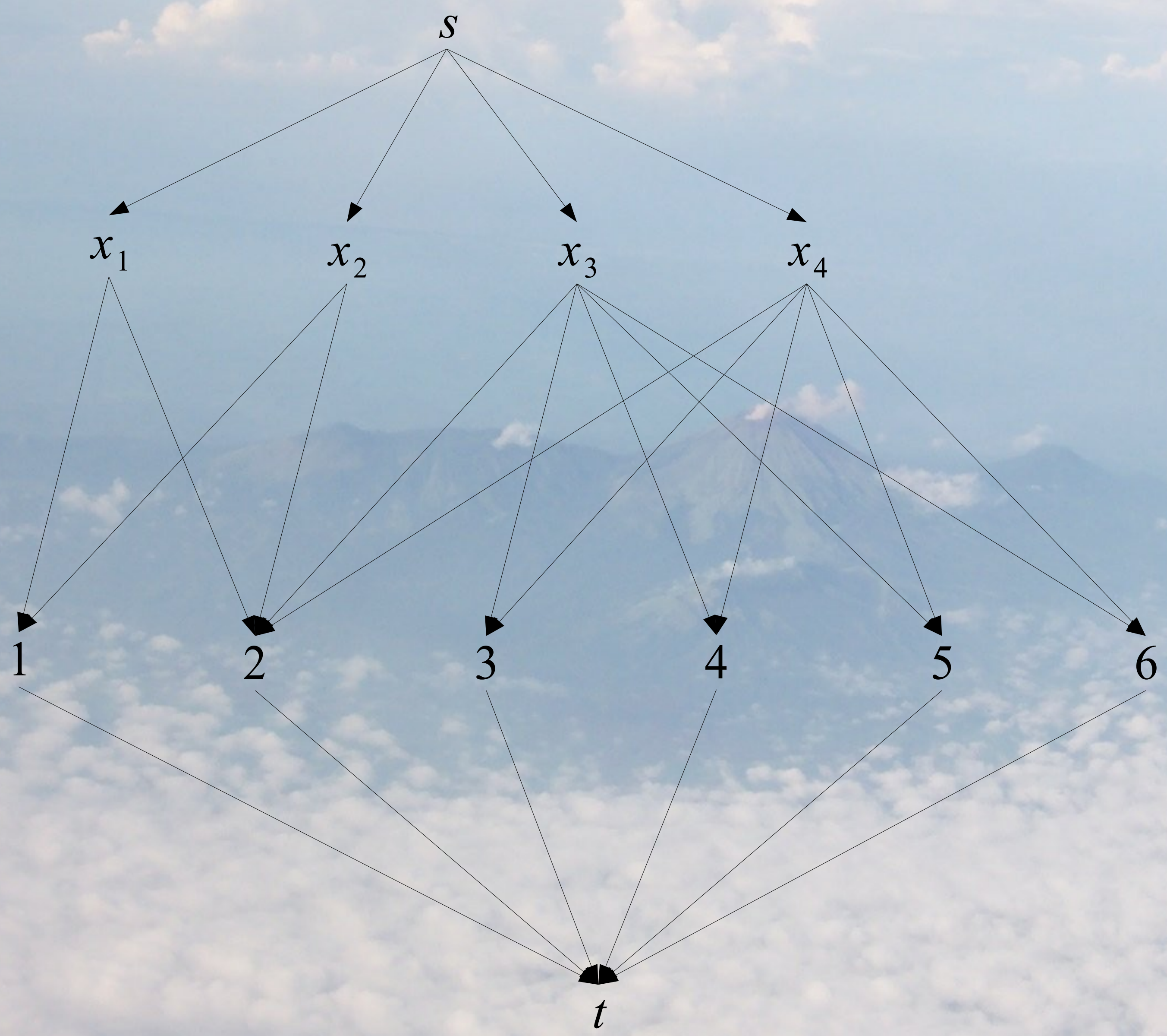
Régin's Algorithm

- Find a maximum matching M from variables to values.
  - Corresponds to a satisfying tuple of the constraint
- If $|M|<r$, the constraint is unsatisfiable
- Construct residual graph R (as described later)
- Edges not in M, and in no cycle in R, correspond to values to prune

Régin's Algorithm

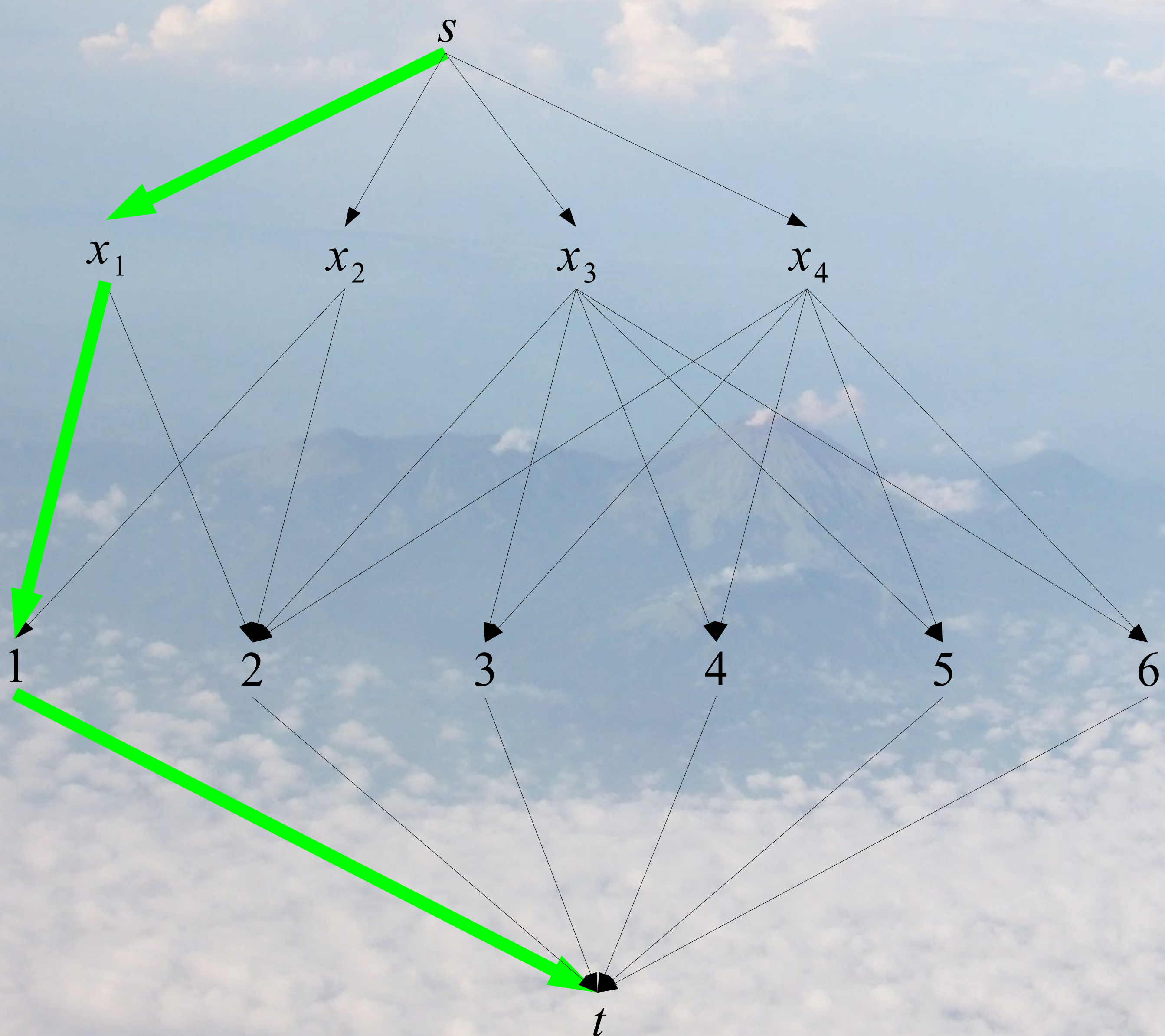
- Described in terms of flow, Ford-Fulkerson BFS algorithm
- Alternative is bipartite graph matching, Hopcroft-Karp or other algorithm

Régin's Algorithm



- Find maximum flow from s to t
- Ford-Fulkerson algorithm

Régin's Algorithm



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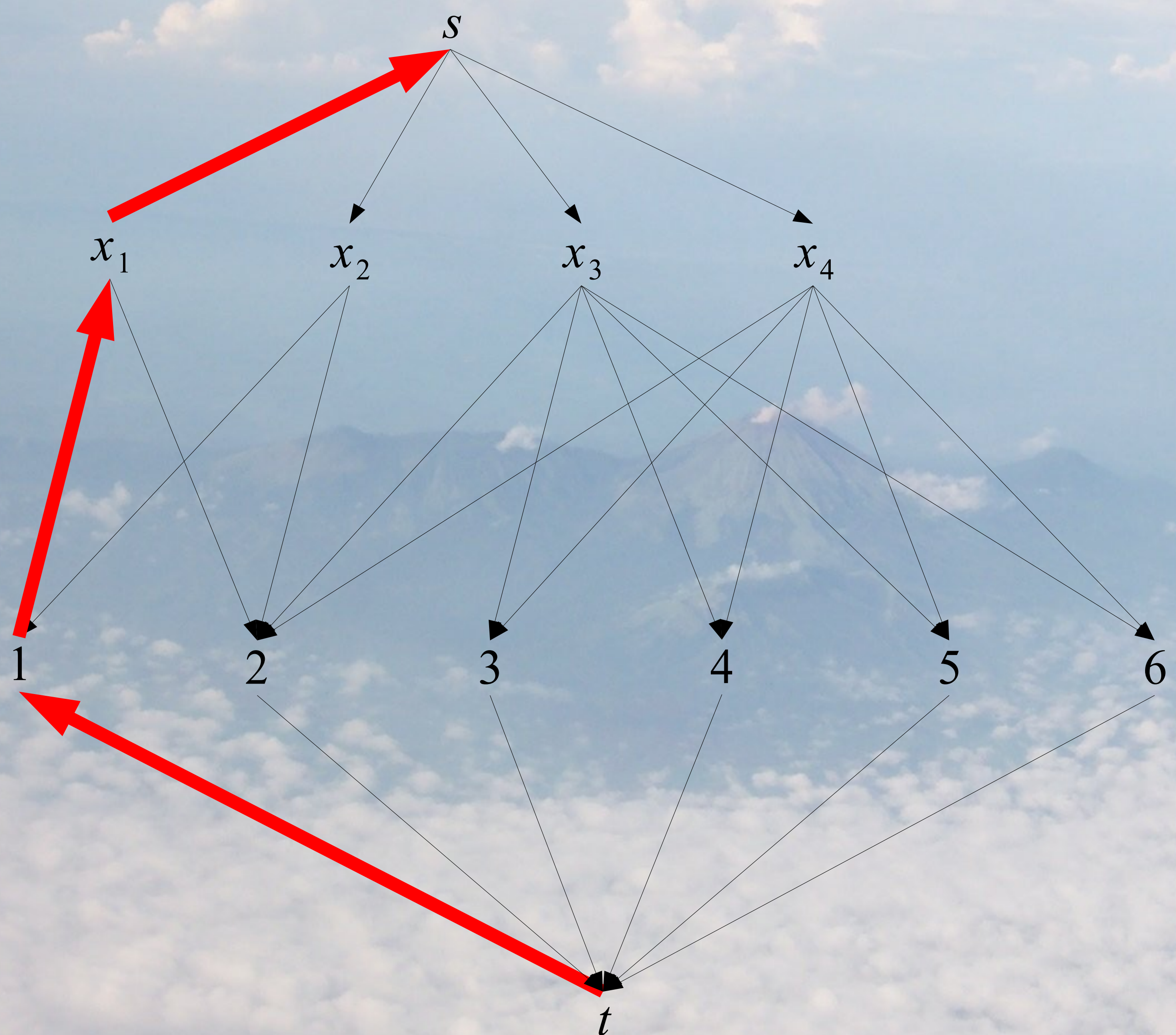

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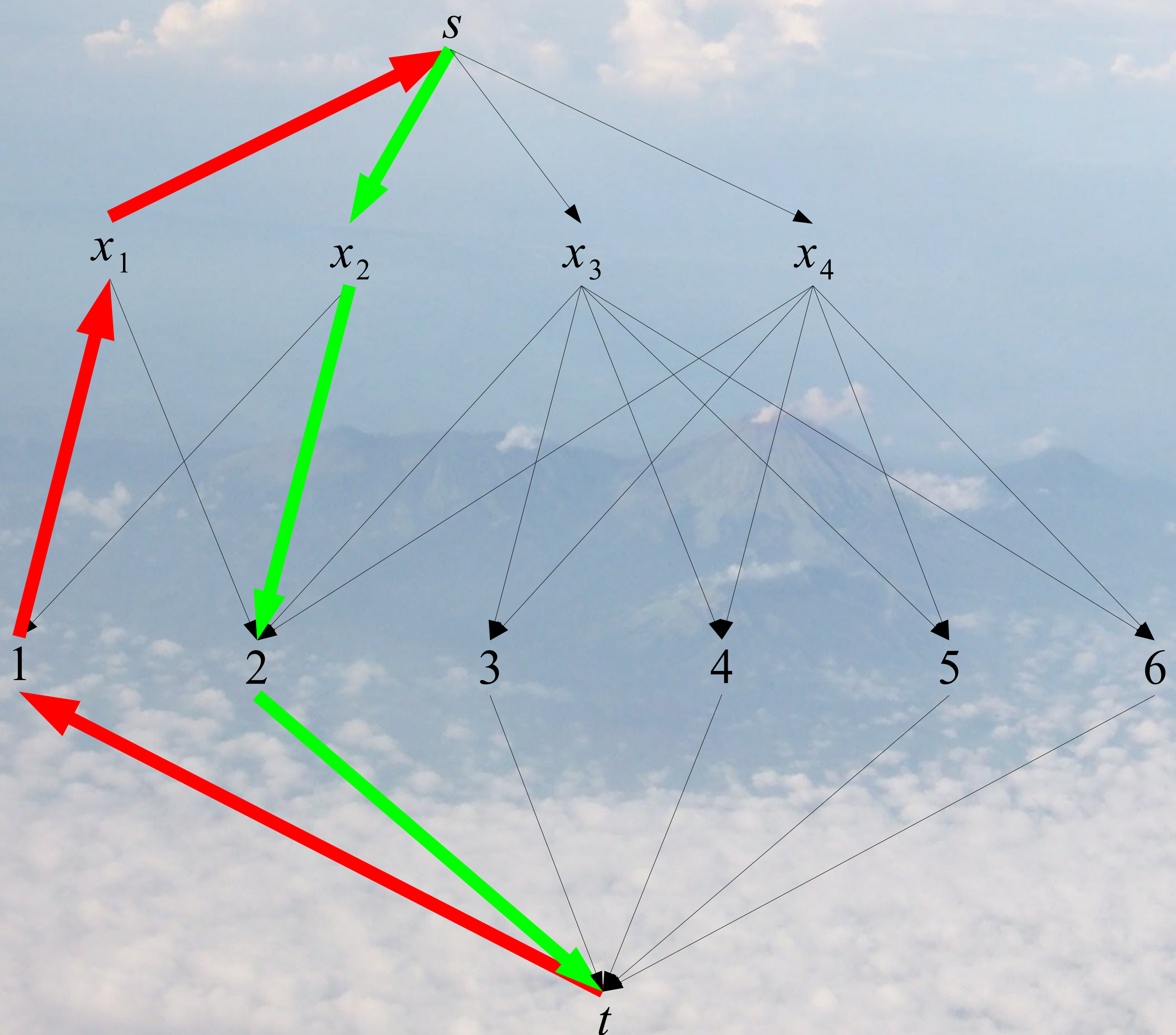

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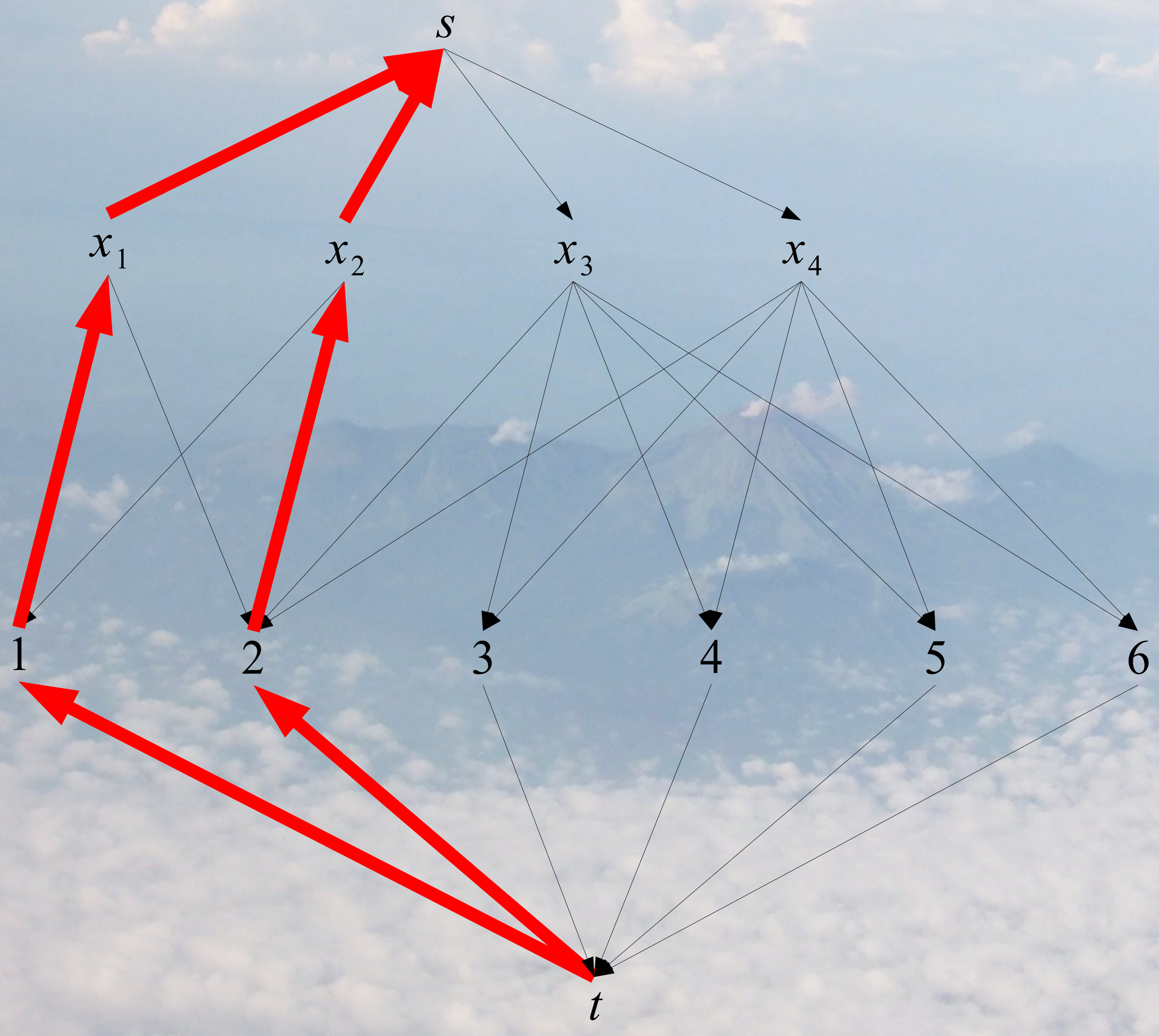

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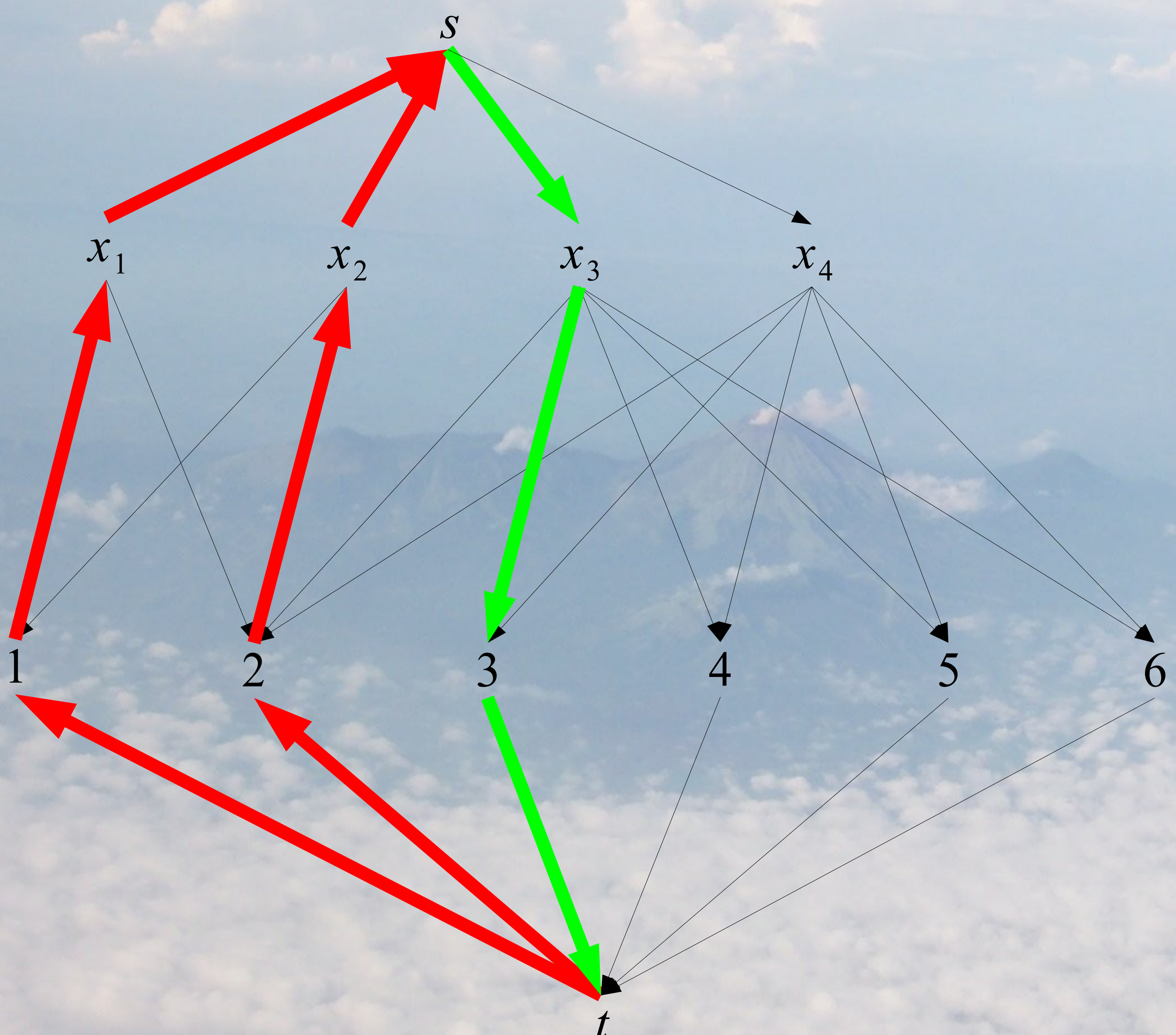

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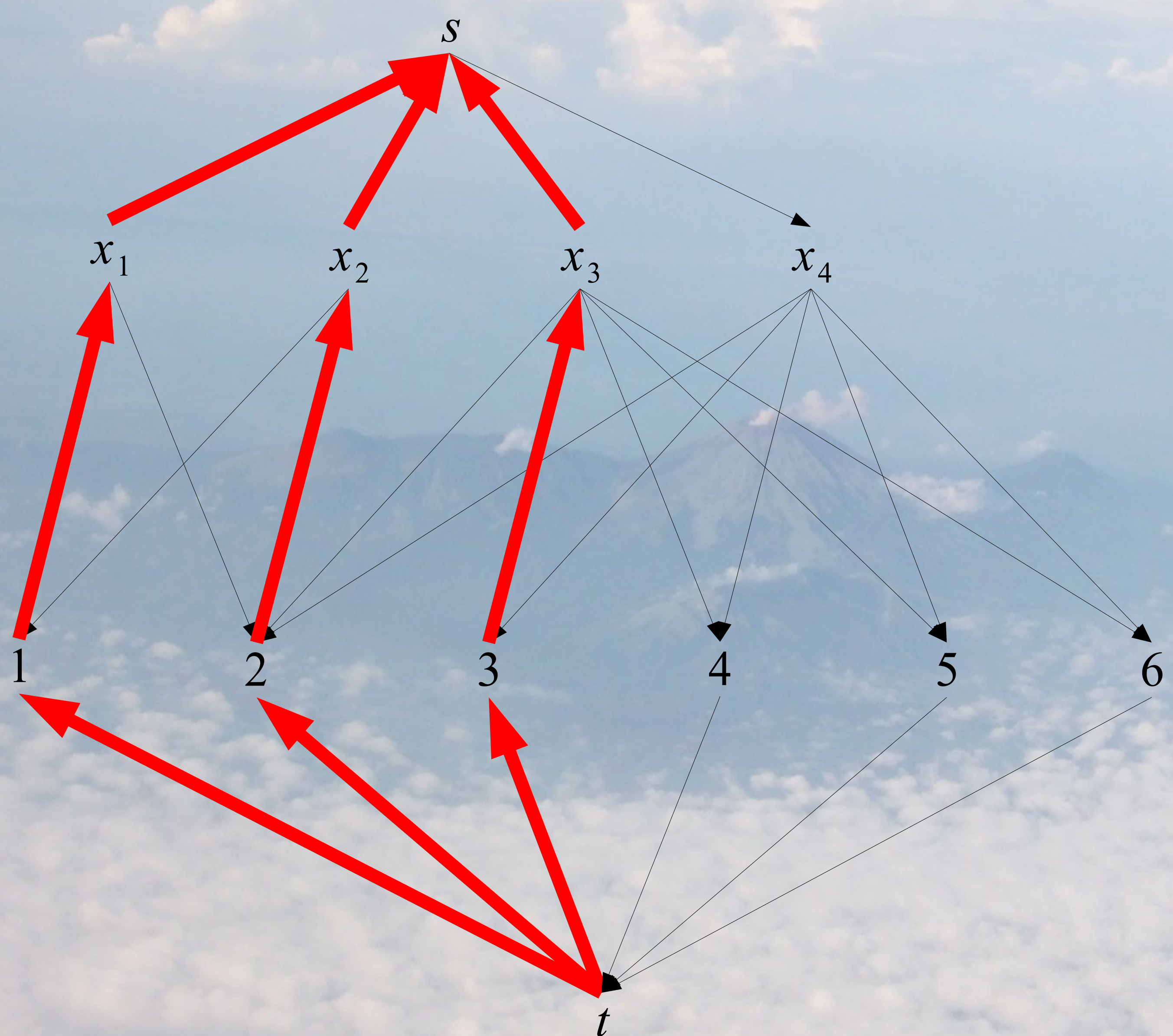

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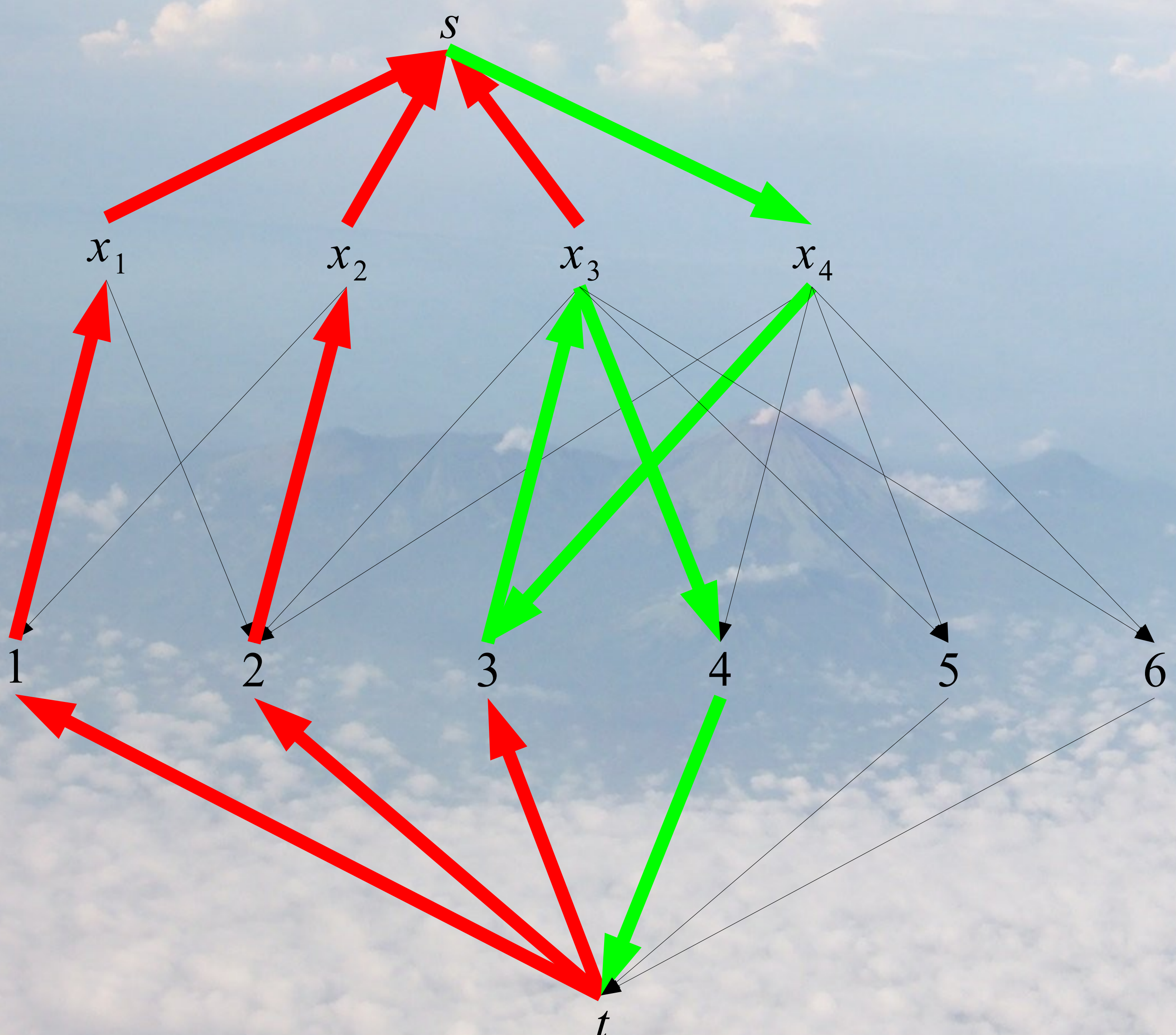

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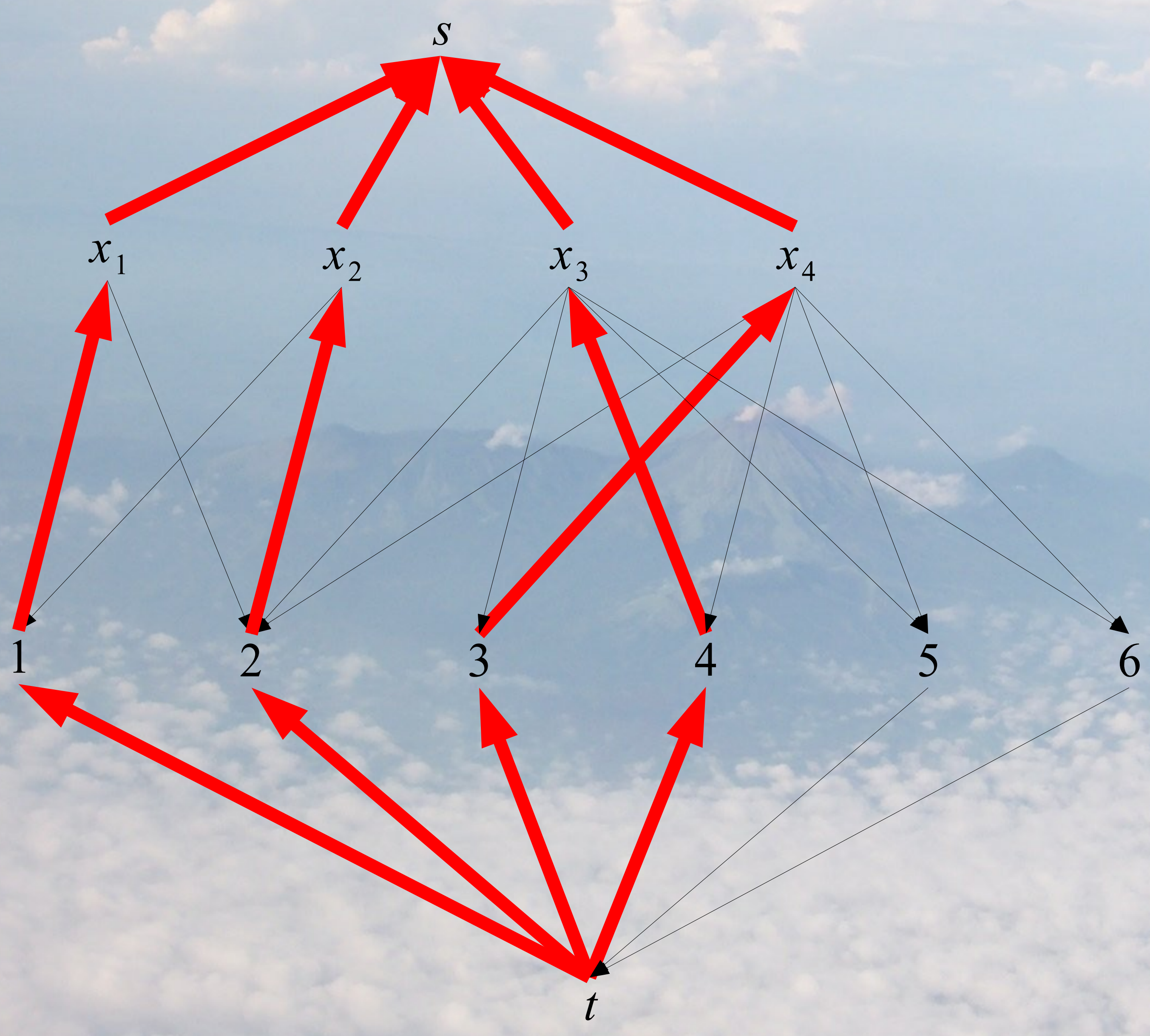
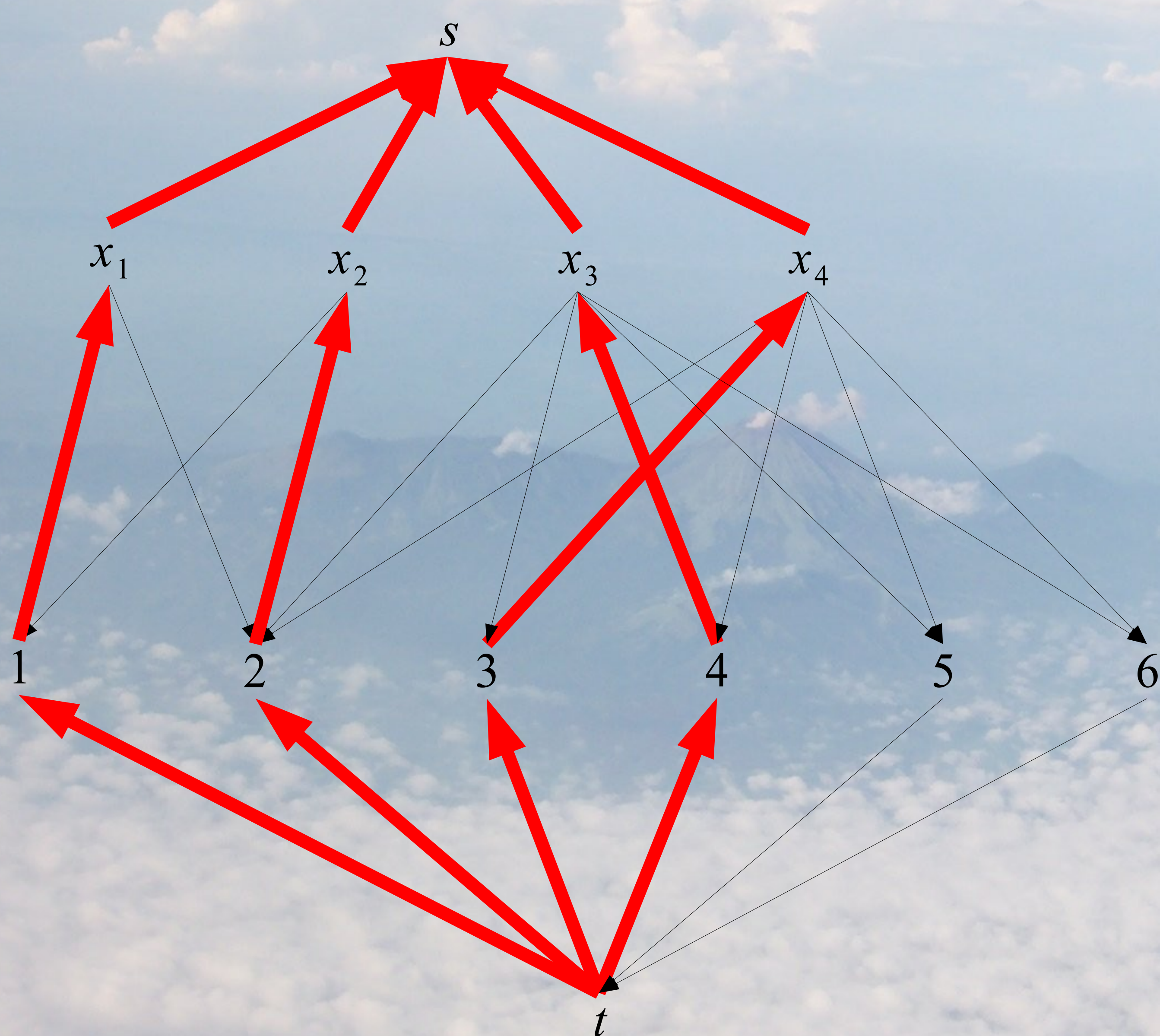


- Completed maximum flow from s to t
- Covers all variables (constraint is satisfiable)
- One of 24

Régin's Algorithm

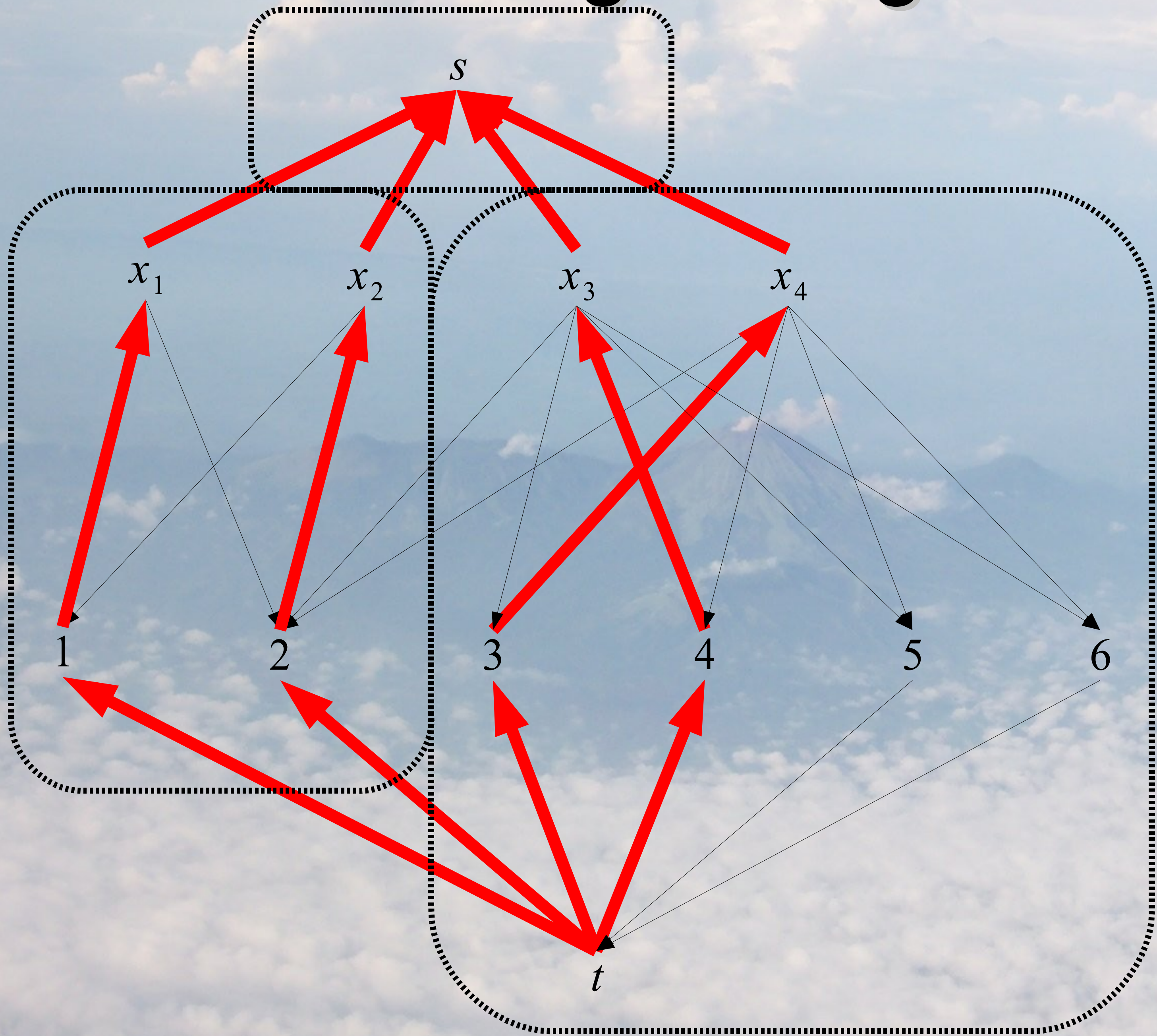


- Find strongly-connected components

Régin's Algorithm

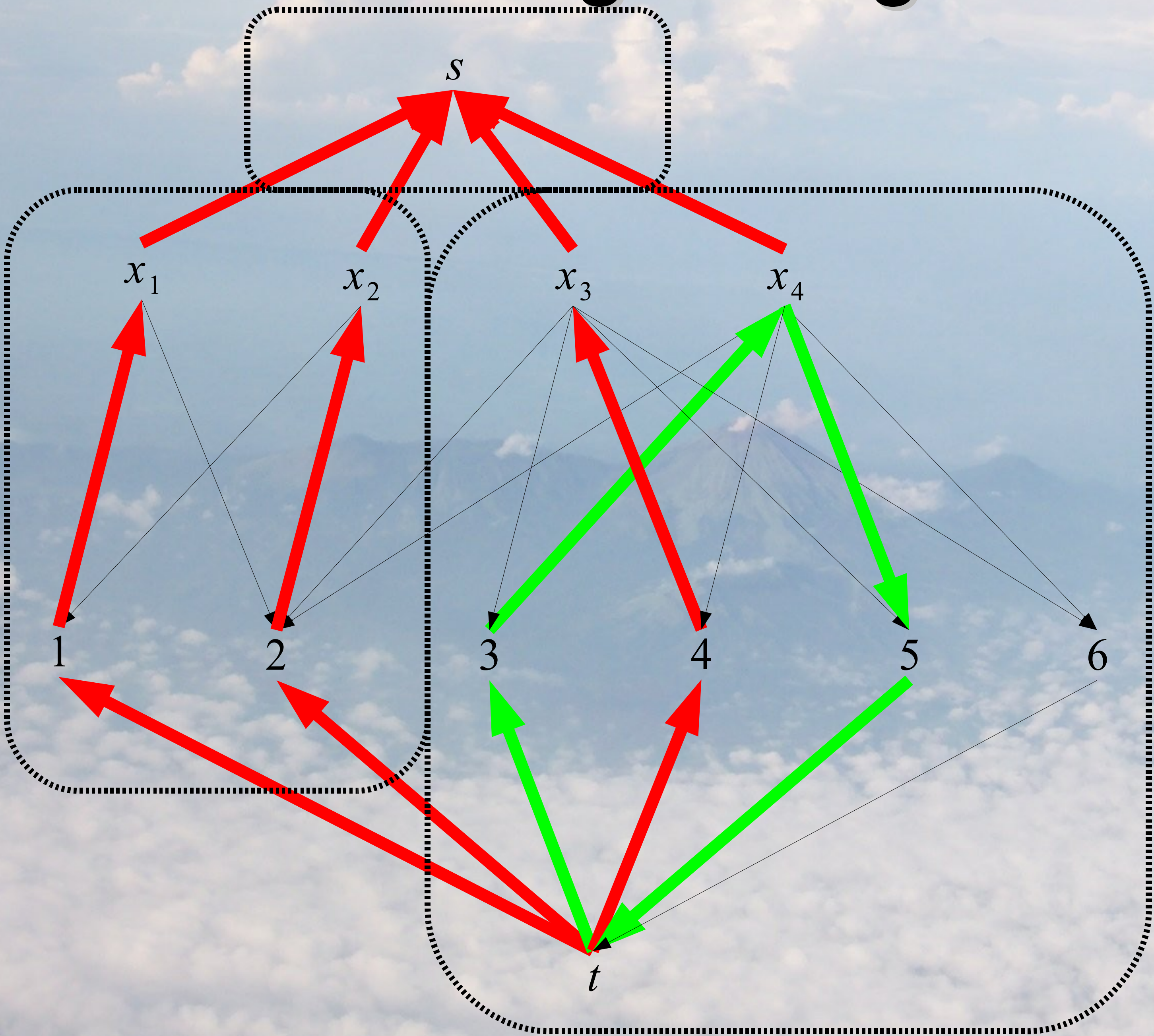
- Strongly-connected components (SCCs)
  - Vertices *i* and *j* in same SCC iff:
    - Path from *i* to *j* and from *j* to *i* in digraph
  - Found by Tarjan's algorithm
    - DFS
  - SCC='Maximal set of cycles'

Régin's Algorithm



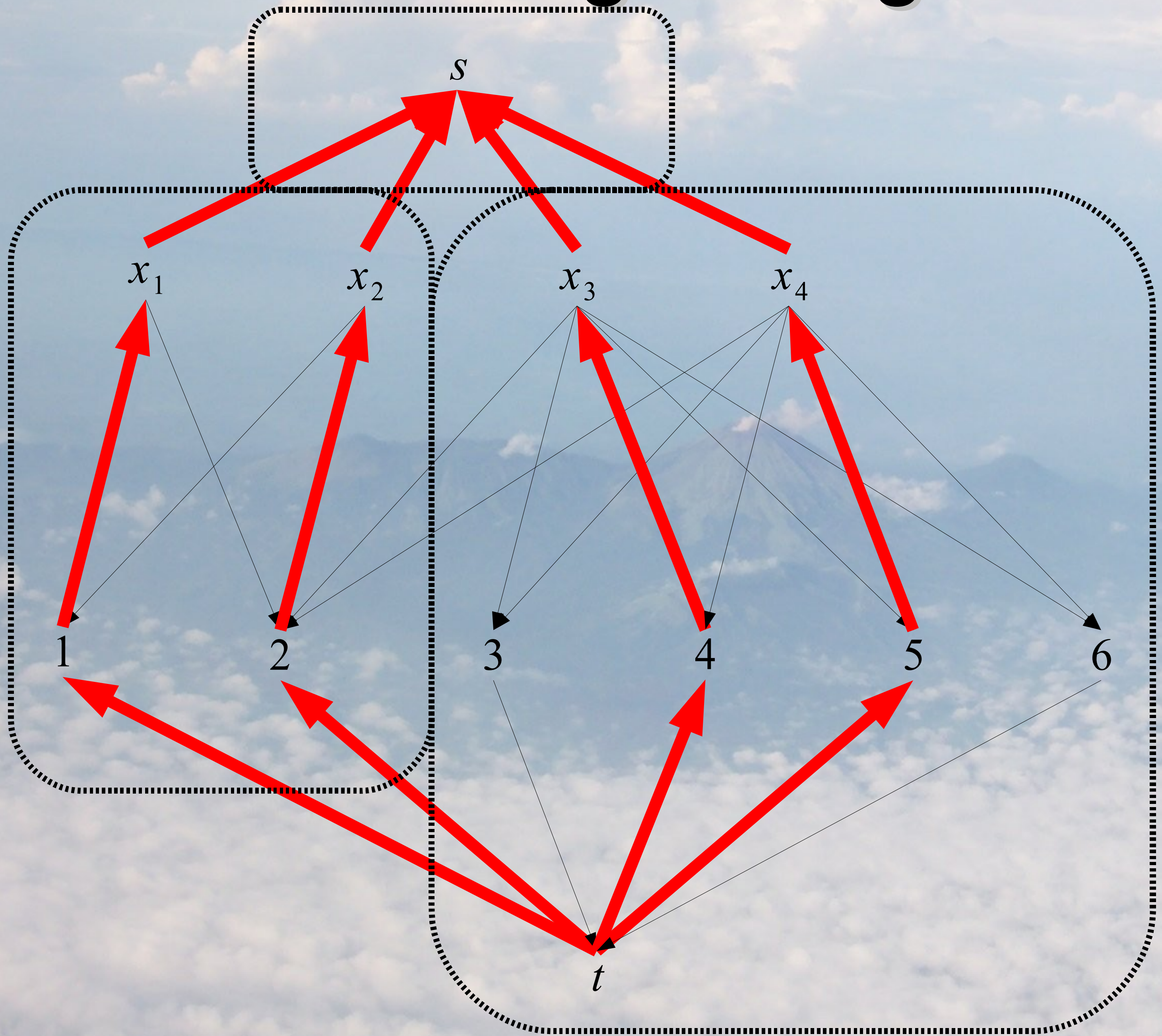
- Find strongly-connected components

Régin's Algorithm



- Cycle within SCC
- Apply cycle to find different maximum flow
- No cycles between SCCs

Régin's Algorithm



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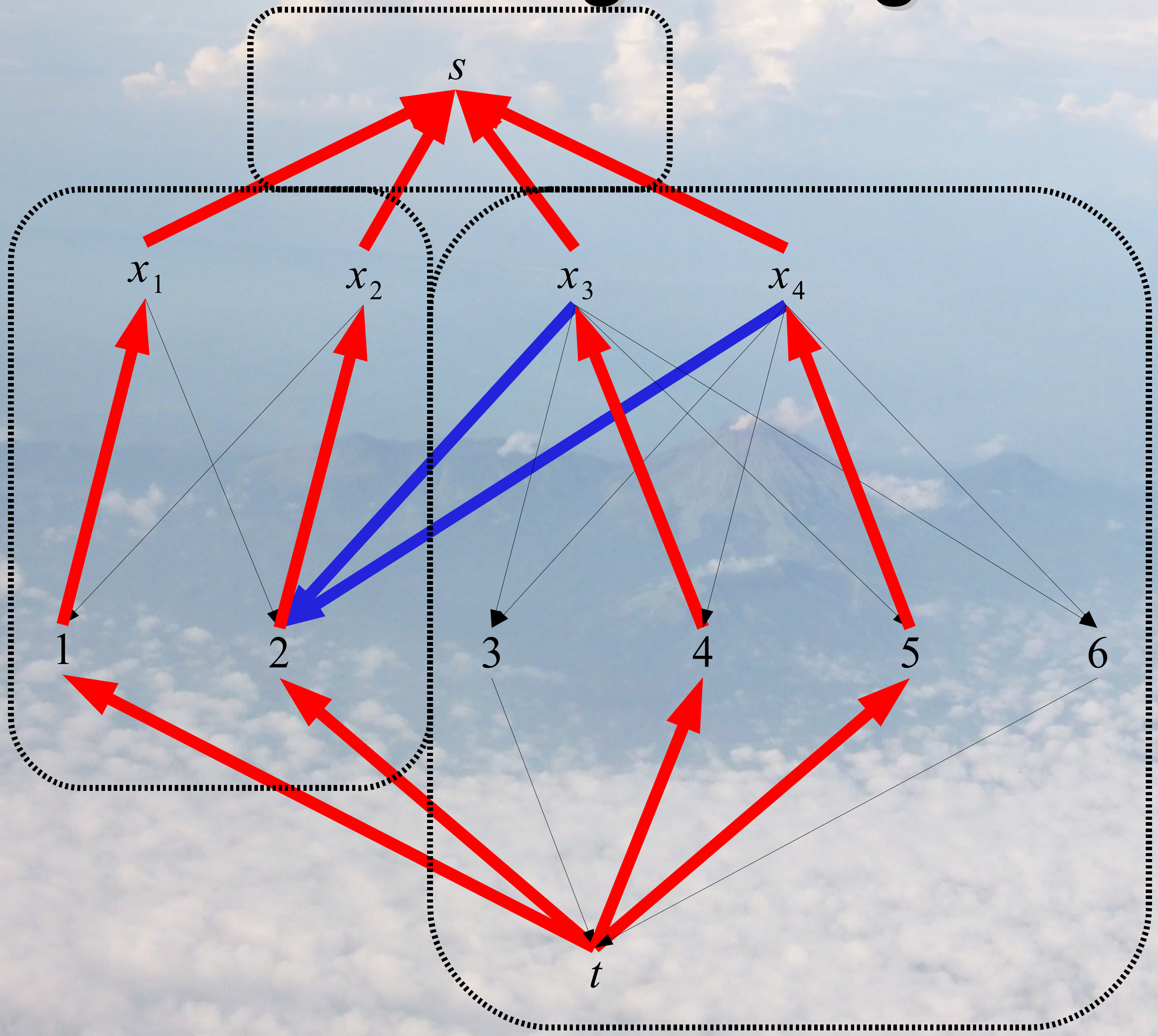
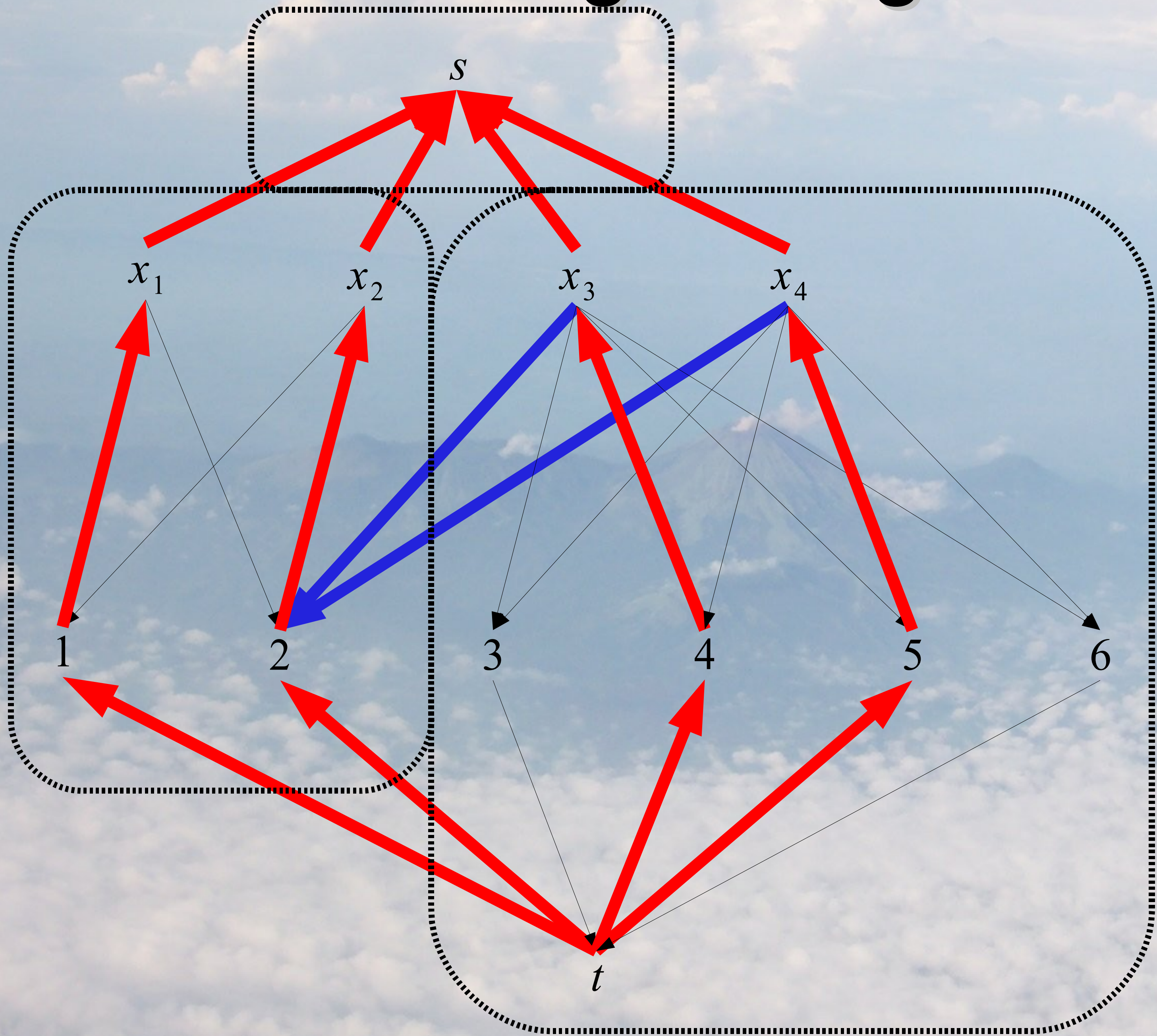


- No cycles between SCCs
- No maximum flows involving x3=2 or x4=2

Régin's Algorithm



- Remove edges which are:
  - Between SCCs
  - Not in flow
- Corresponds to theorem by Berge, 1973

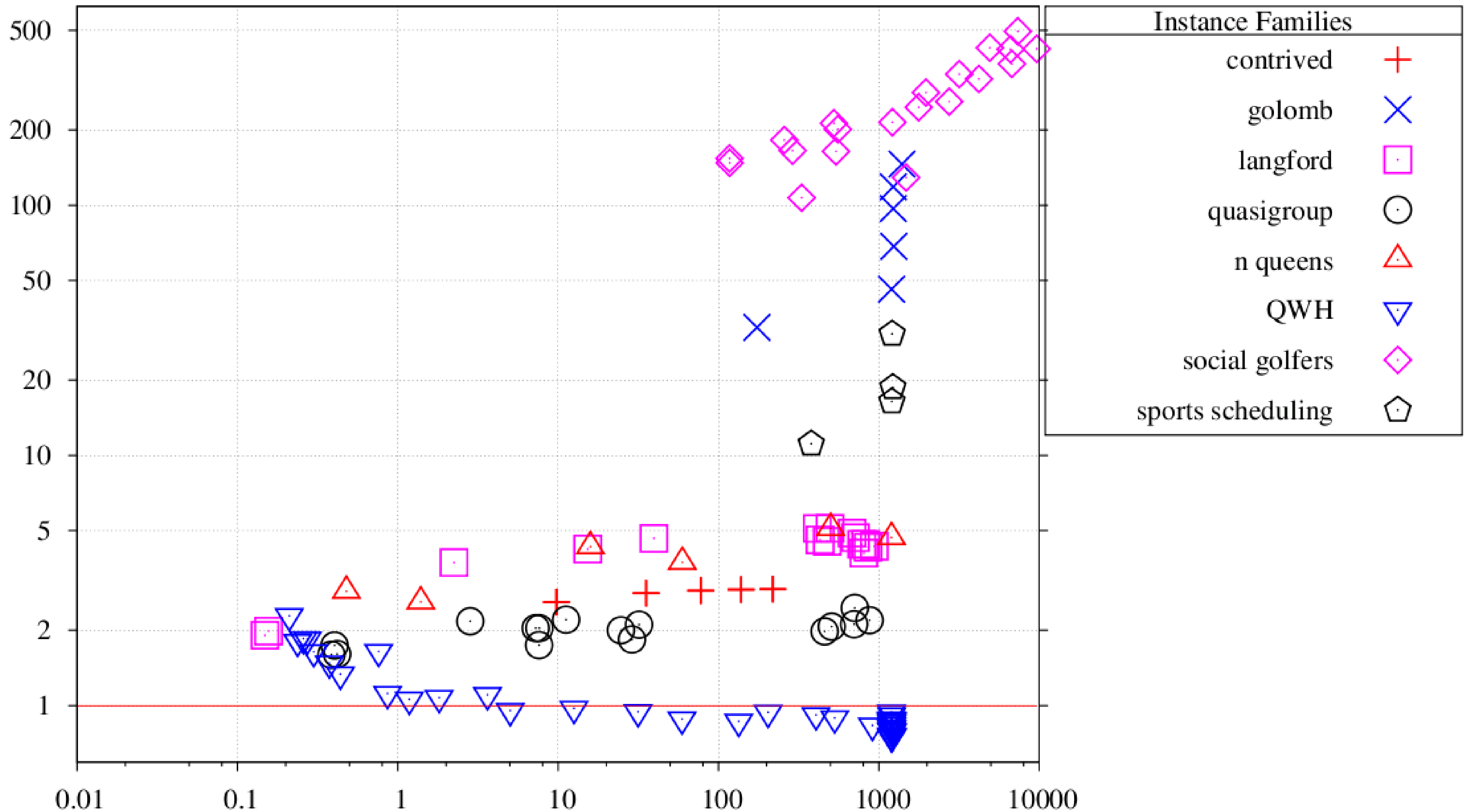
Implementation

- Key assumption: don't maintain the graph, discover it as you traverse
  - Domain queries cheap in Minion
  - Alternative: maintain and BT adjacency lists, size O(rd)
  - We claim this is better without experiment
  - If Patrick reads the paper, I'm in trouble!
  - If the assumption is not true, our experiments are somewhat less reliable, but the big results should still hold

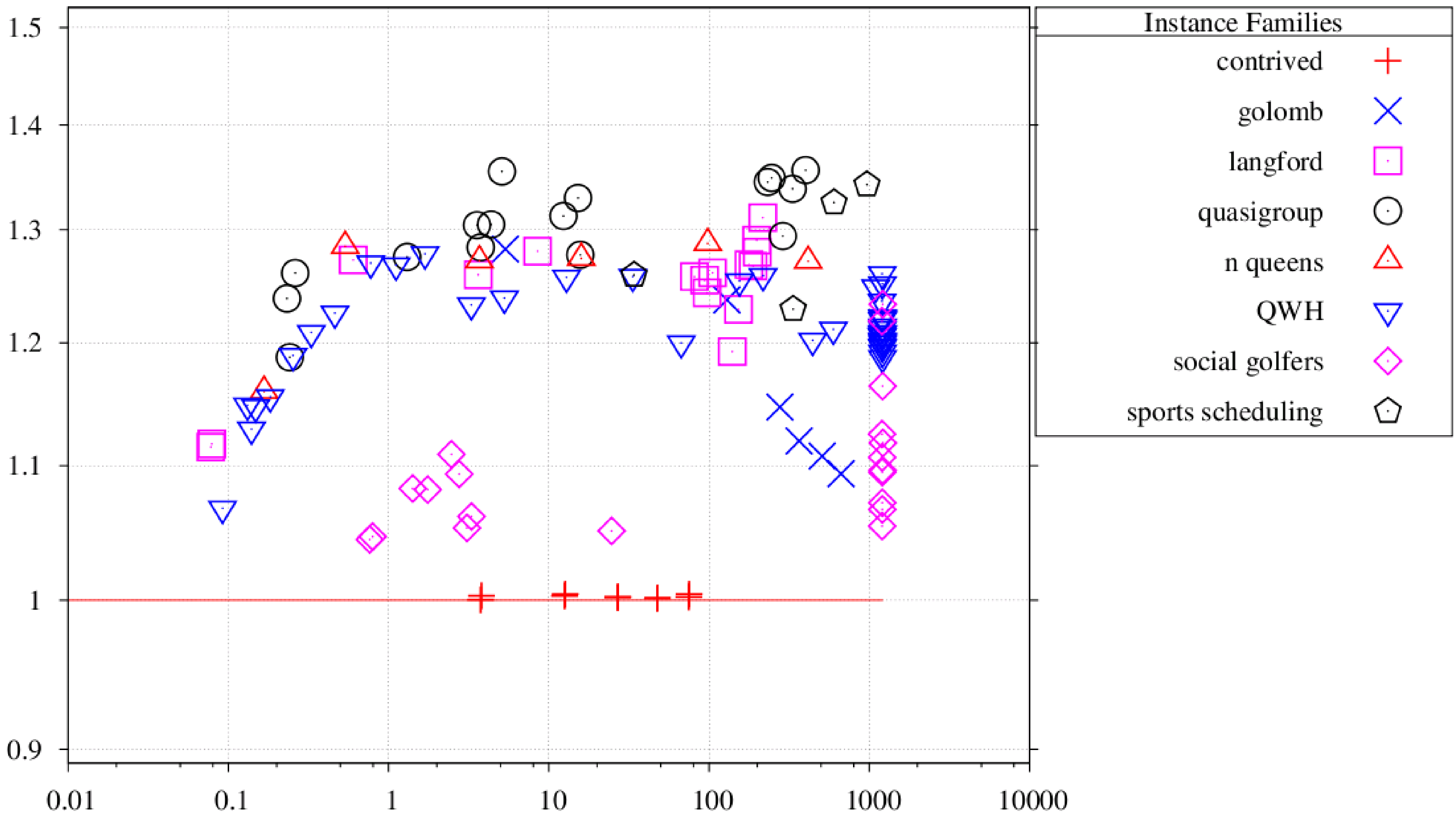
Optimizations in Literature

- Incremental matching (Régin)
- Priority Queue
  - Execute at low priority and with no duplicate events
- Staged propagation (Schulte & Stuckey)
  - Do simple propagation at high priority, GAC at low priority
- Domain counting (Quimper & Walsh)
- Fixpoint reasoning (Schulte & Stuckey)
  - Solves the 'Double Call Problem'
- Advisors (Lagerkvist & Schulte)

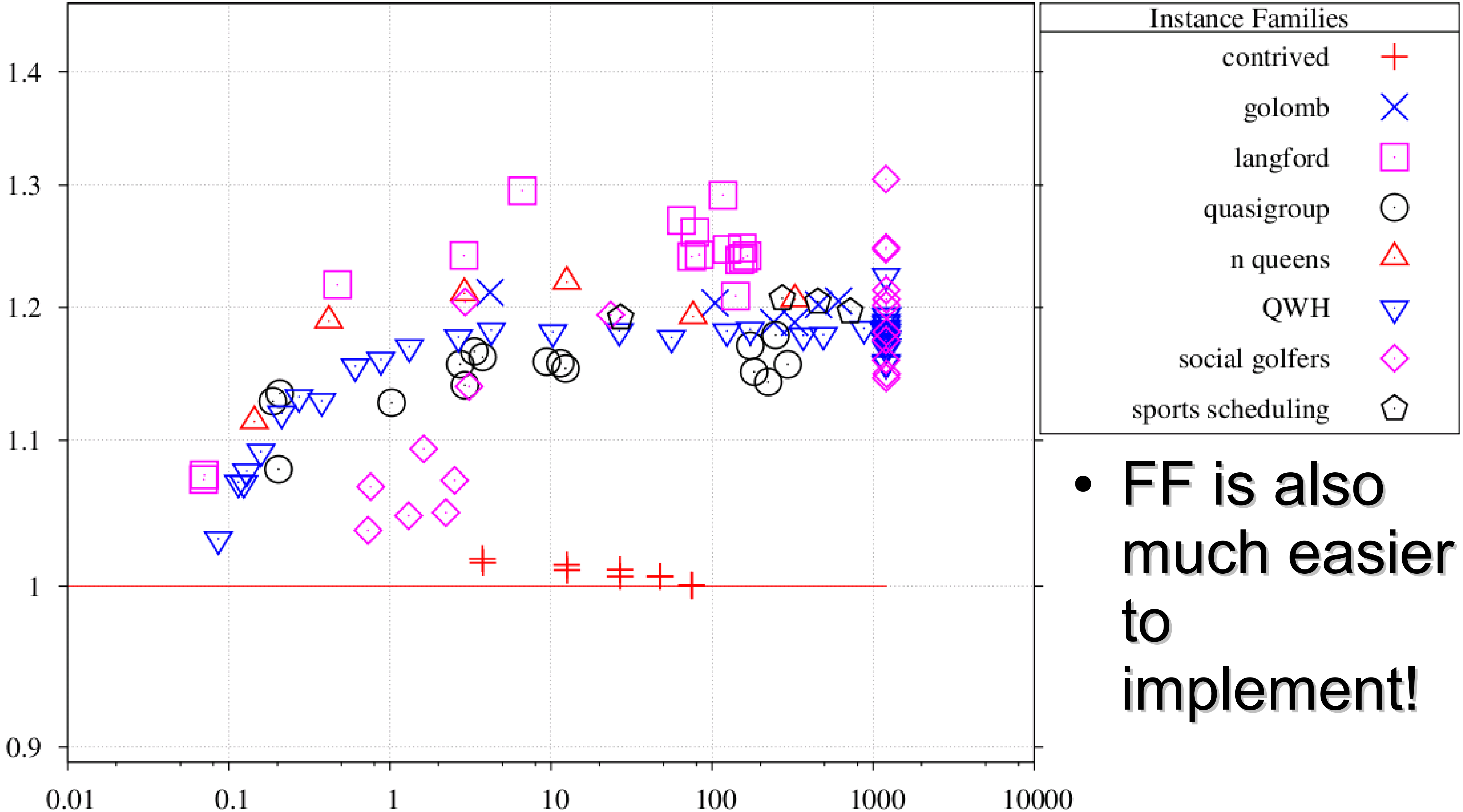
Priority Queue



Incremental Matching

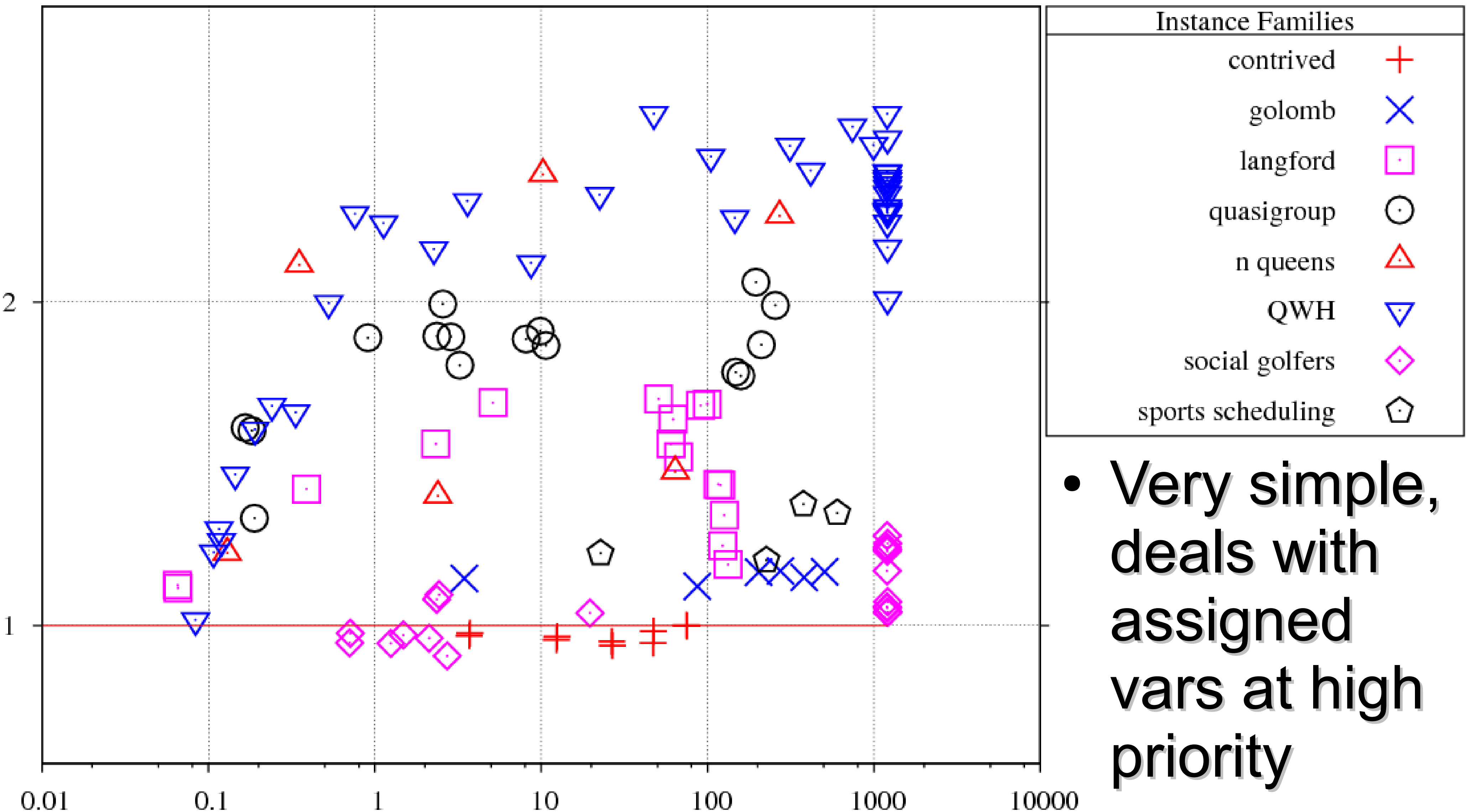


FF-BFS vs HK



- FF is also much easier to implement!

Staged propagation



- Very simple, deals with assigned vars at high priority

Triggering

- Trigger only on relevant values (Dynamic Triggers)
  - It is not necessary to trigger on all domain removals
  - Identify $t \leq 2r+d$ trigger values from rd
  - Doesn't work on our instances!
  - Ratio not low enough

Triggering

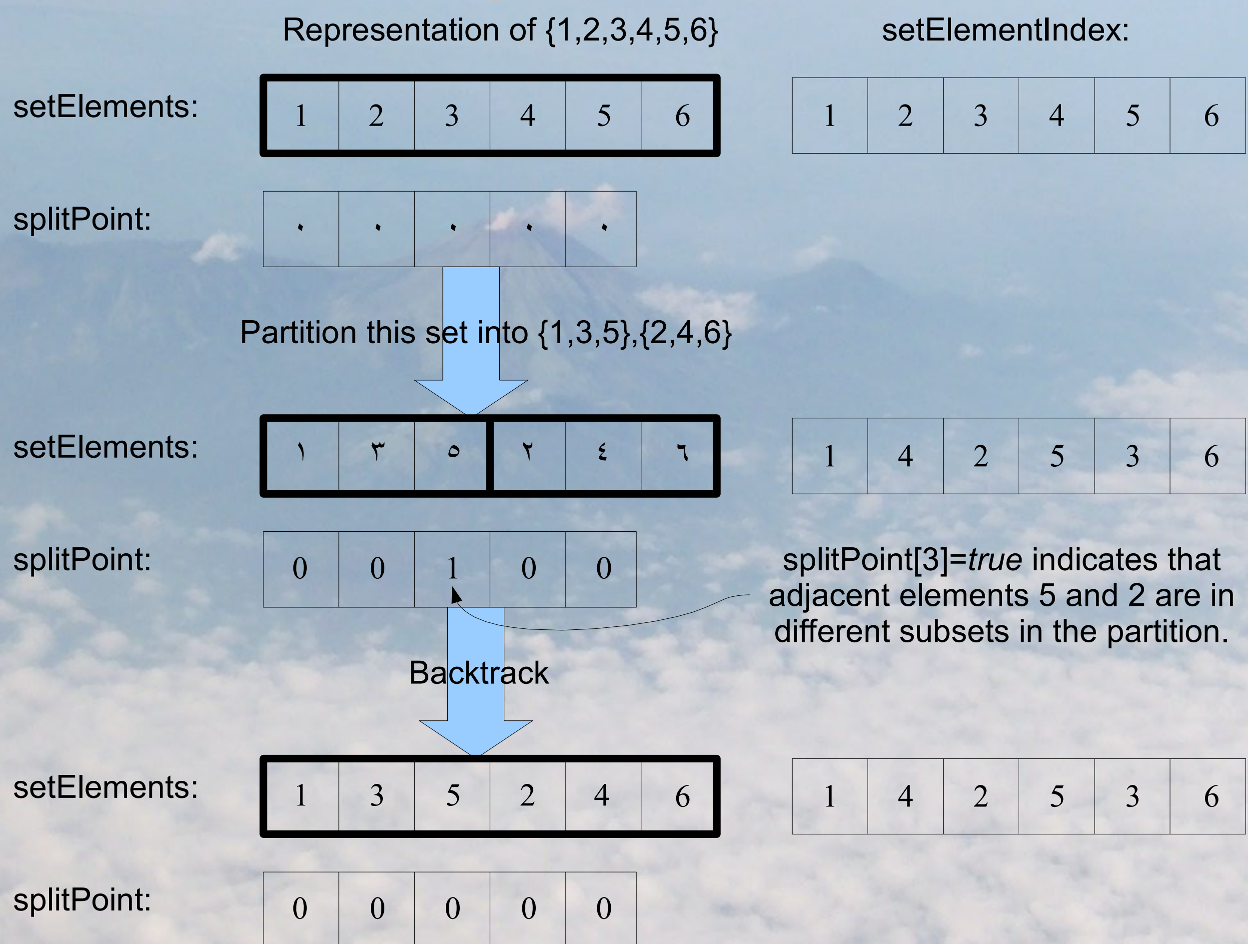
- Domain counting (Lagerkvist & Schulte, variant of Quimper & Walsh)
  - Only trigger when domain size less than r
  - Very cheap but has almost no effect
- Fixpoint reasoning and advisors
  - No claim in original papers that these are useful for AllDifferent
  - DT results suggest fixpoint reasoning is useless
  - We have something like advisors (although more general) – the variable event queue!

Partitioning the constraint

- Partition by SCCs
  - Each SCC corresponds to an independent sub-constraint
  - Store and re-use this partition (of the variables)
  - Run expensive algorithm only on sub-constraint

Partitioning the constraint

- Small incremental data structure which backtracks efficiently



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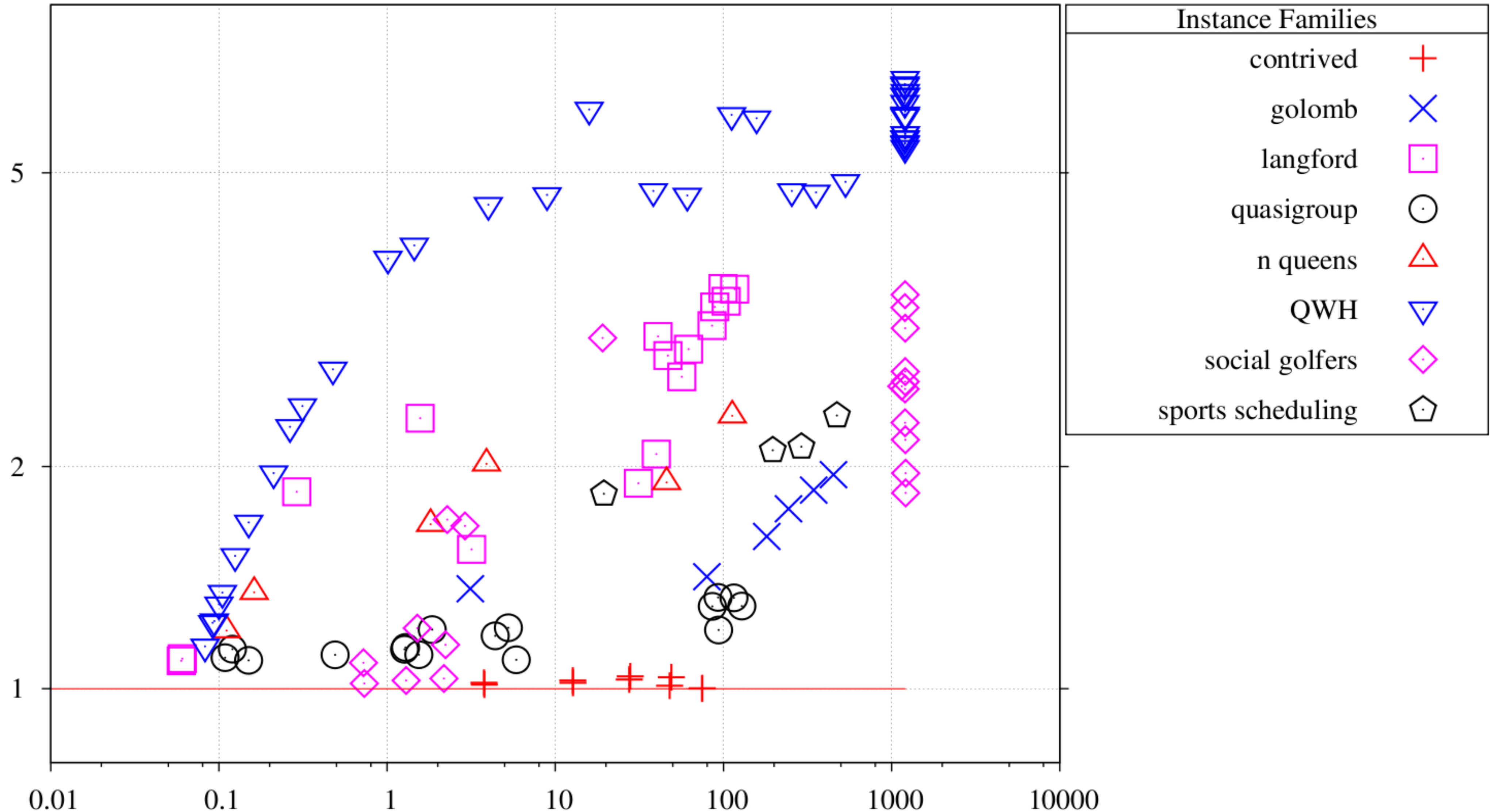


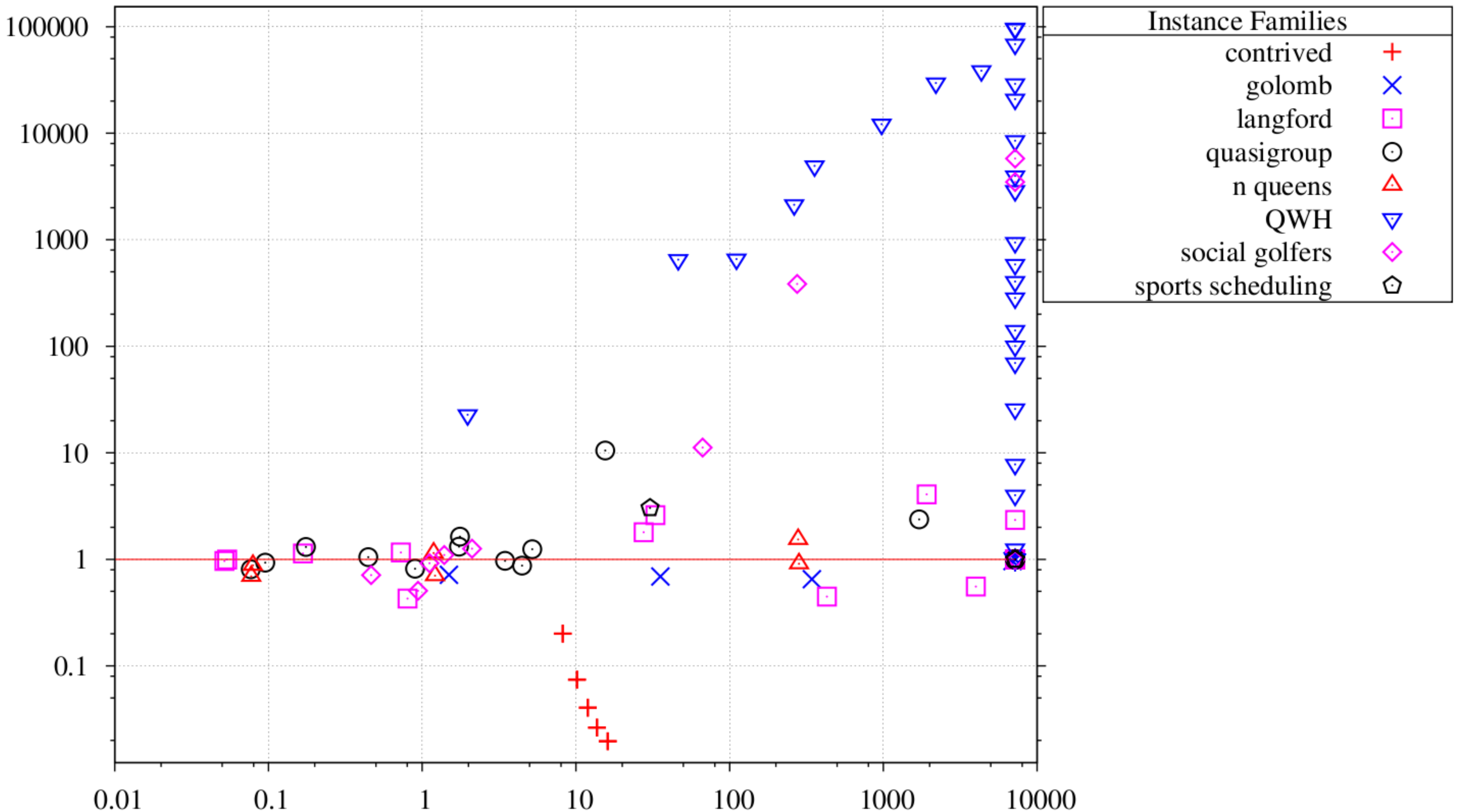
Partitioning the constraint

- Worth considering for other large constraints
  - GAC GCC partitions in the same way
  - Graph connectivity partitions when you find a 'bridge'
  - Sequence constraint?
  - Regular/Slide partition when variables are assigned in middle

Pairwise AllDifferent

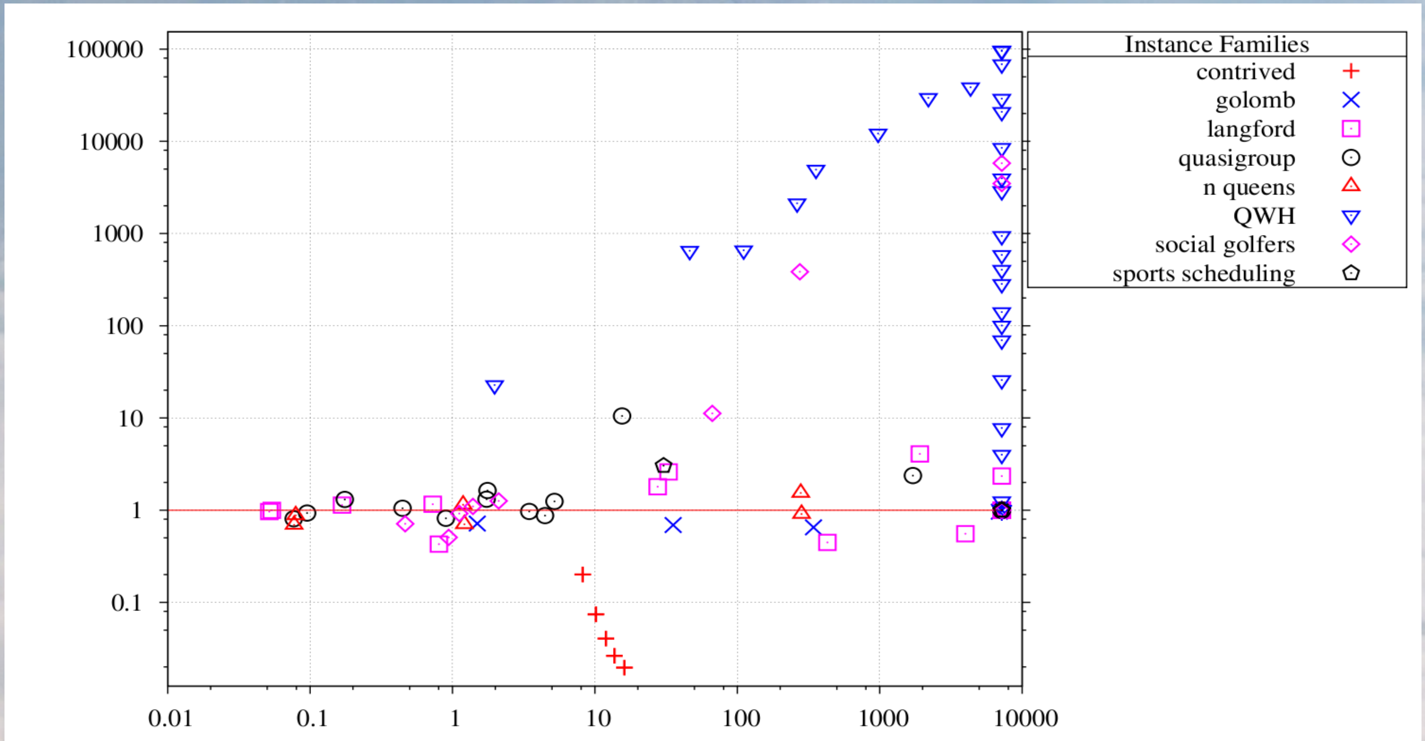
- Trigger only on assignment of a variable
- Remove assigned value from all other variables
- Extremely cheap
- Equivalent to AC on pairwise not-equal constraints
- This is no straw man!

Comparing to Pairwise



Comparing to Pairwise

- GAC AllDifferent never slows down search by more than 2.34 times
- Can be 100,000 times faster
- Most AllDifferent constraints here are tight



Modelling with AllDifferent

- Golomb Ruler
  - Triangular table representing all pairs
  - One AllDifferent constraint
  - Optimization tightens AllDiff
  - Implied constraints

```
Ruler:  0     A     B     C      D   ... (monotonic)
Diffs:  A     B     C     D   ...
        B-A   C-A   D-A   ...
        C-B   D-B   ...
        D-C   ...
        ...
```

AllDifferent

Modelling with AllDifferent

- Langford's problem with 2 instances of each number
  - Model due to Rendl
  - Permutation
  - Represent the indices rather than the actual Langford sequence

```
For Langford sequence of length n with n/2 numbers.
Pos[1..n] -- AllDifferent
Pos[1]+2=Pos[11] // first instance of number 1 is distance
                 // 1 from the second instance of 1
...
```

Modelling with AllDifferent

- Quasigroup and QWH
  - Similar to Sudoku (without the sub squares)
  - n x n matrix of variables with domain 1..n
  - AllDifferent on each row and each column
  - QWH has some values filled in already
    - Well known to show off GAC AllDifferent
  - Quasigroup has various properties (e.g. associativity, idempotence)
    - Colton & Miguel's model and implied constraints

Modelling with AllDifferent

- N Queens problem
  - Model 1
    - Three vectors representing queen position in row, the number of the leading diagonal, and the number of the secondary diagonal
    - These vectors are all different
  - Model 2
    - One vector representing queen position in row (all different)
    - Constraints to forbid diagonals
    - Tailor creates 30 auxiliary variables for n=16

Modelling with AllDifferent

- Sports scheduling
  - Two viewpoints
    - For each week, a vector of the teams (all different)
    - Vector of games (all different)
    - Channelling constraints between the two (table)
    - Symmetry breaking constraints (< for each game, lex on weeks, lex on stadiums)
    - Stadium constraints (each team plays no more than twice in one stadium)

	Stadium 1		Stadium 2		
Week 1:	1	3	2	4	...

Modelling with AllDifferent

- Social Golfers
  - Very similar to sports scheduling
  - Two viewpoints
    - For each week, a vector of the golfers (all different)
    - Vector of pairs who played together (all different but not necessarily a permutation)
    - Channelling constraints between the two (table)
    - Symmetry breaking constraints (< within the groups, lex on weeks, lex between groups)

Week 1:	1	2	4	5	3	6	7	9	...

Modelling with AllDifferent

- As you can see, AllDifferent is widely used! 7 example problems.
- The AllDifferent is tight in all examples
  - In a lot of cases it is worth doing GAC, but not all
  - I think it does depend on tightness, but also on other constraints surrounding the AllDifferent
  - I refuse to offer any advice!

Conclusions

- A bag of useful tricks from the literature
- One new trick which worked: partitioning the constraint
  - Perhaps this is general!
- One new trick which didn't: dynamic triggers from SCC algorithm
- The only modelling advice is to try a couple of different propagators!

Thank You

- Any Questions?